EMBEDDINGS OF 4–MANIFOLDS IN $\mathbb{C}P^3$

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Abstract. In this article we show that every closed orientable smooth 4–manifold admits a smooth embedding in the complex projective 3–space. We also provide a new proof of embeddings of 4–manifolds in $\mathbb{R}^7$.

1. Introduction

A basic question in the field of geometric topology which concerns embeddings of manifolds, can be stated as follows: Given a pair of manifolds M and N, how many smooth embeddings of M exist in N?

Detailed investigations in this regard have led to the discovery of interesting invariants of manifolds. One of the earliest seminal results in this context is due to H. Whitney who showed that every closed manifold of dimension n admits an embedding in $\mathbb{R}^{2n}$. Subsequently, this result has been extensively generalized. Most notably, M. Hirsch showed [15] that every closed orientable odd–dimensional manifold M^{2n-1} admits a smooth embedding in $\mathbb{R}^{4n-3}$. This result, together with those by C.T.C Wall and V. Rokhlin implies that every closed 3–manifold admits an embedding in $\mathbb{R}^5$.

For closed even dimensional manifolds, combining results of A. Haefliger [14], A. Haefliger and M. Hirsch [17], and W. Massey and F. Peterson [21], one knows that every orientable n–manifold embeds in $\mathbb{R}^{2n-1}$ when $n > 4$, and if n is not a power of two, then every n-manifold embeds in $\mathbb{R}^{2n-1}$. For 4–manifolds it was shown by M. Hirsch [16] and C. T. C. Wall[1] that every orientable PL 4–manifold admits a PL embedding in $\mathbb{R}^7$.

It is usually possible to construct an invariant of a manifold M using its embeddings in a manifold N, provided that (1) the topology of N is relatively simple and (2) the co-dimension of the embedding of M in N is small. The importance of these two conditions is evident even from the examples of embeddings of surfaces. We recall that there exists an embedding of a closed smooth surface Σ in $\mathbb{R}^3$ if and only if Σ is orientable. This clearly shows that the orientability of a smooth closed surface can be captured by its embeddability in Euclidean 3–space. Further, the embeddability of every closed surface in $\mathbb{R}^4$ demonstrates the importance of lower co-dimension of embeddings, while the fact that $\mathbb{R}P^3 \# \mathbb{R}P^3$ admits an embedding of every closed surface shows the need for a relatively simple topology for the target space.

It was shown by S. Cappell and J. Shaneson [7] that a smooth 4–manifold admits a smooth embedding in $\mathbb{R}^6$ if and only if it admits a spin structure. We know that a closed orientable 4–manifold is spin if and only if the second Stiefel-Whitney class $w_2(M)$ is zero. In particular, this implies that $\mathbb{C}P^2$ does not smoothly embed in $\mathbb{R}^6$. In this article, we investigate whether there exist topologically simple closed 6–dimensional manifolds which admit embeddings of all smooth 4–manifolds.

Two important classes of closed orientable smooth 4–manifolds are symplectic 4–manifolds and smooth algebraic surfaces. Their embeddings in various complex projective spaces have been extensively examined (see, for instance [2],[9], and [10]), and the question of their embeddability in $\mathbb{C}P^3$ is very important. Furthermore, the topology of $\mathbb{C}P^3$ is very simple and $\mathbb{C}P^2$ naturally embeds in $\mathbb{C}P^3$. We therefore investigate embeddings of 4–manifolds in $\mathbb{C}P^3$ and establish the following:

Theorem 1.1. *Every closed orientable smooth 4–manifold admits a smooth embedding in $\mathbb{C}P^3$.*

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[1] M. Hirsch has mentioned in [16] that C. T. C. Wall had independently proved this result.

To the best of our knowledge, Theorem 1.1 and Theorem 5.4, which establishes embedding of 4–manifolds in certain 6–manifolds of the type $N \times \mathbb{C}P^1$, are the only results establishing the existence of closed 6–manifolds in which all orientable smooth 4–manifolds embed.

The central idea for the proof of Theorem 1.1 is drawn from a well–known fact that given a projective embedding of a smooth algebraic surface, the standard *Lefschetz pencil* of the complex projective space generically induces a Lefschetz pencil structure on the surface. It was established by I. Baykur and O. Saeki [5] that every smooth 4–manifold admits a *simplified broken Lefschetz fibration* (SBLF), which can be regarded as a natural generalization of the Lefschetz pencil for an arbitrary smooth 4–manifold. This decomposition allows us to express any smooth 4–manifold as a singular fiber bundle over $\mathbb{C}P^1$ with a finite number of *Lefschetz singularities* and a unique *fold singularity.* The advantage of this decomposition is that we can associate with any smooth 4–manifold certain data which comprise two constituents. These are: (1) an element of the *mapping class group* of a closed orientable surface of genus g expressed as a product of *Dehn twists*, corresponding to Lefschetz singularities, and (2) a round handle attachment corresponding to the fold singularity.

Let us now briefly outline the argument establishing Theorem 1.1. We need Theorem 5.4 to establish Theorem 1.1. Hence, we begin by outlining a proof of Theorem 5.4.

Consider any closed orientable 4–manifold N which admits an embedding of a Hopf link which is *separable* in the sense of Definition 4.3, by which we mean that N admits a handle decomposition that satisfies the following property: the boundary of a 0–handle has a Hopf link, which is a slice in the complement of the 0–handle. In the following discussion we fix one such 4–manifold N.

Given a closed orientable smooth 4–manifold M, consider the manifold M together with any given SBLF. The first step is to produce an embedding f of M in $N \times \mathbb{C}P^1$ such that the trivial fibration $\pi_2 : N \times \mathbb{C}P^1 \to \mathbb{C}P^1$ of $N \times \mathbb{C}P^1$ induces the given SBLF.

The three important steps for constructing the embedding f are the following: In the first step, using an appropriate generalization of techniques from [22], and a specific local embedding model for a given Lefschetz singularity, we provide an embedding of genus $g + 1$ *Lefschetz sub–fibration* over a disk $\mathbb{D}^2$ in $N \times \mathbb{D}^2$, which is associated with the given SBLF. This embedding is such that the trivial product fibration $\pi_2 : N \times \mathbb{D}^2 \to \mathbb{D}^2$ induces the given *Lefschetz fibration.* This is the most important step in the proof, and is detailed in Section 4. In fact, in Section 4 we show how to embed any Lefschetz fibration over a disk or $\mathbb{C}P^1$ in a trivial fibration over $\mathbb{C}P^1$ with fiber N.

Next, we use a local embedding model for fold singularities to produce an embedding of a sub-manifold $(\widetilde{M}, \partial\widetilde{M}) \subset M$ (having two disjoint boundary components) in $N \times I \times \mathbb{S}^1$. This embedding is constructed such that it agrees with the embedding in the first step near one of the boundary components of $\widetilde{M}$, and is a trivial fibration $\Sigma_g \times S^1$ near the other boundary component of $\widetilde{M}$. Here, Σ_g denotes a surface of genus g. This provides us with a fiber preserving embedding of $M \setminus \Sigma_g \times \mathbb{D}^2$ in $N \times \mathbb{D}^2$. Finally, we extend the embedding of $M \setminus \Sigma_g \times \mathbb{D}^2$ in $N \times \mathbb{D}^2$ using an embedding of $\Sigma_g \times \mathbb{D}^2$ in $N \times \mathbb{D}^2$ to obtain the embedding $f : M \hookrightarrow N \times \mathbb{C}P^1$. These two steps are discussed in Section 5. Embeddings of M in $N \times \mathbb{C}P^1$ is the content of Theorem 5.4. Theorem 5.4 immediately implies Theorem 6.1 which establishes embeddings of smooth closed orientable 4–manifolds in $\mathbb{R}^7$.

Having outlined a proof of Theorem 5.4, let us now discuss how to establish embeddings of 4–manifolds in $\mathbb{C}P^3$ as claimed in Theorem 1.1. Given a smooth, orientable, closed 4–manifold, we first consider the manifold $M\#\mathbb{C}P^2\#\overline{\mathbb{C}P^2}$ together with a specific SBLF. Next, we notice that the the *blow-up* of $\mathbb{C}P^3$ along $\mathbb{C}P^1$ is a fiber bundle over $\mathbb{C}P^1$ with fiber $\mathbb{C}P^2$ with the property that the fiber bundle is trivial in the complement of the *exceptional divisor.*

We embed $M\#\mathbb{C}P^2\#\overline{\mathbb{C}P^2}$ the blow-up of $\mathbb{C}P^3$ using this specific SBLF by observing that $\mathbb{C}P^2$ admits a separable Hopf link, and hence a slight generalization of the argument necessary to establish Theorem 5.4 allows us to embed $M\#\mathbb{C}P^2\#\overline{\mathbb{C}P^2}$ Next, we note that the *blow-up* of $\mathbb{C}P^3$. Further, by ensure certain intersection property of the fiber of the specific SLBF , we ensure that the embedding of $M\#\mathbb{C}P^2\#\overline{\mathbb{C}P^2}$ constructed is such that when we *blow-down* the blow-up of $\mathbb{C}P^3$, we produce a $\mathbb{C}P^3$ that has M as its embedded sub-manifold. The construction of the specific SBLF, blow-up and blow-down procedures, and the proof of Theorem 1.1 are discussed in the final section.

The mathematical preliminaries to carry out these steps are given in Sections 2 and 3. In particular, we discuss relevant aspects of *broken Lefschetz fibrations* in Section 2, and of mapping class groups in Section 3.

Finally, a few remarks on conventions used in this article. By a manifold we mean a compact orientable manifold with or without boundary. We denote manifolds by capital letters M, N, etc. When we need to emphasis that we are working with a manifold with boundary, we use the notation $(M, \partial M)$ consisting of the pair M and the boundary ∂M of M. As usual, the notation Σ or Σ_g is used for denoting a closed orientable surface, with g indicating the genus.

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2. Review of Broken Lefschetz fibrations

Broken Lefschetz fibrations (BLF) were introduced by D. Auroux, S. K. Donaldson, and L. Katzarkov in [1]. These are generalized Lefschetz fibrations. I. Baykur [3] established that every smooth orientable 4–manifold admits a broken Lefschetz fibration. The purpose of this section is to review few definitions and result related to BLF. We refer to [3] and [5] for a detailed discussion on BLF. Let us begin by recalling the definition of Lefschetz singularity.

Definition 2.1 (Lefschetz singularity)**.** Let M be an oriented 4–manifold and Σ an oriented surface. Let $f : M \to \Sigma$ be a smooth map. A point $x \in M$ is said to have a Lefschetz singularity at x for the map f, provided that there is an orientation preserving parameterization $\phi : U \subset M \to \mathbb{C}^2$, and an orientation preserving parameterization $\psi : V \subset \Sigma \to \mathbb{C}$ such that the following properties are satisfied:

(1) $x \in U$, and $\phi(x) = (0, 0) \in \mathbb{C}^2$.
(2) $f(x) \in V$, and $\psi(f(x)) = 0 \in \mathbb{C}$.
(3) For the map $g : \mathbb{C}^2 \to \mathbb{C}$ given by $g(z_1, z_2) = z_1.z_2$, the following diagram commutes:

$$\begin{array}{ccc} U & \xrightarrow{\ \phi\ } & \mathbb{C}^2 \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle g} \\ V & \xrightarrow{\ \psi\ } & \mathbb{C}. \end{array}$$

Remark 2.2.

(a) Observe that both M as well as Σ can have non-empty boundary, however, it follows from Definition 2.1 that the critical point c belongs to the interior $\mathring{M}$ of M, and $f(c) \in \mathring{\Sigma}$.
(b) In case we do not put any condition regarding preservation of orientations by the parameterization around x and $f(x)$ in Definition 2.1 above, the singularity is termed as *achiral Lefschetz singularity.*
(c) Let $f : M \to S$ be a map with an isolated Lefschetz singularity at $c \in M$. It is well known that the fiber over $f(c)$ is obtained by pinching a simple closed curve γ on nearby smooth fiber Σ_g to a point. The curve γ is known as a *vanishing cycle.*
(d) If we take a small closed disk $\mathbb{D}$ around $f(c)$ not containing any other critical value, then the $f^{-1}(\partial\mathbb{D})$ is a mapping torus over the smooth fiber Σ_g with monodromy a positive Dehn twist along the vanishing cycle γ. In case of an achiral Lefschetz singularity, the monodromy could be a positive or a negative Dehn twist along γ.

Next, we recall the definition of 1*–fold singularity.*

Definition 2.3 (1–fold Singularity)**.** Let M be an oriented 4–manifold, and let Σ be an oriented surface. Let $f : M \to \Sigma$ be a smooth surjective map. A point $x \in M$ is said to have a 1–fold singularity at x

provided there is an orientation preserving parameterization $\phi : U \subset M \to \mathbb{R}^4$, and an orientation preserving parameterization $\psi : V \subset \Sigma \to \mathbb{R}^2$ such that the following properties are satisfied:

(1) $x \in U$, and $\phi(x) = (0,0,0,0) \in \mathbb{R}^4$.
(2) $f(x) \in V$, and $\psi(f(x)) = (0,0) \in \mathbb{R}^2$.
(3) For the map $h : \mathbb{R}^4 \to \mathbb{R}^2$ given by $h(t, x_1, x_2, x_3) = (t, -x_1^2 + x_2^2 + x_3^2)$, the following diagram commutes:

$$\begin{array}{ccc} U & \xrightarrow{\phi} & \mathbb{R}^4 \\ \downarrow f & & \downarrow h \\ V & \xrightarrow{\psi} & \mathbb{R}^2. \end{array}$$

Remark 2.4.

(a) If a map $f : M \to \Sigma$ has a 1–fold singularity at x, then $x \in \mathring{M}$, and $f(x) \in \mathring{\Sigma}$.
(b) When the map h in the definition of 1-fold singularity is allowed to have the local model:

$$(t, x_1, x_2, x_3) \to (t, \pm x_1^2 \pm x_2^2 \pm x_3^2),$$

the singularity is termed as a *fold singularity.* In this article, we will only need the local model around 1–fold singularity.
(c) A local singularity model for a smooth function of the form:

$$(t, x_1, x_2, x_3) \to (t, x_1^3 + tx_1 \pm x_2^2 \pm x_3^2)$$

is known as a *cusp singularity.*

We are now in a position to recall the notion of a broken Lefschetz fibration (BLF).

Definition 2.5 (Broken Lefschetz fibration)**.** Let M a smooth oriented 4–manifold. By a broken Lefschetz fibration of M we mean a smooth map $f : M \to \mathbb{C}P^1$ such that f has only 1-fold or Lefschetz singularity.

Remark 2.6.

(a) Given a BLF $f : M \to \mathbb{C}P^1$, the inverse image $f^{-1}(y)$ for any regular value y is called a fiber of BLF.
(b) Generically, the image set of a 1–fold singularity on Σ is an immersed circle in $\mathring{\Sigma}$.

A BLF without 1–fold singularity is called a Lefschetz fibration. These singular fibrations are extremely useful in algebraic geometry [12] and symplectic geometry [10]. Let us now formally define a Lefschetz fibration.

Definition 2.7 (Lefschetz fibration)**.** Let M be a smooth oriented 4–manifold. A smooth map $f : M \to \Sigma$, where Σ is an oriented surface, having its singular points modeled only on Lefschetz singularities is called a Lefschetz fibration of M.

Remark 2.8.

(a) Unlike a fiber bundle or Lefschetz fibration, the fibers of a BLF are typically not diffeomorphic. In fact, the 1-fold singularity in the definition of BLF corresponds to a round 1–handle attachment [11, 5]. Hence, if BLF has points having fold singularity, then the genus of fibers change as we cross the image of an immersed circle coming from a 1–fold singularity.
(b) The fibers of BLF need not be connected. However, it can be shown that every 4–manifold admits a BLF with connected fibers having genus at least 2. This follows from [3, Theorem 1.1].

Observe that a BLF provides us a decomposition of a smooth manifold into simple pieces. A more simplified form of this decomposition of smooth 4–manifold is what we will need for this article. This simplification was introduced by I. Baykur and O. Saeki in [5]. This decomposition is known as a simplified broken Lefschetz fibration. Let us recall the definition of this:

Definition 2.9 (Simplified broken Lefschetz fibration (SBLF))**.** Let $f : M \to \mathbb{C}P^1$ be a BLF. We say that this BLF is a simplified broken Lefschetz fibration (SBLF) provided the function f satisfies the following additional properties:

(1) The set Z_f of all $x \in M$ admitting a 1-fold singularity model is connected.
(2) All fibers are connected.
(3) The map f is injective when restricted to Z_f as well as when restricted to the set, C_f, of Lefschetz singular points.

The definition of SBLF motivates the definition of the following:

Definition 2.10 (Simplified Lefschetz fibration (SLF))**.** Let M be a smooth oriented 4–manifold. Let Σ be $\mathbb{C}P^1$ or a 2–disk $\mathbb{D}^2$. A Lefschetz fibration $f : M \to \Sigma$ is said to be simplified Lefschetz fibration provided all the critical values of f in Σ are isolated, and for any regular value $y \in \Sigma$, the fiber $f^{-1}(y)$ is connected.

Remark 2.11.

(a) A SBLF having no fold singularity is a SLF.
(b) Observe that the definition of SBLF implies that there exists a disk $\mathcal{D}$ contained in $\mathbb{C}P^1$ such that every $y \in \mathcal{D}$ is a regular value, and the genus of the fiber over y is minimum among all fibers of SBLF. We call this fiber *lower genus fiber*.
(c) Topologically, the unique 1–fold singularity of SBLF corresponds to adding 1–handle to a circle worth of lower genus fibers over $\partial\mathcal{D}$. This corresponds to an attachment of a round 1–handle to $f^{-1}(\mathcal{D})$ such that a generic fiber of SBLF over $\mathbb{C}P^1 \setminus \overline{\mathcal{D}}$ has genus one more than the fibers over $\mathcal{D}$.

In [5], it was shown that every orientable smooth 4–manifold admits a SBLF.

Theorem 2.12 (I. Baykur, O. Saeki: Theorem-1 [5])**.** *Given any generic map from a closed, connected, oriented, smooth 4-manifold X to $\mathbb{C}P^1$, there are explicit algorithms to modify it to a SLBF. In particular, every closed orientable smooth* 4*–manifold admits a SBLF. Furthermore, we can always construct a SLBF on M such that the genus of lower genus figure is bigger than* 1.

We would like to point out that Theorem 2.12 is not stated as above in [5]. The statement regarding the lower bound on the genus of a lower genus fiber is not explicitly mentioned in [5, Theorem-1]. However, it follows from the application of [5, Theorem-1] followed by [5, Theorem-2]. For the sake of completeness, we discuss the proof of Theorem 2.12.

Proof. To begin with, recall that by a *trisection* of a smooth orientable closed 4–manifold M, one means a decomposition of M into three 4–dimensional handle-bodies (thickening of a wedge of circles), meeting pairwise in 3–dimensional handle-bodies, and all three 4–dimensional handle-bodies intersect in a surface. Trisections correspond to a Morse 2–function on M. If k' is the number of indefinite folds for the Morse 2–function associated to a given trisection, and g' is the genus of the surface corresponding to the common intersections of three 4–dimensional handle-bodies, then one says that the 4–manifold has a (g', k')–trisection.

In order to produce a SBLF as stated in Theorem 2.12, we observe that given M, according to [5, Theorem-1], there exists a SBLF $f : M \to \mathbb{C}P^1$. Let g be the genus of lower genus fiber of the SLBF. If $g > 1$, then we are through. In case, $g \leq 1$, we apply [5, Theorem-2] to produce a (g', k')–trisection from the given SLBF $f : M \to \mathbb{C}P^1$. According to [5, Theorem-2], we get a (g', k')–trisection with $g' \geq 1$.

Next, we again apply the second part of [5, Theorem-2] to produce from this trisection a new SBLF. Observe that according to [5, Theorem-2], the new SBLF has lower genus fiber having its genus $g' + 2$. Since $g \geq 0$, the Theorem follows. □

3. Mapping class groups of surfaces

In this section we review some results related to mapping class groups of closed orientable surfaces. Good references for the results discussed here are [4] and [19]. Let us begin by recalling the definition of the mapping class group.

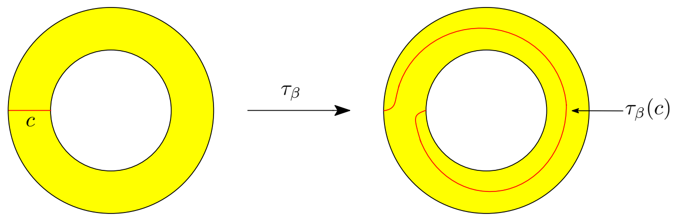

FIGURE 1. The figure is a pictorial description of the Dehn twist τ_β restricted to the neighborhood $\mathcal{A}(\beta) = \mathbb{S}^1 \times [0, 2\pi]$. τ_β is given by $\tau_\beta(\theta, t) = (\theta + t, t)$ when restricted to $\mathcal{A}(\beta)$. It sends the arc c – depicted as a red colored arc in the picture on the left of the figure – to the arc $\tau_\beta(c)$ depicted in the picture on the right of the figure.

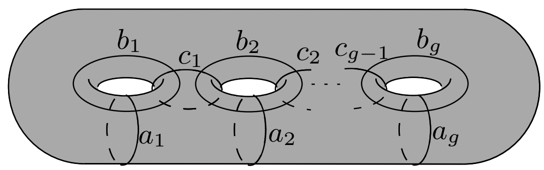

FIGURE 2. Dehn twists along curves a_i's ,b_j's and c_k's generate the mapping class group of an orientable genus g surface.

Definition 3.1 (Mapping class group)**.** Let Σ be a closed orientable surface. By the mapping class group of Σ, we mean the group of orientation preserving self diffeomorphisms of Σ up to isotopy.

We denote the mapping class group of a surface Σ by $\mathcal{MCG}(\Sigma)$. Next, let us discuss the notion of a *Dehn twist* along a simple closed curve embedded in a surface Σ. We refer [4] for a more detailed discussion on Dehn twists.

Definition 3.2 (Dehn twist)**.** Let Σ be an orientable surface. Let β be a simple closed curve embedded in the interior of Σ. By a Dehn twist along β, we mean a diffeomorphism which is identity outside an annulus neighborhood $\mathcal{A}(\beta)$ of β in Σ, and is given by τ_β on $\mathcal{A}(\beta)$ when restricted to $\mathcal{A}(\beta)$, where τ_β is the diffeomorphism of $\mathcal{A}(\beta)$ described in Figure 1.

M. Dehn [8] and W. Lickorish [19] independently established that the mapping class group of an orientable genus g surface Σ_g is generated by Dehn twists along simple closed curves embedded in Σ_g. W. Lickorish further strengthened this result in [20], to show that the mapping class group of a closed orientable surface Σ_g is generated by Dehn twists along the curves a_i's , b_j's and c_k's as depicted in Figure 2. Following [22], we will call these curves as *Lickorish generators*.

We end this section with a proposition which is a consequence of Lemma-3 established in [19]. In order to state this proposition we need a few terminologies from [19].

Let us regard an orientable surface Σ_g of genus g as the boundary of a standard handle-body H_g. Here, a standard handlebody H_g consists of g 1–handles attached the unit 3–ball in $\mathbb{R}^3$ as depicted in Figure 3.

Consider a typical handle H_k, as shown in Figure 3. Following [19], we say that a simple closed curve p *does not meet* the handle H_k provided it does not intersect the curve a_k depicted Figure 3.

Proposition 3.3 (Lickorish: Lemma-3 [19])**.** *Let p be any simple closed curve on Σ_g. There exists a diffeomorphism $\phi : \Sigma_g \to \Sigma_g$ such that $\phi(p)$ does not meet any handle of Σ_g.*

4. Lefschetz fibration embedding

Recall from Definition 2.10 that a Lefschetz fibration $(M, \pi : M \to \Sigma)$, where Σ is either a disk or $\mathbb{C}P^1$, is a SLF, provided that critical values are isolated and fibers are connected. In this section, we show that there exists an embedding of any SLF into certain manifolds of the type $(N^4 \times \mathbb{C}P^1, \pi_2)$, which is fiber preserving

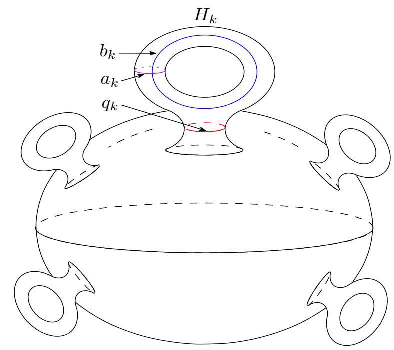

FIGURE 3. The figure shows surface of genus g embedded in $\mathbb{R}^3$ as a boundary of a genus g handle-body considered as a unit ball with g 1–handles attached to it.

in the sense defined in Definition 4.7. This result [Theorem 4.8], can be regarded as the first step towards establishing Theorem 1.1.

4.1. **Flexible embedding in standard position.**

Let us begin this sub-section by reviewing the notion of *flexible embedding.*

Definition 4.1 (Flexible embedding)**.** Let M be an orientable closed smooth manifold. A smooth embedding $\phi : \Sigma_g \hookrightarrow M$ of a closed orientable surface Σ_g is said to be flexible provided for every $f \in \mathcal{MCG}(\Sigma_g)$ there exists a diffeomorphism ψ of M isotopic to the identity which maps Σ_g to itself and satisfies $\phi^{-1} \circ \psi \circ \phi = f$.

Next, we state a lemma regarding a flexible embedding of any surface of genus g into a 4–manifold N, which admits a separable Hopf link. In order to state this lemma, we need to introduce the following definitions:

Definition 4.2 (Embedding in standard position)**.** An embedding $\phi : \Sigma_g \hookrightarrow N$ of a surface Σ_g is said to be in a standard position provided the following properties are satisfied:

(1) Every simple closed curve γ on $\phi(\Sigma)$ is a boundary of a 2–disk $\mathbb{D}^2$ intersecting $\phi(\Sigma_g)$ only in γ.
(2) There exists a tubular neighborhood $\mathcal{N}(\mathbb{D})$ of the disk $\mathbb{D}^2$ having the boundary γ such that $\mathcal{N}(\mathbb{D})$ is the image of a coordinate chart $\phi_\gamma : \mathbb{C}^2 \to \mathcal{N}(\mathbb{D})$ satisfying the following:
$\phi_\gamma^{-1}(\phi(\Sigma_g) \cap \mathcal{N}(\mathbb{D}))$ is $g^{-1}(1)$, where $g : \mathbb{C}^2 \to \mathbb{C}$ is the polynomial map $g(z_1, z_2) = z_1.z_2$.

Definition 4.3 (Separable Hopf link)**.** We say that a link $l_1 \sqcup l_2$ in a 4–manifold N is a separable Hopf link provided following properties are satisfied:

(1) There exist an embedding of a 4–ball $\mathbb{D}^4 = \mathbb{D}^2 \times \mathbb{D}^2$ in N such that $\partial\mathbb{D}^2 \times \{0\} \sqcup \{0\} \times \partial\mathbb{D}^2 = l_1 \sqcup l_2$.
(2) There exists two disjoint properly embedded discs $\mathcal{D}_1$ and $\mathcal{D}_2$ in $N \setminus (\mathbb{D}^2 \times \mathbb{D}^2)^\circ$ such that $\partial\mathcal{D}_1 = l_1$ and $\partial\mathcal{D}_2 = l_2$.

Lemma 4.4. *Let N be a 4–manifold which admits a separable Hopf link. Then there exists an embedding ϕ of any closed orientable surface Σ_g of genus g in N which satisfies the following:*

(1) The embedding is flexible.
(2) The embedding is in a standard position.

Before, we establish this lemma, we would like to point out that the flexible embedding of Σ_g in N was first provided by S. Hirose and A. Yasuhara in [18]. Our main observation is that we can achieve the additional property of the embedding being in a standard position, provided that we use Proposition 3.3 established by Lickorish in [19] in conjunction with the techniques from [18].

Proof of Lemma 4.4. We want to construct an embedding of Σ_g which is both flexible and in a standard position. Let $l_1 \sqcup l_2$ be a separable Hopf link in N. Therefore there exists an embedded 4–ball $\mathbb{D}^4 = \mathbb{D}^2 \times \mathbb{D}^2$ in M such that $\partial\mathbb{D}^2 \times \{0\} \sqcup \{0\} \times \partial\mathbb{D}^2 = l_1 \sqcup l_2$ and there exists two disjoint properly embedded discs $\mathcal{D}_1$

and $\mathcal{D}_2$ in $N \setminus (\mathbb{D}^2 \times \mathbb{D}^2)^\circ$ such that $\partial\mathcal{D}_1 = l_1$ and $\partial\mathcal{D}_2 = l_2$. We regard a 4–ball $\mathbb{D}^4$ as the 4–ball $B^4(0,2)$ of radius 2 in $\mathbb{C}^2$ with its center at the origin. We will also regard $\mathbb{S}^3 \times [1,2]$ as the collar $B^4(0,2) \setminus B^4(0,1)$ contained in N.

Next, Observe that the link $l_1 \times \{\frac{3}{2}\} \sqcup l_2 \times \{\frac{3}{2}\}$ bounds a Hopf band say $\mathcal{H}$ in $\mathbb{S}^3 \times \{\frac{3}{2}\}$. We embed a genus g surface Σ_g in $\mathbb{S}^3 \times \{\frac{3}{2}\} \subset \mathbb{S}^3 \times [1,2] \subset N$ as the boundary of standard genus g handle body H_g and disjoint form $\mathcal{H}$ as depicted in Figure 3. Then we take ambient connected sum of embedded Σ_g and $\mathcal{H}$ in $\mathbb{S}^3 \times \{\frac{3}{2}\}$ to obtain a surface $\widehat{\Sigma_g}$ with two boundary components as shown in Figure 4. Thus by adding two cylinders $l_1 \sqcup l_2 \times [\frac{3}{2}, 2]$ and two disjoint disc $\mathcal{D}_1$,$\mathcal{D}_2$ to $\widehat{\Sigma_g}$, we obtain an embedding of genus g surface. Let us denote this embedding – after smoothing the corners – by ϕ. For a pictorial description of the embedding ϕ we refer the reader to Figure 4. We claim that the embedding $\phi : \Sigma_g \hookrightarrow N$ is both flexible and in standard position. Let us now establish this claim.

The claim that the embedding is flexible is already established in [18, Theorem: 3.1]. Let us briefly review the argument. First of all, notice that every Lickorish generator γ of Σ_g embedded in N via ϕ has – up to an isotopy – a Hopf annulus neighborhood which is contained in $\mathbb{S}^3 \times \{\frac{3}{2}\} \subset N$. Next, recall that the mapping class group of Σ_g is generated by Dehn twists along Lickorish generators, and in $\mathbb{S}^3$ there exists a diffeomorphism isotopic to the identity which induces a Dehn twist on a given Hopf annulus fixing its boundary point wise. In the proof of [18, Theorem: 3.1] it is shown that this implies that there exists a diffeomorphism of N isotopic to the identity which induces a Dehn twist along a Lickorish generator of $\phi(\Sigma_g)$. The claim now follows by successive application of ambient isotopies of N inducing a Dehn twists on Lickorish generators. See also [22] for the necessary details.

Let us now show that the embedding is in a standard position. First of all notice that, by very construction, any simple closed curve on $\phi(\Sigma_g)$ can be isotoped on the surface $\phi(\Sigma_g)$ so that it is contained in $\phi(\Sigma) \cap \mathbb{S}^3 \times \{\frac{3}{2}\}$. We claim that any Lickorish generator of $\phi(\Sigma_g)$ as well as any curve which does not meet handles[2] of $\phi(\Sigma_g)$ satisfy both the properties necessary for an embedding to be in a standard position. This is because:

(1) All curves mentioned in the claim are unknots in $\mathbb{S}^3 \times \{\frac{3}{2}\}$ hence they bound a disk in $\mathbb{S}^3 \times [1, \frac{3}{2}]$, that meets $\phi(\Sigma)$ only in the given curve.
(2) Any curve γ mentioned in the claim admits a neighborhood $\mathcal{N}(C)$ in $\phi(\Sigma_g)$ which is a Hopf band in $\mathbb{S}^3 \times \{\frac{3}{2}\}$.

It follows from both the properties listed above that any curve C, which is either a Lickorish generator or is not meeting any handle, satisfies both the properties necessary for a surface to be in the standard position.

Now, according to Proposition 3.3, given any curve C, there exists a diffeomorphism of $\phi(\Sigma_g)$ which send C to a curve which does not meet any handle. Since the embedding ϕ of Σ_g is flexible in N, given a curve c which not a Lickorish generator and meets some handles can be isotoped so that now it does not meet any handle. Hence, the claim that the embedding is also in a standard position follows.

□

In what follows we will work with embeddings of surfaces in N constructed using the procedure described in the proof of Lemma 4.4. We will used the term *standard embedding* for any such embedding. More precisely, we have the following:

Definition 4.5 (Standard embedding)**.** Let N be a manifold admitting a separable Hopf link. An embedding ψ of a closed orientable surface Σ_g which is isotopic to an embedding obtained following the procedure describe in the proof of Lemma 4.4 will be called a standard embedding of Σ_g

We end this sub-section by establishing an embedding result regarding embeddings of mapping tori in $N \times \mathbb{S}^1$. Recall that given a surface Σ, the mapping torus of Σ with monodromy g, with $g \in \mathcal{MCG}(\Sigma)$, is the quotient space $\Sigma \times [0,1]/$, where $(x,0)$ $(g(x),1)$. We will denote the mapping torus by $\mathcal{MT}(\Sigma, g)$. $\mathcal{MT}(\Sigma, g)$ is a fiber bundle over $\mathbb{S}^1$. Our next lemma establishes a fiber preserving embedding any mapping tours of Σ in $N \times \mathbb{S}^1$. More precisely,

[2] Recall that, we say that a simple closed curve p does not meet the handle H_k provided it does not intersect the curve a_k depicted Figure 3.

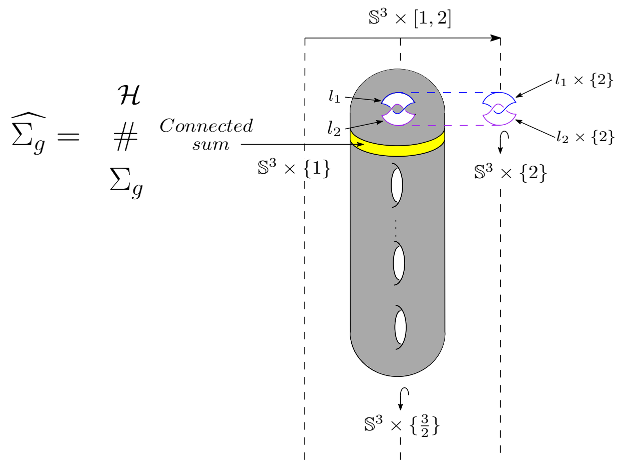

FIGURE 4. The figure depicts the embedding of the surface Σ_g which is flexible as well as in the standard position. Figure depicts the collar $\mathbb{S}^3 \times [1, 2] \subset N$ with dashed lines representing $\mathbb{S}^3$ at levels 1, 2 and $\frac{3}{2}$.

Lemma 4.6. *Let N be a manifold admitting a separable Hopf lank and let $\phi : \Sigma \hookrightarrow N$ be a standard embedding of Σ. Let g be an element of the mapping class group of Σ Let M be the mapping tours of Σ with monodromy g. There exists an embedding of M in $N \times \mathbb{S}^1$ which is fiber preserving.*

Proof. Since the embedding is standard, for $g \in \mathcal{M}CG(\Sigma)$ there exists a family f_t of diffeomorphisms of $\mathbb{N}$ with $f_0 = Id$ and f_1 restricted to Σ satisfies $\phi^{-1} \circ f_1 \circ \phi = g$. This implies $\mathcal{M}T(\Sigma, g)$ is contained in $\mathcal{M}T(N, \phi_1)$. Since f_1 is isotopic to the identity, $\mathcal{M}T(N, f_1) = N \times \mathbb{S}^1$. Hence the lemma. □

Before we proceed. We would like to point out that Lemma 4.6 was implicitly established in [22].

4.2. The existence of Lefschetz fibration embedding.

We are now in a position to state and prove our main result regarding *Lefschetz fibration embeddings*.

As usual, we denote the map $N \times \mathbb{C}P^1$ to $\mathbb{C}P^1$ corresponding to the projection on the second factor by π_2.

Definition 4.7 (Lefschetz fibration embedding)**.** Let $(M, \pi : M \to \Sigma)$ be a Lefschetz fibration, where Σ is 2–disk or $\mathbb{C}P^1$. An embedding $f : M \to N \times \mathbb{C}P^1$ of a manifold M into a manifold $N \times \mathbb{C}P^1$ is said to be a *Lefschetz fibration embedding* provided $\pi_2 \circ f = i \circ \pi$, where i is an inclusion of $\mathbb{D}^2$ in $\mathbb{C}P^1$ when $\partial M \neq \emptyset$, otherwise it is the identity.

Theorem 4.8. *Let M be an orientable smooth 4–manifold. Let N be a 4–manifold which admits a separable Hopf link. If $\pi : M \to \Sigma$, where Σ is either $\mathbb{C}P^1$ or a 2–disk $\mathbb{D}^2$, is a simplified Lefschetz fibration (SLF) of M having genus g fibers with $g \geq 1$, then there exists a Lefschetz fibration embedding of (M, π) in $(N \times \mathbb{C}P^1, \pi_2)$.*

Proof. Let us first provide a proof of the theorem, when $\Sigma = \mathbb{C}P^1$. In this case, M is a closed orientable manifold admitting a SLF $\pi : M \to \mathbb{C}P^1$.

Let $c_1, c_2, \cdots c_k$ be k critical points of the Lefschetz fibration (M, π). Since the Lefschetz fibration is simple, $\pi(c_1) = p_1, \pi(c_2) = p_2, \cdots$, and $\pi(c_k) = p_k$ are distinct points on $\mathbb{C}P^1$. Also, recall that that the genus g of the generic fiber is bigger than or equal to 2. Let γ_i be the vanishing cycle corresponding to the critical point c_i on a generic fiber Σ_g of the SLF.

Let U_i be the open ball in M around c_i such that on U_i we have co-ordinates (z_1, z_2) such that π in this co-ordinates is given by $(z_1, z_2) \to z_1.z_2$. Let $\widetilde{D}_i = \pi(U_i) \subset \mathbb{C}P^1$. Let D_i be an open disk containing p_i with $\overline{D_i} \subset \widetilde{D}_i$.

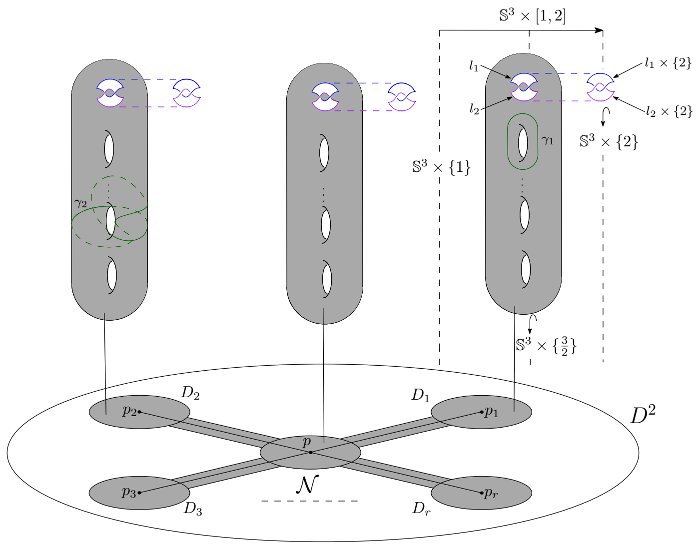

FIGURE 5. The figure depicts part of a Lefschetz fibration (M, π) over a disk embedded as Lefschetz fibration in the (Lefschetz) fibration $N \times D^2 \to D^2$. The embedding is such that the generic fiber of (M, π) is a flexible embedding in the standard position in N. The curves on the surface depicts the vanishing cycles γ_i's.

First of all consider an embedding ϕ of the fiber Σ_g in N which is a standard embedding. Recall that the existence of such an embedding is the content of Lemma 4.4.

Using the flexibility of the embedding ϕ, we first produce an embedding $\widehat{f}$ of $M \setminus \sqcup_{i=1}^k \pi^{-1}(D_i)$ in the manifold $N \times \left(\mathbb{C}P^1 \setminus \sqcup_{i=1}^k D_i\right)$ such that the following diagram commutes:

$$\begin{array}{ccc} M \setminus \sqcup_{i=1}^k \pi^{-1}(D_i) & \xrightarrow{\widehat{f}} & N \times \left(\mathbb{C}P^1 \setminus \sqcup_{i=1}^k D_i\right) \\ \downarrow{\pi} & & \downarrow{\pi_2} \\ \mathbb{C}P^1 \setminus \sqcup_{i=1}^k D_i & \xrightarrow{Id} & \mathbb{C}P^1 \setminus \sqcup_{i=1}^k D_i. \end{array} \tag{1}$$

In order to do this, we observe that the embedding of Σ_g in N is standard. Hence Lemma 4.6 implies given an element $\psi \in \mathcal{MCG}(\Sigma_g)$ there exists an embedding Ψ of the mapping torus, $\mathcal{MT}(\Sigma_g, \psi)$, in the trivial fiber bundle $\pi_2 : M \times \mathbb{S}^1 \to \mathbb{S}^1$ such that the following diagram commutes:

$$\begin{array}{ccc} \mathcal{MT}(\Sigma_g, \psi) & \xrightarrow{\Psi} & N \times \mathbb{S}^1 \\ \downarrow{\pi} & & \downarrow \\ \mathbb{S}^1 & \xrightarrow{Id} & \mathbb{S}^1. \end{array} \tag{2}$$

Next, considering $\partial D_i \subset \mathbb{C}P^1 = \mathbb{S}^1$ then it follows from the existence of an embedding ϕ satisfying the diagram (2) that there is an embedding of the mapping torus $\mathcal{MT}(\Sigma_g, \tau_{\gamma_i})$ in $N \times \partial D_i$, where τ_{γ_i} denotes the Dehn twist along the curve γ_i. Now take arcs connecting a point on ∂D_i to a fixed regular point p for the map π in $\mathbb{C}P^1$ as depicted in Figure 5.

Since the Lefschetz fibration (M, π) restricted to a regular neighborhood $\mathcal{N}$ of D_i's together with arcs connecting them satisfies the following:

(1) $\pi^{-1}(\partial D_i)$ is the mapping torus $\mathcal{MT}(\Sigma_g, \tau_{\gamma_i})$,

(2) M restricted to $\partial\mathcal{N}$ is the manifold $\Sigma_g \times \mathbb{S}^1$ as $\prod_{i=1}^{k} \tau_{\gamma_i} = Id$ in $\mathcal{MCG}(\Sigma_g)$,
(3) the complement of $\mathcal{N}$ is a disk in $\mathbb{C}P^1$,
(4) and the genus $g \geq 2$,

we get the required embedding $\widehat{f}$ such that the diagram (1) commutes.

Our next step is to show how to extend this embedding to produce a Lefschetz fibration embedding of f of M in $N \times \mathbb{C}P^1$. For this the property that the embedding ϕ of Σ_g is also in the standard position is required.

Since the embedding ϕ is in a standard position – by the definition of an embedding in a standard position given in 4.2 – there exists an embedding of $\phi_{\gamma_i} : \mathbb{C}^2 \hookrightarrow N$ which satisfies the second property listed in Definition 4.2.

Next, for each critical point c_i, we claim that, we have following commutative diagram:

$$\begin{array}{ccccccc} U_i \subset M & \xrightarrow{\phi_i} & \mathbb{C}^2 & \xrightarrow{i} & \mathbb{C}^2 \times \mathbb{C} & \xrightarrow{f_{c_i}} & N \times \mathbb{C}P^1 \\ \downarrow \pi & & \downarrow g & & \downarrow P & & \downarrow \pi_2 \\ \widetilde{D}_i & \xrightarrow{\phi} & \mathbb{C} & \xrightarrow{Id} & \mathbb{C} & \xrightarrow{\phi^{-1}} & \widetilde{D}_i, \end{array} \tag{3}$$

where the definitions of the maps appearing in the diagram are as follows:

(1) $\phi_i : U_i \subset M \to \mathbb{C}^2$ and $\phi : \widetilde{D}_i \subset \mathbb{C}P^1 \to \mathbb{C}$ are orientation preserving parameterizations around critical point c_i of π and $\pi(c_i)$ respectively such that left square commutes in the diagram above,
(2) $i : \mathbb{C}^2 \to \mathbb{C}^2 \times \mathbb{C}$ and $g : \mathbb{C}^2 \to \mathbb{C}$ are defined as $i(z_1, z_2) = (z_1, z_2, 0)$ and $g(z_1, z_2) = z_1.z_2$,
(3) $f_{c_i} : \mathbb{C}^2 \times \mathbb{C} \to N \times \mathbb{C}P^1$ and $P : \mathbb{C}^2 \times \mathbb{C} \to \mathbb{C}$ are defined as
$f_{c_i}(z_1, z_2, z_3) = (\phi_{\gamma_i}(z_1, z_2), \phi^{-1}(z_1.z_2 + z_3))$ and $P(z_1, z_2, z_3) = z_1.z_2 + z_3$.

The commutativity of the middle square is follows directly from definitions of maps g, i and P. Also the commutativity of the last square is clear by the definition of the map f_{c_i}. Next, we observe that the commutative diagram 3 allows us to extend the embedding $\widehat{f}$ to the embedding $\widehat{f}_{c_i}$ of $M \setminus \sqcup_{i=1}^{k} \pi^{-1}(D_i) \cup U_i$. This is possible because $\widehat{f}$ and $f_{c_i} \circ i \circ \phi_i$ agree on the overlapping region of the domain. Hence, $\widehat{f}$ and $f_{c_i} \circ i \circ \phi_i$ together defines a map $\widehat{f}_{c_i}$.

Let us now notice that this allows us to extend the embedding $\widehat{f}_{c_i}$ to an embedding $\widehat{f}_{c_i}$ of $W_{c_i} = M \setminus \left(\bigcup_{l=1}^{i-1} \pi^{-1}(D_l) \bigcup_{l=i+1}^{k} \pi^{-1}(D_l) \right)$ in $N \times \mathbb{C}P^1$ such that the following diagram commutes:

$$\begin{array}{ccc} W_{c_i} & \xrightarrow{\widehat{f}_{c_i}} & \widehat{f}_{c_i}(W_{c_i}) \subset N \times \mathbb{C}P^1 \\ \downarrow \pi & & \downarrow \pi_2 \\ \pi(W_{c_i}) \subset \mathbb{C}P^1 & \xrightarrow{Id} & \pi_2(\widehat{f}_{c_i}) = \pi(W_{c_i}). \end{array} \tag{4}$$

Observe that by construction the embeddings $\widehat{f}_{c_i}$ and $\widehat{f}_{c_j}$ agree on on $W_{c_i} \cap W_{c_j}$. Since $M = \cup_{i=1}^{k} W_{c_i}$ we get an embedding f of M with required properties. This completes our argument in case when $\Sigma = \mathbb{C}P^1$.

The case, when $\Sigma = \mathbb{D}^2$ the argument is essentially same. The only difference is that the product $\prod_{i=1}^{k} \tau_{\gamma_i}$ need not be the identity. However, notice that since $\Sigma = \mathbb{D}^2$ the same argument produces an embedding such that the monodromy along $\partial\mathbb{D}^2$ is precisely $\prod_{i=1}^{k} \tau_{\gamma_i}$.

□

5. Embedding of orientable 4-manifolds via SBLF

The purpose of this section is to establish a class of 6–manifolds in which all closed smooth orientable 4–manifolds embed. As mentioned earlier, we will use the SBLF decomposition of a closed orientable smooth 4–manifold for constructing embeddings. We first need the following:

Definition 5.1 (1–fold simple singular fibration)**.** Let $(M, \partial M)$ be an orientable smooth 4–manifold with boundary and let $f : M \to [-1, 1] \times \mathbb{S}^1$ be a smooth surjective map which satisfies the following:

(1) There exists a unique embedded circle Z_f in M of 1-fold singularity for f such that $f(Z_f)$ is an embedded circle in $[-1, 1] \times \mathbb{S}^1$ which is ambiently isotopic to the circle $\{0\} \times \mathbb{S}^1$.
(2) every $x \in M \setminus Z_f$ is a regular value for the map f
(3) $\partial M = f^{-1}\left(\{-1\} \times \mathbb{S}^1 \sqcup f^{-1}\{1\} \times \mathbb{S}^1\right)$.

Then, we say that $f : M \to [-1, 1] \times \mathbb{S}^1$ is a 1*–fold simple singular fibration.*

Remark 5.2.

(a) Since $f : M \to [-1, 1] \times \mathbb{S}^1$ has a unique embedded singular locus Z_f which projects to a circle C isotopic to $\{0\} \times \mathbb{S}^1$ the inverse image of any regular value is a closed surface Σ whose genus is either g or $g + 1$ for some $g \in \mathbb{N} \cup \{0\}$. We call a fiber with genus k as a lower genus fiber.
(b) Observe that as we cross the $f(Z_f)$ a round 1–handle is added to a manifold diffeomorphic to $\Sigma_g \times A$, where A is an annulus.
(c) We will always use the convention that fiber over $\{-1\} \times \mathbb{S}^1$ have lower genus.

Lemma 5.3. *Let $(M, \partial M)$ be an orientable smooth 4–manifold with boundary and $f : M \to [-1, 1] \times \mathbb{S}^1$ be a 1–fold simple singular fibration. Let N be a 4–manifold which admits a separable Hopf link. Then there exists an embedding $\psi : M \to N \times [-1, 1] \times \mathbb{S}^1$ such that following properties are satisfied:*

(1) The following diagram commutes:

$$\begin{array}{ccc} M & \xrightarrow{\psi} & N \times [-1,1] \times \mathbb{S}^1 \\ \downarrow f & & \downarrow \pi_2 \\ [-1,1] \times \mathbb{S}^1 & \xrightarrow{Id} & [-1,1] \times \mathbb{S}^1. \end{array} \tag{5}$$

(2) Given a standard embedding ϕ of a surface of genus $g + 1$ in N, we can ensure that ψ restricted to any higher genus fiber send the fiber to a surface in N which is isotopic to the given embedding ϕ.

Proof. Let us denote by $M_0 = f^{-1}(\{-1\} \times \mathbb{S}^1)$, and $M_1 = f^{-1}(\{1\} \times \mathbb{S}^1$. We know that $\partial M = M_0 \sqcup M_1$. Observe that M_1 is a mapping torus over $\mathbb{S}^1$ with fiber Σ_{g+1}. Recall that any mapping torus over $\mathbb{S}^1$ is determined by its monodromy – an element of $\mathcal{MCG}(\Sigma_g)$. Let ϕ be the monodromy for the fiber bundle M_1 over $\mathbb{S}^1$. Further, since $f : (M, \partial M) \to [-1, 1] \times \mathbb{S}^1$ is a 1–fold simple singular fibration, we have the following: there exists a curve c in Σ_{g+1} which is mapped to itself by ϕ [5, p. 10895], and the boundary component M_0 is obtained from M_1 by the following procedure:

First cut Σ_{g+1} along c, and attach to the resulting surface a pair of disks – say D_1 and D_2. Now form the mapping torus of the resulting surface Σ_g with monodromy the map ϕ restricted to Σ_g.

This also implies that we can obtain $(M, \partial M)$ by suitably adding a round 1–handle along a pair of points times $\mathbb{S}^1$ such that each disk $D_i \times \mathbb{S}^1$ contains a circle of the round attaching sphere.

Now, let $i : \Sigma_{g+1} \subset N$ be a standard embedding of Σ_{g+1} in N Since the embedding is standard, we know every simple close curve γ on Σ_{g+1} bounds a disk in D in N such that the intersection of this disk with N is γ. Furthermore, recall that any simple closed curve in a standard embedding of Σ_{g+1} can be assumed to be disjoint from the separable Hopf link, and the pair of disjoint disks that the link bounds. This implies that there exist a 4–ball B containing the disk D such that $\Sigma_{g+1} \cap B^4$ is an annulus A and ∂A is a pair of unlinked unknots in ∂B^4.

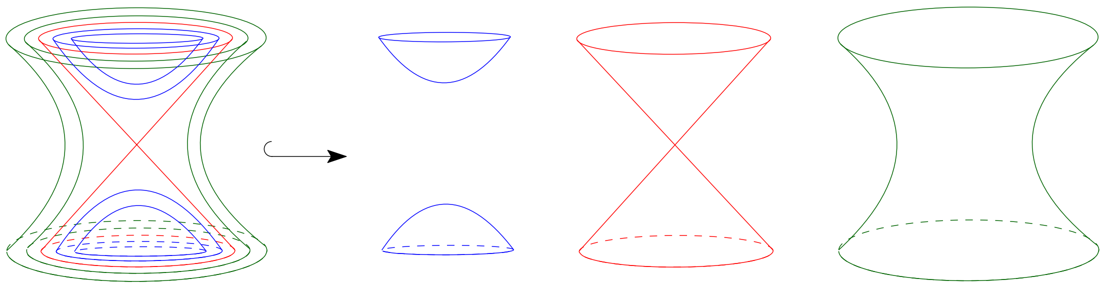

FIGURE 6. Simple Lefschetz fibration embedding

Since the embedding is standard, from Lemma 4.6 it follows that there exist a fiber preserving embedding of M_1 in $N \times \{1\} \times \mathbb{S}^1$. Since ϕ send c to itself $\phi(c) = \pm c$. Since the curve c bounds disk in Σ_g, without loss of generality we can assume that $\phi(c) = c$.

We know that the embedding of a surface Σ_g obtained by cutting Σ_{g+1} along the curve c agrees with Σ_{g+1} everywhere except in the ball B^4. Since the ball B^4 is disjoint form the separable Hopf link and the pair of disjoint disks that the link bounds, we get that the embedding of Σ_g given by by cutting Σ_g is also standard. Hence, applying Lemma 4.6, we get an embedding of M_0 in $N \times \{-1\} \times \mathbb{S}^1$ which is also fiber preserving.

Observe that by very construction the embedding of $\partial M = M_0 \sqcup M_1$ can be extended to an embedding $\widehat{\psi}$ of $(M, \partial M) \setminus \mathcal{N}$ in $N \setminus B^4 \times [-1,1] \times \mathbb{S}^1$, where N is a neighborhood of 1–fold singularity. Furthermore, we can assume that the following diagram commutes:

$$\begin{array}{ccc} M \setminus \mathcal{N} & \xrightarrow{\widehat{\psi}} & N \setminus B^4 \times [-1,1] \times \mathbb{S}^1 \\ \downarrow f & & \downarrow \pi_2 \\ [-1,1] \times \mathbb{S}^1 & \xrightarrow{Id} & [-1,1] \times \mathbb{S}^1. \end{array} \tag{6}$$

Hence, in order to establish the lemma, we need to extend the embedding constructed so far in the region $\mathcal{N}$. We can assume that $\mathcal{N}$ is a tubular neighborhood of the 1-fold critical locus, and hence can be identified with $B^3 \times \mathbb{S}^1$.

Let (x, y, z, θ) be co-ordinates on a tubular neighborhood $\mathcal{N} = B^3 \times \mathbb{S}^1$ of the singular locus Z_f of f such that the map f sends (x, y, z, θ) to $(-x^2 + y^2 + z^2, \theta)$. Let us embed $B^3 \times \mathbb{S}^1$ in $B^4(0,1) \times [-1,1] \times \mathbb{S}^1$. The embedding $\widehat{\psi_1} : B^3 \times \mathbb{S}^1 \to B^4(0,1) \times [-1,1] \times \mathbb{S}^1$ is defined as $\widehat{\psi_1}(x, y, z, \theta) = (x, y, z, 0, -x^2 + y^2 + z^2, \theta)$. We can see $\widehat{\psi_1}$ is defined such that following diagram commutes:

$$\begin{array}{ccc} B^3 \times \mathbb{S}^1 \subset M & \xrightarrow{\widehat{\psi_1}} & B^4(0,1) \times [-1,1] \times \mathbb{S}^1 \subset N \times [-1,1] \times \mathbb{S}^1 \\ \downarrow f & & \downarrow \pi_2 \\ [-1,1] \times \mathbb{S}^1 & \xrightarrow{Id} & [-1,1] \times \mathbb{S}^1. \end{array} \tag{7}$$

Observe that the embedding $\widehat{\psi_1}$ has the property that for each $\{t\} \times \mathbb{S}^1$, the intersection of $f^{-1}(\{t\} \times \mathbb{S}^1$ with $\partial B^4 \times \{t\} \times \mathbb{S}^1$ is a pair of unliked unknot. This implies that by perturbing the embedding $\widehat{\psi}$ slightly, we can assume that both embeddings agrees near the boundary to produce an embedding ψ of M in $N \times [-1,1] \times \mathbb{S}^1$.

Clearly, ψ is the required embedding. This shows that we can produce an embedding of $(M, \partial M)$ in N satisfying the property (2). Since there always exists a standard embedding of Σ_{g+1}, the lemma follows. □

Theorem 5.4. *Let M be an orientable closed smooth 4–manifold. Let N be a 4–manifold which admits a separable Hopf link. Then there exists an embedding $\psi : M \to N \times \mathbb{C}P^1$.*

Proof. Let M be a closed oriented 4–manifold. By Theorem 2.12 there exists a smooth map $f : M \to \mathbb{C}P^1$ which defines SBLF such that the lower genus fiber Σ_g of f has genus bigger than 1. Therefore, We have a decomposition of M, $M = X_1 \sqcup X_2 \sqcup \Sigma_g \times D_2$, satisfying the following:

(1) $X_1 = f^{-1}(D_1)$ with where D_1 is a disc in $\mathbb{C}P^1$ such that in the interior of D_1 contains all Lefschetz critical values of f.
(2) f restricted to X_2 is 1-fold singular fibration.
(3) $\Sigma_g \times D_2 = f^{-1}(D_2)$, where D_2 is a disc in $\mathbb{C}P^1$ containing no critical points of f with $\{-1\} \times \mathbb{S}^1 = \partial D_2$.
(4) Identifications along boundaries of adjacent regions is always via the identity map.

It follows from Theorem 4.8 and Lemma 5.3 that each piece of M embeds in $N \times \mathbb{C}P^1$. Also, it is clear from the second property listed in the statement of Lemma 5.3 that embeddings of each piece can be arranged such that in the overlapping region they agree. This clearly implies that we have an embedding of M in $N \times \mathbb{C}P^1$ as claimed. □

Remark 5.5.

(a) The embedding $\psi : M \to N \times \mathbb{C}P^1$ produced in Theorem 5.4 satisfies $\psi \circ \pi_2 = f$, where $f : M \to \mathbb{C}P^1$ is SBLF associated to M and $\pi_2 : N \times \mathbb{C}P^1 \to \mathbb{C}P^1$ is projection onto second factor of $N \times \mathbb{C}P^1$. In this case, the embedding ψ is termed as *SBLF embedding*.
(b) In general, given a fiber bundle $\pi : X^6 \to \mathbb{C}P^1$ and an embedding of M^4 in X^6 such that π restricted to M induces a SBLF on M will also be termed as an SBLF embedding.

6. Embeddings in $\mathbb{R}^7$

In this section we give a new proof of the fact that every closed smooth orientable 4–manifold admits a smooth embedding in $\mathbb{R}^7$.

Theorem 6.1. *Every 4–manifold admits a smooth embedding in $\mathbb{R}^7$.*

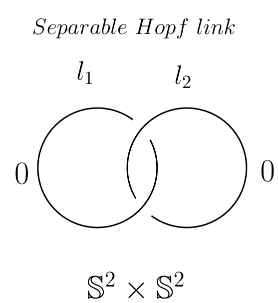

Figure 7. Figure depicts the kirby diagram of $\mathbb{S}^2 \times \mathbb{S}^2$. Observe that attaching circles of 2–handles form a Hopf link in the boundary of the unique 0–handle, and they bound disjoint disk corresponding to attaching disks in $\mathbb{S}^2 \times \mathbb{S}^2$.

Proof. Consider the 4–manifold $\mathbb{S}^2 \times \mathbb{S}^2$. We observe that $\mathbb{S}^2 \times \mathbb{S}^2$ admits a separable Hopf link. This is because $\mathbb{S}^2 \times \mathbb{S}^2$ admits a handle decomposition consisting of a unique 0–handle H_0 one which a pair of two 2–handles are attached such that the attaching circles form a Hopf link in ∂H_0. For a pictorial description of this handle decomposition, we refer to Figure 7, where we have presented a Kirby diagram of $\mathbb{S}^2 \times \mathbb{S}^2$. This clearly implies that the Hopf link consisting of the pair of attaching circles is a separable Hopf link. Thus by Theorem 5.4, every 4–manifold embeds in $\mathbb{S}^2 \times \mathbb{S}^2 \times \mathbb{C}P^1 = \mathbb{S}^2 \times \mathbb{S}^2 \times \mathbb{S}^2$. Now as $\mathbb{S}^2 \times \mathbb{S}^2 \times \mathbb{S}^2$ embeds in $\mathbb{R}^7$, proof of the corollary follows. □

7. Embeddings in $\mathbb{C}P^3$

Let us now establish Theorem 1.1. As mentioned in the introduction, the first step of the proof involves construction of a specific SBLF on $M\#\mathbb{C}P^2\#\overline{\mathbb{C}P^2}$. We then use this SBLF to produce an embedding of $M\#\mathbb{C}P^2\#\overline{\mathbb{C}P^2}$ in the blow-up $\mathcal{B}_{\mathbb{C}P^1}(\mathbb{C}P^3)$ of $\mathbb{C}P^3$ along $\mathbb{C}P^1$. hence there is an embedding Furthermore, we show that this embedding can be constructed such that when we blow-down $\mathcal{B}_{\mathbb{C}P^1}(\mathbb{C}P^3)$, we get an embedding of M in $\mathbb{C}P^3$. We begin by reviewing notions related to blow-up and blow-down.

7.1. Generalized Lefschetz pencil.

Definition 7.1 (Generalized Lefschetz pencil)**.** Let M be an orientable smooth 4–manifold. A *generalized Lefschetz pencil* associated to M is a map $\pi : M \setminus B \to \mathbb{C}P^1$ such that the following properties are satisfied:

(1) B is finite.
(2) $\pi : M \setminus B \to \mathbb{C}P^1$ is a Lefschetz fibration.
(3) For every point $b \in B$ there are parameterizations – *not necessarily preserving orientations* – $\phi : U \subset M \to \mathbb{C}^2$ that satisfies the following:
  (a) $b \in U$ and $\phi(b) = 0 \in \mathbb{C}^2$
  (b) For the map $g : \mathbb{C}^2 \to \mathbb{C}P^1$ given by $g(z_1, z_2) = \frac{z_2}{z_1}$, the following diagram commutes:

$$\begin{array}{ccc} U & \xrightarrow{\phi} & \mathbb{C}^2 \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle g} \\ \mathbb{C}P^1 & \xrightarrow{Id} & \mathbb{C}P^1 \end{array} . \tag{8}$$

In this case, we call B as a base locus of a generalized Lefschetz pencil associated to M.

Remark 7.2.

(a) We would like to emphasis that the notion of generalized Lefschetz pencil defined above is weaker than the notion of Lefschetz pencil. Generally one demands that M and $\mathbb{C}P^1$ are oriented and the parameterizations $\phi : U \subset M \to \mathbb{C}^2$ and $\psi : V \subset \mathbb{C}P^1 \to \mathbb{C}$ are orientation preserving in Definition 7.1.
(b) If the fibration $\pi : M \setminus B \to \mathbb{C}P^1$ is simplified Lefschetz fibration, the pencil is termed as *generalized simplified broken Lefschetz pencil* or generalized SBLF in short.
(c) If the fibration $\pi : M \setminus B \to \mathbb{C}P^1$ is simplified broken Lefschetz fibration and the parameterizations $\phi : U \subset M \to \mathbb{C}^2$ and $\psi : V \subset \mathbb{C}P^1 \to \mathbb{C}$ are orientation preserving, the pencil is termed as *simplified broken Lefschetz pencil* (SBLP).

7.2. Topological blow-up and blow-down of 4–manifolds.

We begin by recalling few standard facts from [13] about the tautological line bundle over $\mathbb{C}P^1$ and the bundle (complex) dual to this bundle.

Consider the tautological line bundle $\tau_{\mathbb{C}P}$ over $\mathbb{C}P^1$, and the bundle $\tau^*_{\mathbb{C}P^1}$ dual to the bundle $\tau_{\mathbb{C}P^1}$. Let $\mathcal{Z}_\tau$ denote the zero section of the bundle $\tau_{\mathbb{C}P^1}$, while $\mathcal{Z}_{\tau^*}$ denote the zero section of the bundle $\tau^*_{\mathbb{C}P^1}$.

We know that $\tau_{\mathbb{C}P^1} \setminus \mathcal{Z}_\tau$, and $\tau^*_{\mathbb{C}P^1} \setminus \mathcal{Z}_{\tau^*}$ are diffeomorphic to $\mathbb{R}^4 \setminus \{0\}$ by diffeomorphisms coming from the restrictions of the projection of second factor for the corresponding bundles. We fix this identification of the complement of zero sections with $\mathbb{R}^4 \setminus \{0\}$ for both these bundles.

Definition 7.3 (Topological blow-up)**.** Let M a smooth 4–manifolds. Let p be a point in M. Let U be a neighborhood of p diffeomorphic to $\mathbb{R}^4$ via a diffeomorphism which sends p to $0 \in \mathbb{R}^4$ The manifold $\widehat{M}$ obtained by removing p from U and identifying $U \setminus \{p\}$ with either $\tau^*_{\mathbb{C}P^1} \setminus \mathcal{Z}_{\tau^*}$ or with $\tau_{\mathbb{C}P^1} \setminus \mathcal{Z}_\tau$ is called a topological blow-up of M along p.

Remark 7.4.

(a) The operation of topological blow-up of a manifold along a point corresponds to its connected sum with $\mathbb{C}P^2$ or $\overline{\mathbb{C}P^2}$. While performing a topological blow-up, if we use the tautological line bundle $\tau_{\mathbb{C}P^1}$, then we get $M\#\overline{\mathbb{C}P^2}$. On the other hand, if we use the dual bundle to $\tau_{\mathbb{C}P^2}$, then we get $M\#\mathbb{C}P^2$.

(b) Topological blow-up of M along p produces a manifold $\widehat{M}$ admitting an embedded $\mathbb{C}P^1$ with self intersection number ± 1. Recall that the usual blow-up always produces an embedded $\mathbb{C}P^1$ with self intersection -1.

(c) Throughout this discussion, an embedded $\mathbb{C}P^1$ in a 4–manifold M with self intersection number ± 1 will be called an *exceptional sphere* in M.

Definition 7.5 (Topological blow-down)**.** Let $\widehat{M}$ be a smooth 4–manifold admitting an embedded $\mathbb{C}P^1$ whose normal bundle is isomorphic to $\tau_{\mathbb{C}P^1}$ or $\tau^*_{\mathbb{C}P^1}$. That is the embedded $\mathbb{C}P^1$ is an exception sphere in $\widehat{M}$. In this case, we can carry out the process exact opposite of the one describe in the definition of blow-up, where we remove a tubular neighborhood of $\mathbb{C}P^1$ and replace it with a 4-ball. The resulting manifold M that we obtain as a result of this process is called a topological blow-down of $\widehat{M}$.

Remark 7.6.

(1) Observe that given a manifold M admitting an embedding $\mathbb{C}P^1$ with its self intersection number ± 1, we can perform topological blow-down operation.

(2) Suppose we are given a manifold $M\#\mathbb{C}P^2\#\overline{\mathbb{C}P^2}$. Let E_1 and E_{-1} be two embedded $\mathbb{C}P^1$'s corresponding to zero sections of $\tau^*_{\mathbb{C}P^1}$ and $\tau_{\mathbb{C}P^1}$ respectively. Suppose we have $f : M\#\mathbb{C}P^2\#\overline{\mathbb{C}P^2} \to \mathbb{C}P^1$ be a SBLF such that the intersection number of a fiber with E_1 is 1, and the intersection with E_{-1} is -1, then the two operations of blow-downs corresponding to removal of E_{-1} and E_1 produces a generalized SBLP on M

7.3. Construction of SBLF on $M\#\mathbb{C}P^2\#\overline{\mathbb{C}P^2}$.

The purpose of this subsection is to establish a SBLF on $M\#\mathbb{C}P^2\#\overline{\mathbb{C}P^2}$ which satisfies the property that intersection of each fiber with two exceptional spheres E_1 and E_{-1} corresponding to zero sections is -1 and $+1$ respectively.

Lemma 7.7. *Consider a closed orientable smooth manifold $M\#\mathbb{C}P^2\#\overline{\mathbb{C}P^2}$. There exists a SBLF f : $M\#\mathbb{C}P^2\#\overline{\mathbb{C}P^2} \to \mathbb{C}P^1$ which satisfies the following:*

(1) The lower genus fiber has its genus bigger than 1.

(2) The fibration agrees with the standard fibration in a tubular neighborhood of both exception spheres E_1 and E_{-1}.

In particular, bowling down the SBLF $f : M\#\mathbb{C}P2\#\overline{\mathbb{C}P^2} \to \mathbb{C}P^1$ produces a generalized SBLP on M.

In [6, Theorem:6.5], I. Baykur and O. Saeki established the existence of simplified broken Lefschetz pencil for any near symplectic manifold for any near symplectic manifold admitting connected singular locus for near symplectic structure. It is easy to see that following the proof of [6, Theorem:6.5] – essentially verbatim – provides a proof of Lemma 7.7.

Proof. To begin with, notice that there exists an embedded surface Σ in $M\#\mathbb{C}P^2\#\overline{\mathbb{C}P^2}$ which satisfy the following properties:

- The self intersection of Σ is 0
- $\Sigma \cap E_1 = +1$ and $\Sigma \cap E_{-1} = -1$.
- Σ is connected and the genus of Sigma is bigger than three.

Observe that since the self inter section of E_1 is $+1$ and $E_{-1} = -1$ it is easy to construct a disconnected surface consisting of disjoint union of two spheres. By making connected sums of these two sphere with an embedded surface bounding a 3–dimensional handle-body and embedded in B^4, it is easy to construct such a surface.

Consider the map $\pi : \Sigma \times \mathbb{D}^2 \to \mathbb{D}^2$, corresponding to the projection on the second factor, and regard $\mathbb{D}^2$ as embedded in $\mathbb{C}P^1$ as a southern hemisphere. This allows us to regard π as a map from a tubular neighborhood

$\mathcal{N}(\Sigma)$ to southern hemisphere. Next, construct map to a map from $g : \mathcal{N}(\Sigma) \cup \mathcal{N}(E_1) \cup \mathcal{N}(E_{-1}) \to \mathbb{C}P^1$ which satisfies the following: properties:

(1) The map when restricted $\mathcal{N}(E_1)$ and $\mathcal{N}(E_{-1}$ is the surjection on $\mathbb{C}P^1$ coming from the bundle projections $\pi_{E_1} : \mathcal{N}(E_1) \to E_1$, and $\pi_{E_{-1}} : \mathcal{N}(E_{-1}) \to E_{-1}$.
(2) The map agrees with π when restricted to $\mathcal{N}(\Sigma)$.

Next, extend the map g to a generic smooth map $\widehat{f} : M\#\mathbb{C}P^2\#\overline{\mathbb{C}P^2} \to \mathbb{C}P^1$. According to [6, Remark:4.5], this map can be modified to produce a SBLF $\widehat{f} : M\#\mathbb{C}P2\#\overline{\mathbb{C}P^2} \to \mathbb{C}P^1$ such that all the modification performed while obtaining the SBLF from g are performed alway from the region where g is defined.

Next, we convert the SLBF $\widehat{f} : M\#\mathbb{C}P^2\#\overline{\mathbb{C}P^2} \to \mathbb{C}P^1$ to an SLBF $f : M\#\mathbb{C}P2\#\overline{\mathbb{C}P^2} \to \mathbb{C}P^1$ whose lower genus fiber is bigger than 3 by applying the technique from the proof of Theorem 2.12. The SLBF $f : M\#\mathbb{C}P2\#\overline{\mathbb{C}P^2} \to \mathbb{C}P^1$ can be ensured to satisfy the required because every fiber of f is homologous to the original fiber Σ and hence the intersection of fibers of f has same property that Σ had. This completes our argument.

□

Remark 7.8.
From now on the SBLF on $M\#\ CP^2\#\ \overline{CP^2}$described in the statement of Lemma 7.7 will be denoted by the notation $\pi_{spl} : M\#\mathbb{C}P^2\#\overline{\mathbb{C}P^2} \to \mathbb{C}P^1$.

7.4. Blow-up and blow-down of $\mathbb{C}P^3$ along $\mathbb{C}P^1$.

Let us begin this sub-section by making a convention. By a standard $\mathbb{C}P^1$ in $\overline{\mathbb{C}P^2}$, we mean a $\mathbb{C}P^1$ embedded in $\overline{\mathbb{C}P^2}$ with its normal bundle isomorphic to the dual of the tautological line bundle over $\mathbb{C}P^1$. On the other hand, by a standard $\mathbb{C}P^1$ in $\mathbb{C}P^n$, we mean $\{[z_1, z_2, \cdots, z_n] | z_i = 0\ \forall\ i \geq 3\}$, where $[z_1, z_2, \cdots, z_n]$ denotes the homogeneous coordinates of $\mathbb{C}P^n$.

Consider $\mathbb{C}P^3$ and a standard $\mathbb{C}P^1$ embedded in it. Fix a local trivialization $\mathbb{D}^2 \times \mathbb{C}^2$ of the normal bundle $\mathcal{N}(\mathbb{C}P^1)$ of $\mathbb{C}P^1$ in $\mathbb{C}P^3$.

Now consider $\mathbb{D}^2 \times \mathbb{C}^2 \times \mathbb{C}P^1$ and a subset V of $\mathbb{D}^2 \times \mathbb{C}^2 \times \mathbb{C}P^1$ given by,

$$V = \{(w, z_1, z_2, l)|\ \|z_1^2\| + \|z_2^2\| \leq 1 \text{ and } (z_1, z_2) \in l\},$$

where a point l in $\mathbb{C}P^1$ is identified with the complex linear subspace corresponding to that point.

Now, observe that the complement of $\mathbb{D}^2 \times \{(0, 0)\} \times \mathbb{C}P^1$ in V can be identified with the complement of $\mathbb{D}^2 \times \{(0, 0)\}$ in $\mathbb{D}^2 \times \mathbb{C}^2$.

Choose two local trivializations $U_1 \times \mathbb{C}^2$ and $U_2 \times \mathbb{C}^2$ over open set U_1 and U_2 such that U_1 and U_2 cover $\mathbb{C}P^1$. By the (topological) blow-up of $\mathbb{C}P^3$ along $\mathbb{C}P^1$ we mean the operation of removing $U_i \times \{(0, 0)\}$ from $U_i \times \mathbb{C}^2$, for each i, and replacing it with the interior of V as discussed in the previous paragraph.

Remark 7.9.

(1) First of all, observe that since the real normal bundle of $\mathbb{C}P^1$ in $\mathbb{C}P^3$ is trivial, the manifold $\mathcal{B}_{\mathbb{C}P^1}(\mathbb{C}P^3)$ is diffeomorphic to $\mathbb{C}P^1 \times \mathbb{C}P^2$.
(2) *Exceptional divisor* of $\mathcal{B}_{\mathbb{C}P^1}(\mathbb{C}P^3)$ is the union of $\mathbb{D}^2 \times \{(0, 0)\} \times \mathbb{C}P^1$ over a finite collection V_s of trivializations of the bundle $\mathcal{N}(\mathbb{C}P^1)$. Again notice that the triviality of the normal bundle of $\mathbb{C}P^1$ in $\mathbb{C}P^3$ implies that the exceptional divisors is diffeomorphic to $\mathbb{C}P^1 \times \mathbb{C}P^1$.

The notion of blow-up discussed above is a particular case of blow-up of a manifold along a sub-manifold. We refer [12, p. 196,602] for a detailed discussion on blow-ups.

By a blow-down of $\mathcal{B}_{\mathbb{C}P^1}(\mathbb{C}P^3)$ we will mean the process exactly opposite to the process of blow-up. More precisely, let $\mathcal{B}_{\mathbb{C}P^1}(\mathbb{C}P^3)$ be obtained by blowing up a $\mathbb{C}P^1$. Let E be the exceptional divisor obtained as a result of the blow-up. By blow-down of $\mathcal{B}_{\mathbb{C}P^1}(\mathbb{C}P^3)$, we mean removal of a tubular neighborhood of E and replacing it by a tubular neighborhood of $\mathbb{C}P^1$ in $\mathbb{C}P^3$.

We say that $\mathbb{C}P^3$ is obtained from $\mathcal{B}_{\mathbb{C}P^1}(\mathbb{C}P^3)$ by blowing down along E. Since E is diffeomorphic to $\mathbb{C}P^1 \times \mathbb{C}P^1$, we sometimes do not distinguish between E and $\mathbb{C}P^1 \times \mathbb{C}P^1$ and say that $\mathbb{C}P^3$ is obtained from $\mathcal{B}_{\mathbb{C}P^1}(\mathbb{C}P^3)$ by blowing down along $\mathbb{C}P^1 \times \mathbb{C}P^1$.

We end this subsection with the following:

Lemma 7.10. *Let $M\#\mathbb{C}P^2\#\overline{\mathbb{C}P^2}$ be a smooth manifold. Let $\pi_{spl} : M\#\mathbb{C}P^2\#\overline{\mathbb{C}P^2} \to \mathbb{C}P^1$ be the SBLF on $M\#\mathbb{C}P^2\#\overline{\mathbb{C}P^2}$ as in the statement of Lemma 7.7 . If there exists a SBLF embedding of $M\#\mathbb{C}P^2\#\overline{\mathbb{C}P^2}$ in $\mathcal{B}_{\mathbb{C}P^1}(\mathbb{C}P^3)$ such that each fiber of SBLF intersects the standard $\mathbb{C}P^1$ of the fiber $\mathbb{C}P^2$ of $\mathcal{B}_{\mathbb{C}P^1}(\mathbb{C}P^3)$ in two distinct but fixed points, then there exist an embedding of M in $\mathbb{C}P^3$ such that the standard pencil of $\mathbb{C}P^3$ induces the generalized SBLP of M corresponding to the SBLF of $M\#\mathbb{C}P^2\#\overline{\mathbb{C}P^2}$*

Proof. Let E_1 and E_{-1} be two exceptional divisors of $M\#\mathbb{C}P^2\overline{\mathbb{C}P^2}$. Recall the exceptional divisor of $\mathcal{B}_{\mathbb{C}P^1}(\ CP^3)$ consist of a union of two local exceptional divisors of the type $U_i \times W$, where $W \subset \mathbb{C}P^1 \times \mathbb{C}^2$ consist of $\{(l, z_1, z_2)|(z_1, z_2) \in l\}$. Since by hypothesis the fiber of π_{spl} intersects the standard $\mathbb{C}P^1$ inside $\mathbb{C}P^2$ in a pair of fixed point, we can assume that of a tubular neighborhoods of an exceptional divisors $E_{\pm 1}$ is contained in $U_i \times W$, and since the embedding is fiber preserving it consist of $\{p_\pm\} \times W \subset U_1 \times W$.

Furthermore, by the definition of the blow-up, the fibration on $\mathcal{B}_{\mathbb{C}P^1}(\mathbb{C}P^3)$ restricted to $U_1 \times W$ can be assumed to be given by $(u, l, z_1, z_2) \to l$. This clearly implies the when we blow-down $\mathcal{B}_{\mathbb{C}P^1}(\mathbb{C}P^3)$ along the exceptional divisor $\mathbb{C}P^1 \times \mathbb{C}P^1$ we get $M \subset \mathbb{C}P^3$ with standard pencil of $\mathbb{C}P^3$ inducing the generalized SBLP on M associated to SBLF $\pi_{spl} : M\#\mathbb{C}P^2\overline{\mathbb{C}P^2} \to \mathbb{C}P^1$.

□

7.5. Embeddings in $\mathcal{B}_{\mathbb{C}P^1}(\mathbb{C}P^3)$.

In this sub-section we establish SLBF embedding of the special SLBF $\pi_{spl} : M\#\mathbb{C}P^2\#\overline{\mathbb{C}P^2} \to \mathbb{C}P^1$ in $\mathcal{B}_{\mathbb{C}P^1}(\mathbb{C}P^3)$.

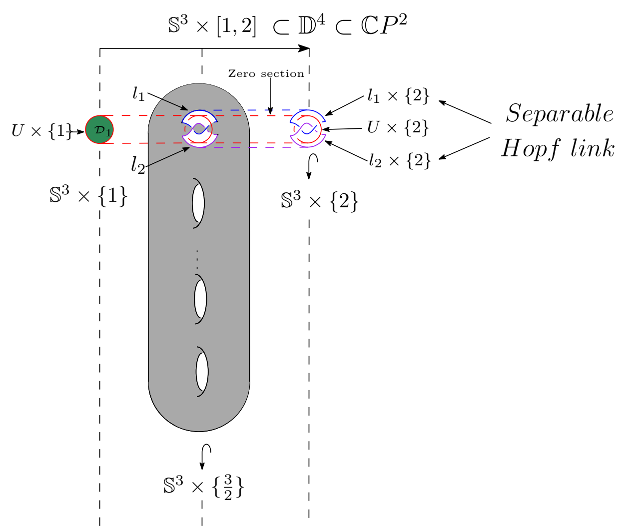

FIGURE 8. Figure depicts an embedded surface in $\mathbb{C}P^3$ which is flexible and in a standard position. The diagram focus on a collar $\mathbb{S}^3 \times [1, 2]$ of a 4–ball $\mathbb{D}^4$ regarded as the unique zero handle H_0 of $\mathbb{C}P^2$. The circle U is the attaching circle of the unique 2–handle H_2. $U \times [1, 2]$ with the core disk attached at $U \times \{2\}$ and the green disk at $U \times \{1\}$ forms the standard $\mathbb{C}P^1$ embedded in $\mathbb{C}P^2$

Proposition 7.11. *Let M be a closed orientable smooth 4-manifold. Let $f : M \to \mathbb{C}P^1$ be a SBLF with the lower genus fiber having genus bigger than 1. There exists a SLBF embedding of M in $\mathcal{B}_{\mathbb{C}P^1}(\mathbb{C}P^3)$ such that each fiber of SLBF intersect the standard $\mathbb{C}P^1$ in the fibre $\mathbb{C}P^2$ in a pair of cancelling intersection points.*

Proof. We will follow the line of argument we used to establish Theorem 5.4. The only difference is that the fibration $\mathcal{B}_{\mathbb{C}P^1}(\mathbb{C}P^3)$ is not a trivial fibration. However, since the normal bundle of $\mathbb{C}P^1$ in $\mathbb{C}P^3$ is trivial, we note that this bundle as a real bundle is trivial provided we remove the section of the fiber bundle $\mathcal{B}_{\mathbb{C}P^1}(\mathbb{C}P^3) \to \mathbb{C}P^1$ corresponding to the exceptional divisor. Hence, we first consider neighborhoods of exceptional divisors E_1 and E_{-1} of $M\#\mathbb{C}P^2\#\overline{\mathbb{C}P^2}$, and embed them in a tubular neighborhood of the exceptional divisor $\mathbb{C}P^1 \times \mathbb{C}P^1$ of $\mathcal{B}_{\mathbb{C}P^1}(\mathbb{C}P^3)$ such that the embedding is fiber preserving.

In order to produce this embedding recall that a tubular neighborhood of the exceptional divisor $\mathbb{C}P^1 \times \mathbb{C}P^1$ is union of two open sets $U_i \times W, i = 1, 2$. Consider $U_1 \times W$, and let us denote by π the fibration $\pi : \mathcal{B}_{\mathbb{C}P^1}(\mathbb{C}P^3) \to \mathbb{C}P^1$ obtained via blow-up of the standard pencil of $\mathbb{C}P^3$.

Next, consider a pair of points p_+, p_- in U_1, and consider spheres $\{p_\pm\} \times \mathbb{C}P^1$ embedded in $U_1 \times W$ Since tubular neighborhood of $E_{\pm 1}$ is isomorphic to tubular neighborhood of any sphere in $U_1 \times W$ of the form $\{p\} \times \mathbb{C}P^1$, where p is a point in U_1, we get the there exist an embedding of small neighborhoods of $E_{\pm 1}$ in a neighborhood of the exceptional divisor $\mathbb{C}P^1 \times \mathbb{C}P^1$ such that π_{spl} restricted to this neighborhood agrees with restriction of π on the embedded neighborhoods.

Observe that the intersection of the embedded neighborhoods of $E_{\pm 1}$ with a fiber of the fibration π : $\mathcal{B}_{\mathbb{C}P^1}(\mathbb{C}P^3)$ is a pair of disk satisfying the property that the intersection of this pair of disk with the boundary of a small tubular neighborhood of $\mathbb{C}P^1 \subset \mathbb{C}P^2$ is a Hopf link. Furthermore, observe that the since the embedding of the neighborhood of E_1 with tubular neighborhood of $\{p_+\} \times \mathbb{C}P^1$ is orientation reversing, and the embedding of neighborhood of E_{+1} with $\{p_- \times \mathbb{C}P^1$ is orientation preserving. This implies that if we establish the following:

(1) $\mathbb{C}P^2$ admits a separable Hopf link,
(2) there exists an embedding of any surface of genus g in $\mathbb{C}P^2$ which is standard embedding,
(3) the embedded surface Σ_g intersects the standard $\mathbb{C}P^1$ contained in $\mathbb{C}P^2$ in a pair of algebraically cancelling point, and $\Sigma_g \cap \partial\mathcal{N}(\mathbb{C}P^1)$ is a Hopf link in $\partial\mathcal{N}(\mathbb{C}P^1)$, where $\mathcal{N}(\mathbb{C}P^1)$ is a fixed open tubular neighborhood of $\mathbb{C}P^1$ in $\mathbb{C}P^2$,

then the triviality of the fibration $\pi : \mathcal{B}_{\mathbb{C}P^1}(\mathbb{C}P^3)$ an argument similar to the one which establishTheorem 5.4 implies required SBLF embedding of $M\#\mathbb{C}P^2\#\overline{\mathbb{C}P^2}$ in $\mathcal{B}_{\mathbb{C}P^1}(\mathbb{C}P^3)$.

Hence, the task at our hand is to establish an embedding of a surface satisfying the three properties listed above. We now proceed to produce such an embedding.

To begin with, we regard $\overline{\mathbb{C}P^2}$ as a handle-body with the 0–handle H_0 corresponding to $B^4(0, 2)$ – the 4–ball of radius 2 in $\mathbb{C}^2$ with its center at the origin – to which a 2–handle H_2 is attached along an unknot with framing +1. Finally a 4–handle H_4 is attached to the 4–manifold, which is the union of the 0–handle $B^4(0, 2)$ and the 2–handle H_2. Regarding H_0 as a ball. Let $\mathbb{S}^3 \times [1, 2]$ be a collar of ∂H_0. Let $U \times \{2\}$ be the attaching circle of H_2. Observe that any Hopf link consisting of a parallel copy of the attaching circle – say $l_1 \times \{2\}$ and a circle $l_2 \times \{2\}$ which links both the attaching circle and l_1 once as depicted in Figure 8 constitute a Hopf link which is separable. This is because $l_1 \times \{2\}$ bounds a parallel copy of the core of 2–handle, and $l_2 \times \{2\}$ bound a disk in the unique 4–handle.

Next, consider cylinders $l_i \times [\frac{3}{2}, 2], i = 1, 2$. They intersect $\mathbb{S}^3 \times \{\frac{3}{2}\}$ in $l_i \times \{\frac{3}{2}\}$. Observe that there exists a surface Σ_g with two boundary component whose boundary is the Hopf link $l_1 \times \{\frac{3}{2}\} \sqcup l_2 \times \{\frac{3}{2}\}$. See Figure 8. It follows from an argument similar to the one used in establishing Lemma 4.4 that the embedding is both flexible and in a standard position.

Regarding the standard $\mathbb{C}P^1$ as the union of core of 2–handle H_2 with a disk $\mathbb{D}$ that $U \times \{2\}$ bounds, we see that the embedded Σ_g intersects $\mathbb{C}P^1$ in a pair of points. This pair has to be algebraically cancelling as we can push the disk $\mathbb{D}$ down to produce an isotopy of $\mathbb{C}P^1$ that sends the $\mathbb{C}P^1$ to a new $\mathbb{C}P^1$ which consist of union of core of H_2, $U \times [1, 2]$, and a disk $\mathbb{D}$ that $U \times \{1\}$ bounds. The disk that $U \times \{1\}$ bounds is denoted by a blue disk in Figure 8. Notice that the isotoped $\mathbb{C}P^1$ is disjoint from Σ_g implying that the algebraic intersection of Σ_g with the standard $\mathbb{C}P^1$ is zero.

This completes our argument.

□

Now we have established all the results necessary to establish Theorem 1.1. We now proceed and supply a proof of Theorem 1.1.

7.6. **Proof of Theorem 1.1.** Recall that we need to prove that every smooth orientable closed 4–manifold admits an embedding in $\mathbb{C}P^3$.

Proof of Theorem 1.1. Let M be the given closed orientable 4–manifold. Consider the manifold $\widehat{M} = M\#\mathbb{C}P^2\#\overline{\mathbb{C}P^2}$ thought as a blow-up of M done at two distinct points p_1 and p_2. Recall that $\widehat{M}$ admits a pair of exceptional divisors – say E_1 and E_{-1} such that $E_1 \cap E_1 = 1$ while $E_{-1} \cap E_{-1} = -1$.

Next, apply Lemma 7.7 to produce a SBLF on $\widehat{M}$ which satisfies the following:

(1) The lower genus fiber has its genus bigger than 1.
(2) The fibration agrees with the standard fibration in a tubular neighborhood of both exception spheres E_1 and E_{-1}.

Now, by Proposition 7.11 there exist SBLF embedding of $\widehat{M}$ in $\mathbb{C}P^2 \times \mathbb{C}P^1$. Since $\mathbb{C}P^2 \times \mathbb{C}P^1$ is diffeomorphic to $\mathcal{B}_{\mathbb{C}P^1}(\mathbb{C}P^3)$, we get an embedding of $M\#\mathbb{C}P^2\#\overline{\mathbb{C}P^2}$ in $\mathcal{B}_{\mathbb{C}P^1}(\mathbb{C}P^3)$.

Also notice that the intersection property of the embedded fiber of SLBF with with standard $\mathbb{C}P^1$ contained in $\mathbb{C}P^2$ stated in Proposition 7.11 implies that the embedding is such that each fiber of the SBLF associated to $M\#\mathbb{C}P^2\#\overline{\mathbb{C}P^2}$ intersects the standard $\mathbb{C}P^1$ of a fiber $\mathbb{C}P^2$ of the trivial fibration $\mathcal{B}_{\mathbb{C}P^1}(\mathbb{C}P^3) \to \mathbb{C}P^1$ in a pair of algebraically cancelling points.

Finally, blow-down $\mathcal{B}_{\mathbb{C}P^1}(\mathbb{C}P^3)$ along its exceptional divisor. Observe that Lemma 7.10 implies that blow-down produces an embedding of M in $\mathbb{C}P^3$ such that the standard Lefschetz pencil of $\mathbb{C}P^3$ induces a SBLP on M.

□

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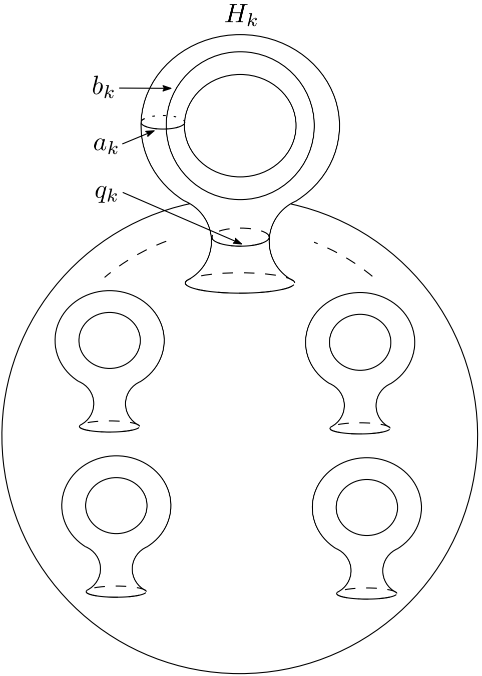
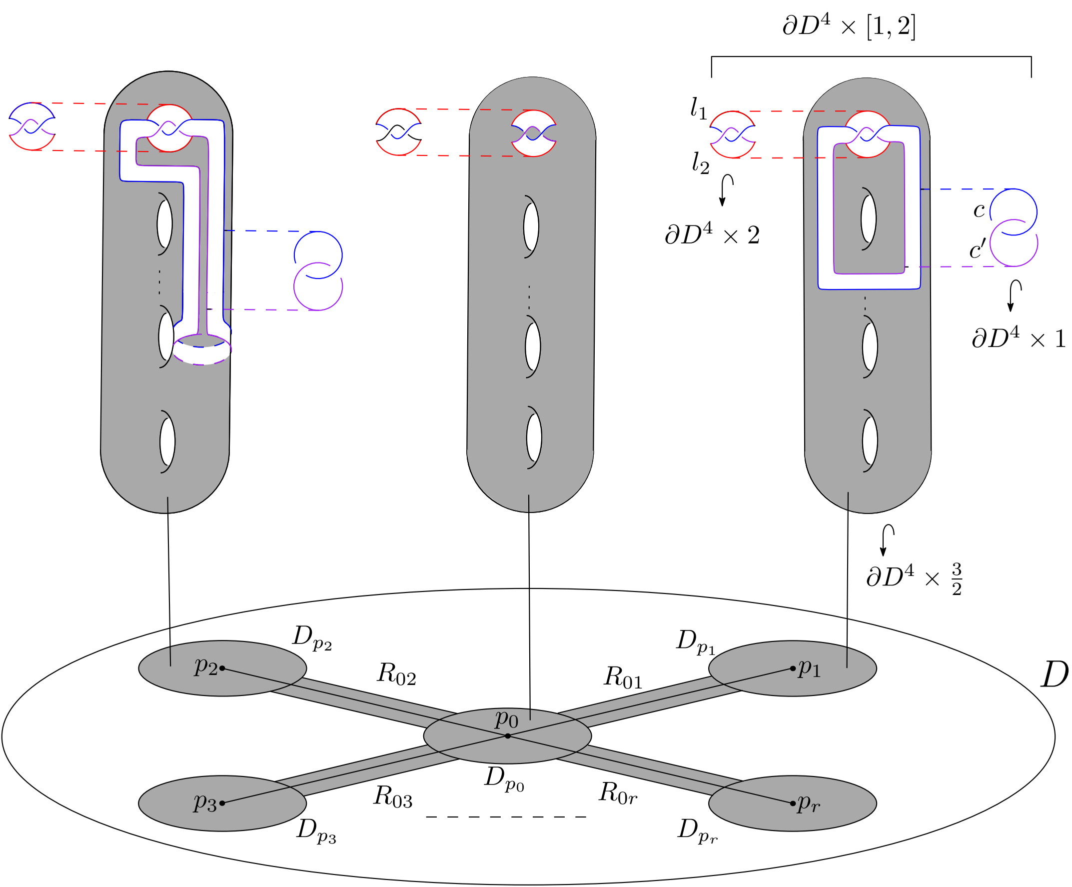


H_k
b_k
a_k
q_k



$\partial D^4 \times [1,2]$
l_1
l_2
$\partial D^4 \times 2$
c
c'
$\partial D^4 \times 1$
$\partial D^4 \times \frac{3}{2}$
D
D_{p_2}
p_2
R_{02}
R_{01}
D_{p_1}
p_1
p_0
D_{p_0}
p_3
R_{03}
R_{0r}
p_r
D_{p_3}
D_{p_r}