Functional Bell inequalities can serve as a stronger entanglement witness

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We consider a Bell inequality for a *continuous* range of settings of the apparatus at each site. This "functional" Bell inequality gives a better range of violation for generalized GHZ states. Also a family of N-qubit bound entangled states violate this inequality for $N > 5$.

A remarkable feature of entanglement is that it gives rise to correlations that cannot be explained by any local realistic theory. This is the statement of the Bell theorem [1]. This theorem usually utilizes some inequalities that are satisfied by any local realistic theory but are violated by quantum correlations, between two or more systems. Modulo the well known loopholes, such violations have actually been experimentally demonstrated. Violation of such an inequality is a signature of entanglement. However it is not known whether the correlations in all entangled states are strong enough to violate a Bell inequality.

The usual formulation of the Bell theorem is for two apparatus settings at each site [2, 3, 4]. However there are several reasons for generalizing to more than two settings [5, 6, 7]. The simplest of them is that new Bell inequalities could reveal violation of local realism for cases when the standard inequalities fail. Or they could be more appropriate to some experimental situations [8].

In this paper, we consider a multipartite Bell inequality that involves a *continuous* range of settings at each site (which we call the "functional" Bell inequality). We show that the functional Bell inequality is stronger in many cases than the standard Bell inequalities. This shows that this inequality may be a useful tool for classification of states with respect to violation of local realism.

All bipartite pure entangled states violate a Bell inequality [9]. The multipartite situation is however more complicated. For example, using the Bell inequalities for correlation functions, which involve the usual choice between two observables for each of the local observers, the N-qubit generalised GHZ states

$$|\psi_N\rangle = sin\beta \, |0\rangle^{\otimes N} + cos\beta \, |1\rangle^{\otimes N} \tag{1}$$

(with $0 \leq \beta \leq \pi/4$) do not violate any such inequalities *for N separated qubits without postselection* for $sin2\beta \leq \frac{1}{\sqrt{2^{N-1}}}$ for odd N [10, 11]. This is quite surprising, considering the fact that these states are a generalization of the GHZ state [12] $\frac{1}{\sqrt{2}}\left(|0\rangle^{\otimes N} + |1\rangle^{\otimes N}\right)$ which strongly violates the standard Bell inequalities. For the functional Bell inequality that we consider in this paper, a violation is obtained for $sin2\beta \geq 2\left(\frac{2}{\pi}\right)^N$ which is better than the previous bounds for Bell violation for odd $N \geq 5$.

One of the open questions in quantum information is whether bound entangled states violate any Bell inequality. Since the seminal works of dense coding [13] and teleportation [14], the maximally entangled states have acquired special significance. However there exist entangled states which cannot be transformed into a maximally entangled state when the parties, sharing the entangled state, are separated. Such states has been called bound entangled states [15]. It is intriguing to consider whether such states can violate local realism. Indeed it has been conjectured that bound entangled states with positive partial transpose (PPT) [16, 17] cannot violate local realism [18]. Further work in this direction has been carried out in Refs. [19, 20]. In a recent paper, Dür [21], considers this question in the multipartite scenario. It is shown there that an N-qubit state

$$\rho_N = \frac{1}{N+1}\left(|GHZ\rangle\langle GHZ| + \frac{1}{2}\sum_{k=1}^{N}\left(P_k + \overline{P_k}\right)\right) \tag{2}$$

violates a Mermin-Klyshko inequality [3] for $N \geq 8$, despite being PPT in all $1 : N-1$ party cuts. Here

$$|GHZ\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle^{\otimes N} + e^{i\alpha_N}|1\rangle^{\otimes N}\right),$$

with α_N being a phase. And $P_k = |\phi_k\rangle\langle\phi_k|$, $|\phi_k\rangle = |0\rangle_1 \ldots |0\rangle_{k-1}|1\rangle_k|0\rangle_{k+1}\ldots|0\rangle_N$ with $\overline{P_k}$ obtained from P_k by interchanging 0s and 1s in P_k. The Bell violation in Ref. [21] was exhibited for $\alpha_N = \frac{\pi}{4(N-1)}$ [22]. Further work was recently done in Ref. [23], where violation of local realism was obtained for $N \geq 7$ for all values of the parameter α_N, by using Bell inequalities that involve three settings per observer.

We show here that the functional Bell inequality is violated by the state ρ_N for $N \geq 6$ irrespective of the value of the parameter α_N.

To begin, let us discuss the functional Bell inequality [6], which essentially follows from a simple geometric observation that in any real vector space, if for two vectors h and q one has $\langle h \mid q\rangle < \parallel q \parallel^2$, then this immediately implies that $h \neq q$. In simple words, if the scalar product of two vectors has a lower value than the length of one of them, then the two vectors cannot be equal.

Let ϱ_N be a state shared between N separated parties. Let O_n be an arbitrary observable at the nth location ($n = 1, \ldots, N$). The quantum mechanical prediction E_{QM} for the correlation in the state ϱ_N, when these observables are measured, is

$$E_{QM}(\xi_1, \ldots, \xi_N) = Tr\left(O_1 \ldots O_N \varrho_N\right), \tag{3}$$

where ξ_n is the aggregate of the local parameters at the nth site. Our object is to see whether this prediction can be reproduced in a local hidden variable theory. A local hidden variable correlation in the present scenario must be of the form

$$E_{LHV}(\xi_1, \ldots, \xi_N) = \int d\lambda \rho(\lambda) \Pi_{n=1}^N I_n(\xi_n, \lambda), \qquad (4)$$

where $\rho(\lambda)$ is the distribution of the local hidden variables and $I_n(\xi_n, \lambda)$ is the predetermined measurement-result of the observable $O_n(\xi_n)$ corresponding to the hidden variable λ.

Consider now the scalar product

$$\begin{aligned} \langle E_{QM} \mid E_{LHV} \rangle = & \int d\xi_1 \ldots d\xi_N \\ & \times \; E_{QM}(\xi_1, \ldots, \xi_N) \, E_{LHV}(\xi_1, \ldots, \xi_N) \end{aligned} \qquad (5)$$

and the norm

$$\parallel E_{QM} \parallel^2 = \int d\xi_1 \ldots d\xi_N \left(E_{QM}(\xi_1, \ldots, \xi_N)\right)^2. \qquad (6)$$

If we can prove that a strict inequality holds, namely for all possible E_{LHV}, one has

$$\langle E_{QM} \mid E_{LHV} \rangle \leq B, \qquad (7)$$

with the number $B < \parallel E_{QM} \parallel^2$, we would immediately have $E_{QM} \neq E_{LHV}$, indicating that the correlations in the state ϱ_N are of a different character than in any local realistic theory. We then could say that the state ϱ_N violates the "functional" Bell inequality (7), as this Bell inequality is expressed in terms of a typical scalar product for square integrable functions. Note that the value of the product depends on a continuous range of parameters (of the measuring apparatuses) at each site.

Let us first consider the case of generalized GHZ states $|\psi_N\rangle$ given by (1), and experiments in which each observer is allowed to measure the local observables

$$O_n(\phi_n) = |+, \phi_n\rangle \langle +, \phi_n| - |-, \phi_n\rangle \langle -, \phi_n|, \qquad (8)$$

where

$$|\pm, \phi_n\rangle = \frac{1}{\sqrt{2}} \left(|0\rangle \pm e^{i\phi_n} |1\rangle\right). \qquad (9)$$

The aggregate ξ_n of local parameters at the nth site is just the single parameter ϕ_n here.

With the notations introduced before, one can find that the correlation function E_{QM} for the generalised GHZ state is given by

$$\begin{aligned} E_{QM}(\phi_1, \ldots, \phi_N) &= \langle \psi_N| O_1 \ldots O_N |\psi_N\rangle \\ &= sin(2\beta) \, cos\left(\sum_{n=1}^N \phi_n\right). \end{aligned}$$

Suppose we want to reproduce this correlation function by local hidden variables. The corresponding correlation function must be of the form (4). As the allowed values of the observable $O_n(\phi_n)$ are ± 1, we must correspondingly have $I_n(\phi_n, \lambda) = \pm 1$.

Defining the inner product between (the real-valued functions) $f(\phi_1, \ldots, \phi_N)$ and $g(\phi_1, \ldots, \phi_N)$ by

$$\langle f \mid g\rangle = \Pi_{n=1}^N \left(\int_0^{2\pi} d\phi_n\right) f(\phi_1, \ldots, \phi_N) g(\phi_1, \ldots, \phi_N), \qquad (10)$$

we have (for the generalised GHZ state $|\psi_N\rangle$)

$$\parallel E_{QM} \parallel^2 = \frac{(2\pi)^N}{2} sin^2 2\beta,$$

while

$$\begin{aligned} \langle E_{QM} \mid E_{LHV}\rangle \;=\; & \Pi_{i=1}^N \left(\int_0^{2\pi} d\phi_i\right) \int d\lambda \rho(\lambda) \\ & \times \; \Pi_{j=1}^N I_j(\phi_j, \lambda) cos(\textstyle\sum_{k=1}^N \phi_k) sin 2\beta. \end{aligned}$$

It has been shown in Refs. [6, 7] that the modulus of

$$\Pi_{i=1}^N \left(\int_0^{2\pi} d\phi_i\right) \int d\lambda \rho(\lambda) \Pi_{j=1}^N I_j(\phi_j, \lambda) cos(\sum_{k=1}^N \phi_k)$$

is less than or equal to 4^N (see eqns. (20-23) of Ref. [6] or eqns. (A9-A18) of [7]). Consequently we have $|\langle E_{QM} \mid E_{LHV}\rangle| \leq 4^N sin 2\beta$ for the generalised GHZ state $|\psi_N\rangle$.

Therefore $|\langle E_{QM} \mid E_{LHV}\rangle|$ is strictly less than $\parallel E_{QM} \parallel^2$ whenever

$$sin 2\beta > 2\left(\frac{2}{\pi}\right)^N. \qquad (11)$$

So whenever eq. (11) is satisfied, the generalised GHZ state $|\psi_N\rangle$ violates the functional Bell inequality. Using the WWWZB Bell inequalities [4] for two settings per observer, the range of violation is given by $sin 2\beta > \frac{1}{\sqrt{2^{N-1}}}$ for odd N [10, 11], and for even N violation is obtained for the whole range of β [11]. Note that $\frac{1}{\sqrt{2^{N-1}}} > 2\left(\frac{2}{\pi}\right)^N \Leftrightarrow \left(\frac{\pi}{2^{3/2}}\right)^N > \sqrt{2}$ which holds for all $N \geq 4$. Therefore the functional Bell inequality shows a better range of violation for *odd* $N \geq 5$. And the difference between the two limits of β (for the standard Bell inequalities and the functional Bell inequality) grows with N. The region of violation covers the whole range of β as $N \to \infty$ as in Ref. [11]. However, note that the functional inequality is less restrictive in the case of N even. Perhaps in this case a different version of such an inequality must be used. We leave this question open.

Note that we only consider here violations of the inequality with all the N parties separated and without postselection. The state $|\psi_N\rangle$ is just $sin\beta\,|00\rangle + cos\beta\,|11\rangle$ in any bipartite cut with all the parties on any side of the cut being together. Such a bipartite entangled state always violate a Bell inequality [9]. Also, allowing postselection would always result in a Bell violation as shown by Popescu and Rohrlich [24]. Similar result can be shown using numerical approach [25].

We now go over to our second example of states given by (2). Dür [21] obtained the following interesting result in the multi-qubit case. The state ρ_N of eq.(2) is PPT [16] in all $1 : N-1$ party cuts. A bipartite state ρ_{AB} which is PPT can be either bound entangled or separable [15, 17]. If the state has negative partial transposition [16], it is always entangled [17]. The state ρ_N has a negative partial transpose for all $2 : N-2$ party cuts for $N \geq 4$. Consequently the state ρ_N (for $N \geq 4$) is a bound entangled state as long as all the parties are separated. Nevertheless, Dür [21] showed that such states violate Mermin-Klyshko inequalities [3], with the allowed observables being between σ_x and σ_y at all the N locations (i.e. for the type given by our (8)), for $N \geq 8$. Interestingly, Acín [26] showed that if a state violates the Mermin-Klyshko inequality, it would be possible to create a maximally entangled state in at least one bipartite cut. Further results were obtained in Ref. [27].

In the case of states given by (2), again we allow each observer to make the measurements corresponding to the observables $O_n(\phi_n)$ defined in eq. (8). Then

$$E_{QM} = Tr\,(O_1 \dots O_N \rho_N) = \frac{1}{N+1} cos(\alpha_N - \sum_{i=1}^{N} \phi_i)$$

Consequently (with the same inner product as in the previous case)

$$\parallel E_{QM} \parallel^2 = \frac{1}{(N+1)^2}\frac{1}{2}(2\pi)^N$$

and

$$|\langle E_{QM} \mid E_{LHV}\rangle| \leq \frac{4^N}{N+1},$$

which is again obtained from the results in Refs. [6, 7]. And therefore

$$\parallel E_{QM} \parallel^2 = \frac{1}{(N+1)^2}\frac{1}{2}(2\pi)^N > \frac{4^N}{N+1} \geq |\langle E_{QM} \mid E_{LHV}\rangle|$$

i.e., $E_{QM} \neq E_{LHV}$ for $N \geq 6$ irrespective of the value of the parameter α_N. Thus the state ρ_N violates the functional Bell inequality for $N \geq 6$ for all values of the parameter α_N.

We have considered in this paper, a Bell inequality which seems to be effective when the standard Bell inequalities are not. Contrary to the standard ones, this Bell inequality is for a *continuous* range of settings of the apparatus at each site. We have shown that this inequality shows Bell violation of the generalised GHZ state $sin\beta\,|0\rangle^{\otimes N} + cos\beta\,|1\rangle^{\otimes N}$ (with N separated parties without postselection) for a larger range of the parameter β (for odd $N \geq 5$) than by any of the standard Bell inequalities. Further a 6-qubit one-parameter family of bound entangled states is shown to violate this inequality.

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