Vacuum-field Rabi oscillations of atoms in a cavity

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The quantum electrodynamics of an atom in a cavity is reexamined from the view point of the vacuum-field Rabi oscillations. Many features of such oscillations in a single atom as well as in a cooperative system are discussed. Various physical quantities that carry the signature of these vacuum-field oscillations are calculated. Even oscillator systems are shown to exhibit such vacuum-field Rabi oscillations, which is in contrast to the external-field-induced Rabi oscillations. Such oscillations are found to occur in situations involving both single-photon and multiphoton transitions.

1. INTRODUCTION

The study of Rydberg atoms[1–6] has proved to be exciting from several points of view. For example, one now has the possibility of testing the predictions of the various simplified quantum-electrodynamic models of radiation–matter interaction. Among the various possible quantum-electrodynamic effects produced by atoms in cavities, one finds the existence of Rabi oscillations induced by the *vacuum* of the radiation field. Such vacuum-field Rabi oscillations have been shown, for example, to lead to a splitting in the spontaneous-emission spectra of atoms in cavity.[6] Preliminary experiments[2] on these Rabi oscillations have already been performed.

The purpose of the present study is to discuss the general features of the vacuum-field Rabi oscillations and to examine the effects of the environment on such oscillations. The organization of the paper is as follows: In Section 2, we discuss the general characteristics of vacuum-field Rabi oscillations. We discuss the possibility of several other observables, such as higher-order intensity correlations that exhibit these oscillations. In Section 3, we examine the question of whether the nonlinearity of the atomic-operator algebra (cf. $[S^+, S^-] = 2S^z$) is essential for the existence of such oscillations. Sections 3 and 4 are devoted to the study of the effects of the transition to a neighboring level and the cooperativity of the atomic system on vacuum-field Rabi oscillations. In Section 6, we treat the case in which the two states of the atom are coupled by a multiphoton transition.

2. VACUUM-FIELD RABI OSCILLATIONS

The Rabi oscillations of the atoms induced by the external field are well known.[7,8] For example, consider the interaction of a two-level system, with states $|e\rangle$ and $|g\rangle$ separated by the frequency ω, with a plane monochromatic field of frequency ω. Then the probability $p_{e\to g}$ or $p_{g\to e}$ for making a transition from excited state to ground state and vice versa is given by

$$p_{e\to g} = p_{g\to e} = \sin^2 \frac{\Omega t}{2}, \qquad \Omega = \frac{2}{\hbar}\mathbf{d}\cdot\epsilon. \tag{2.1}$$

Thus the atomic population oscillates, with frequency Ω, back and forth between the ground and the excited states. The oscillation frequency is proportional to the electric field. Such external-field-induced oscillations have been studied extensively and are known to lead to important observable effects[8] in resonance fluorescence, Raman scattering, etc.

The transition probabilities in Eqs. (2.1) have been calculated by treating the external field classically. The question now arises about what happens if the external field is extremely weak. For example, the field may contain few photons. This would certainly be the case in an empty cavity. In such a case, the field ϵ is to be treated quantum mechanically. Jaynes and Cummings[9] consider a simple-model Hamiltonian characterizing the interaction of a two-level system with a single mode of the quantized electromagnetic field of frequency ω:

$$H = \hbar\omega_0 S^z + \hbar\omega a^+ a + \hbar(g S^+ a + g^* S^- a^+),$$
$$g = i\mathbf{d}\cdot\hat{\epsilon}\left(\frac{2\pi\omega}{\hbar V}\right)^{1/2}. \tag{2.2}$$

Here, the two-level system has been represented by spin ½ operators $S^\pm$ and S^z, V is the effective volume of the cavity, and $\hat{\epsilon}$ is the polarization vector of the field. Field-creation and -annihilation operators are represented by a^+ and a. This model has been extensively studied.[10–16] Jaynes and Cummings diagonalized Eqs. (2.2) with the results

$$H|0, g\rangle = \hbar\left(-\frac{\omega_0}{2}\right)|0, g\rangle,$$
$$H|\psi_n{}^\pm\rangle = \hbar\left[\omega(n + 1/2) \pm \frac{\Omega_{n\Delta}}{2}\right]|\psi_n{}^\pm\rangle = \hbar\omega_n{}^\pm|\psi_n{}^\pm\rangle,$$
$$\Omega_{n\Delta}{}^2 = 4|g|^2(n+1) + \Delta^2, \qquad \Delta = (\omega_0 - \omega),$$
$$|\psi_n{}^\pm\rangle = \begin{pmatrix}\cos\theta_n\\ -\sin\theta_n\end{pmatrix}|n+1, g\rangle + \begin{pmatrix}\sin\theta_n\\ \cos\theta_n\end{pmatrix}|n, e\rangle,$$
$$n = 0, 1, 2, \ldots, \infty,$$
$$\tan\theta_n = 2g\sqrt{n+1}/(\Omega_{n\Delta} - \Delta). \tag{2.3}$$

Here, the coupling coefficient g has been made real by a choice of the phase of the dipole-matrix element. Since the eigenvalues and eigenfunctions of Eqs. (2.2) are known in closed form, all the dynamical questions can be answered. The time-evolution operator can be written as

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$$\exp(-iHt/\hbar) = \exp[-it(\omega - \omega_0/2)]|0, g\rangle\langle 0, g| + \sum_{n=0}^{\infty} [\exp(-i\omega_n{}^+t)|\psi_n{}^+\rangle\langle\psi_n{}^+| + \exp(-i\omega_n{}^-t)|\psi_n{}^-\rangle\langle\psi_n{}^-|], \quad (2.4)$$

and hence

$$\exp(-iHt/\hbar)|n, e\rangle = A_{ne}(t)|n, e\rangle + B_{ne}(t)|n+1, g\rangle, \quad (2.5)$$

$$\exp(-iHt/\hbar)|n+1, g\rangle = A_{n+1g}(t)|n+1, g\rangle + B_{n+1,g}(t)|n, e\rangle. \quad (2.6)$$

Here, the coefficients A and B are easily calculated from Eq. (2.4):

$$A_{ne}(t) = \sin^2\theta_n \exp(-i\omega_n{}^+t) + \cos^2\theta_n \exp(-i\omega_n{}^-t), \quad (2.7)$$

$$A_{n+1,g}(t) = \cos^2\theta_n \exp(-i\omega_n{}^+t) + \sin^2\theta_n \exp(-i\omega_n{}^-t),$$
$$A_{0g}(t) = \exp(i\omega_0 t/2), \qquad B_{0g}(t) = 0, \quad (2.8)$$

$$B_{ne}(t) = B_{n+1,g} = \cos\theta_n \sin\theta_n[\exp(-i\omega_n{}^+t) - \exp(-i\omega_n{}^-t)]. \quad (2.9)$$

The coefficient $B_{ne}(t)$ gives the transition amplitude for finding the atom in the ground state given that it was in the excited state at $t = 0$. In the transition, the excitation of the field mode changes by unity. The transition probability $p_{en\to g,n+1}$ is

$$P_{en\to g,n+1}(t) = \frac{4g^2(n+1)}{\Omega_{n\Delta}{}^2}\sin^2 t\Omega_{n\Delta}/2, \quad (2.10)$$

where the oscillation frequency $\Omega_{n\Delta}$ is

$$\Omega_{n\Delta}{}^2 = \Delta^2 + 4g^2(n+1). \quad (2.11)$$

The probability [Eq. (2.10)] resembles Eqs. (2.1) (for $\Delta = 0$) but is now valid for arbitrary excitations of the field. Thus it holds also if the field is initially in the vacuum state. Hence even when the field is initially in vacuum, the atomic population oscillates between excited and ground states. Such oscillations are known as vacuum-field Rabi oscillations or self-induced Rabi oscillations. These oscillations characterize the exchange of the energy between the radiation field and the atom, i.e., the atomic excitation is transferred to the field and vice versa. The purpose of the present paper is to explore various consequences of such vacuum-field Rabi oscillations and how such oscillations are affected by the changes in the environment in which the atom is situated.

Vacuum-field Rabi oscillations show up in a variety of the atomic-correlation functions, which in turn yield the characteristics of the emitted radiation. The two time-correlation functions $\langle S^+(t)S^-(\tau)\rangle$ of the dipole-moment operators give the spectrum of the emitted radiation.[17] Such a correlation has been calculated by Sanchez-Mondragon *et al.* using the solution of nonlinear Heisenberg equations of motion. We show how $\langle S^+(t)S^-(\tau)\rangle$ can be calculated by using Eq. (2.5) and summarize some of its properties. We assume that the atom was initially in the excited state and the field mode had arbitrary distribution, i.e., the initial density matrix was

$$\rho(0) = \sum_{nn'} p_{nn'}|n, e\rangle\langle n', e|. \quad (2.12)$$

From the definition of Heisenberg operators we get

$$\Gamma(t, \tau) \equiv \langle S^+(t+\tau)S^-(t)\rangle$$
$$= \sum_{nn'} p_{nn'}\langle n', e|\exp[iH(t+\tau)/\hbar]S^+ \times \exp(-iH\tau/\hbar)S^-\exp(-iHt/\hbar)|n, e\rangle$$
$$= \sum_{nn'} p_{nn'}A_{ne}(t)A_{n'e}{}^*(t+\tau)\langle n', g| \times \exp(-iH\tau/\hbar)|n, g\rangle$$
$$= \sum_{nn'} p_{nn'}A_{ne}(t)A_{n'e}{}^*(t+\tau)A_{ng}(\tau)\langle n'g|ng\rangle$$

and hence

$$\Gamma(t, \tau) = \sum_n p_{nn}A_{ne}(t)A_{ne}{}^*(t+\tau)A_{ng}(\tau). \quad (2.13)$$

The coefficients A are given by Eqs. (2.7) and (2.8). If initially the field is in Fock state $|n\rangle$, then

$$\Gamma(t, \tau) = A_{ne}(t)A_{ne}{}^*(t+\tau)A_{ng}(\tau). \quad (2.14)$$

For other states of the field, one has to average the result [Eq. (2.14)] with respect to the distribution p_{nn}. If we let $\Delta \to 0$, then Eq. (2.14) simplifies considerably:

$$\Gamma(t, \tau) = e^{i\omega\tau}\left\{\frac{1}{2}\cos\frac{\Omega_{n0}}{2}(2t+\tau)\cos\frac{\Omega_{n-1,0}}{2}\tau + \frac{1}{4}\left[\cos\frac{\tau}{2}(\Omega_{n0} + \Omega_{n-1,0}) + \cos\frac{\tau}{2}(\Omega_{n0} - \Omega_{n-1,0})\right]\right\}, \quad n \neq 0 \quad (2.15)$$

$$\Gamma(t, \tau) = e^{i\omega\tau}\frac{1}{2}\left[\cos\frac{\Omega_{00}}{2}(2t+\tau) + \cos\frac{\Omega_{00}\tau}{2}\right]. \quad n = 0 \quad (2.16)$$

The time-averaged correlation function has an interesting structure:

$$\bar{\Gamma}(t, \tau) = \frac{e^{i\omega\tau}}{4}[1 + \cos 2g\sqrt{n}\tau], \qquad n \gg 1, \quad (2.17)$$

$$\bar{\Gamma}(t, \tau) = \frac{e^{i\omega\tau}}{2}\cos(g\tau), \qquad n = 0. \quad (2.18)$$

The Fourier transform of Γ, which yields the spectrum of the emitted radiation, will have (1) three peaks at ω, $\omega \pm 2g\sqrt{n}$ if $n \gg 1$, the central peak being twice as intense as side peaks; (2) two peaks at $\omega \pm g$ if $n = 0$. Note that $n = 0$ case corresponds to the spontaneous emission, and hence the spontaneous-emission spectra will exhibit a doublet. The separation of the doublet is just equal to the frequency of the vacuum-field Rabi oscillation. Some of the above results are in agreement with the numerical results of Sanchez-Mondragon *et al.* Note that the derivation [Eq. (2.17)] uses the Fock state as initial the state of the field, whereas the numerical results of Sanchez-Mondragon *et al.* are for the initial coherent state of the field.

Higher-order correlations of the atomic operators are also expected to carry information on the vacuum-field Rabi oscillations. Let us consider the correlation function $\langle S^+(t)S^+(t+\tau)S^-(t+\tau)S^-(t)\rangle \equiv \Gamma^{(2)}(t, \tau)$ which describes the joint probability of detecting one photon at the time t and another photon at $t + \tau$. Assuming the initial state [Eq. (2.12)] of the atom-field system, we write

$$\begin{aligned}\Gamma^{(2)}(t,\tau) &= \sum_{nn'} \langle n', e| \exp(iHt/\hbar)S^+ \exp(iH\tau/\hbar)(S^+S^-) \\ &\quad \times \exp(-iH\tau/\hbar)S^- \exp(-iHt/\hbar)|n,e\rangle p_{nn'} \\ &= \sum_{nn'} p_{nn'}A_{ne}(t)A_{n'e}^*(t)\langle n', g| \exp(iH\tau/\hbar) \\ &\quad \times S^+S^- \exp(-iH\tau/\hbar)|ng\rangle \\ &= \sum_{nn'} p_{nn'}A_{ne}(t)A_{n'e}^*(t)B_{ng}(\tau)B_{n'g}^*(\tau). \\ &\quad \times \langle n'-1, e|S^+S^-|n-1, e\rangle \\ &= \sum_n p_{nn}|A_{ne}(t)|^2|B_{ng}(\tau)|^2 \to 0 \qquad \text{if } \tau \to 0.\end{aligned} \tag{2.19}$$

For the initial Fock-state excitation[18]

$$\Gamma^{(2)}(t,\tau) = |A_{ne}(t)|^2|B_{ng}(\tau)|^2 \to 0 \qquad \text{if } n \to 0, \tag{2.20}$$

which, for $\Delta = 0$, reduces to

$$\Gamma^{(2)}(t,\tau) = \cos^2\left(\frac{\Omega_{n0}t}{2}\right)\sin^2\left(\frac{\Omega_{n-1,0}\tau}{2}\right), \qquad n \neq 0. \tag{2.21}$$

The intensity correlation for the Fock-state excitation has a simple factorization property in terms of the probabilities for the two events. $|A_{ne}(t)|^2$ is the probability of finding the field-atom system in the state $|n, e\rangle$ if it was at $t = 0$ in the state $|n, e\rangle$. Note that t is the time when the first photon is detected; then the atom goes to the ground state. The total number of photons in the field is $(n + 1)$, from which the detected photon must have leaked out of the cavity, thus leaving n photons in the cavity. Thus, after the detection of the first photon, the field-atom state is $|n, g\rangle$. The probability of finding the atom-field system in the state $|n - 1, e\rangle$ at time τ, given that at $\tau = 0$ it was in the state $|n, g\rangle$, is $|B_{n,g}(\tau)|^2$. The situation is more complex for other states of the radiation field. Finally note that the higher-order correlations such as $\langle S^+(t)S^+(t + \tau_1) \ldots S^+(t + \tau_n)S^-(t + \tau_n) \ldots S^-(t)\rangle$ also be calculated by using the above procedure.

3. IS NONLINEARITY ESSENTIAL FOR VACUUM-FIELD RABI OSCILLATIONS?

In this section, we examine whether the nonlinearity of the radiation–matter interaction is essential for the existence of the vacuum-field Rabi oscillations. The usual Rabi oscillations [Eqs. (2.1)] are associated with spin systems in external fields. The existence of the Rabi oscillations is due to the fact that the commutator $[S^+, S^-] = 2S^z$ depends on the atomic-inversion operator. An oscillator system in an external field does not show Rabi oscillations. However, we prove the remarkable property that the nonlinearity of the radiation–matter interaction is *not* essential for vacuum-field Rabi oscillations, and, in fact, even an oscillator can exhibit such vacuum-field Rabi oscillations.

Consider the interaction between an atomic oscillator (with annihilation and creation operators b and b^+) and the field oscillator

$$H = \hbar\omega_0 b^+b + \hbar\omega a^+a + (\hbar g b^+a + \text{H.c.}). \tag{3.1}$$

If we write the solution for the Heisenberg equations of motion as

$$b(t) = \beta(t)b(0) + \alpha(t)a(0), \tag{3.2}$$

then the correlation function $\Gamma(t, \tau) = \langle b^+(t + \tau)b(t)\rangle$ of the (oscillator) dipole operators will be

$$\Gamma(t,\tau) = \beta^*(t+\tau)\beta(t)\langle b^+(0)b(0)\rangle. \tag{3.3}$$

Here, we have assumed that the oscillators a and b are uncorrelated and that the field is in the vacuum state at $t = 0$. The function $\beta(t)$ is

$$\begin{aligned}\beta(t) &= \frac{\exp(-i\omega t - i\Delta t/2)}{2\nu}\left[\left(\frac{\Delta}{2} + \nu\right)\exp(-i\nu t)\right. \\ &\quad \left. - (\Delta/2 - \nu)\exp(i\nu t)\right], \\ \nu &= [g^2 + (\Delta/2)^2]^{1/2} \equiv \Omega_{0\Delta}/2.\end{aligned} \tag{3.4}$$

We thus find that even the oscillator model of the atomic system exhibits vacuum-field Rabi oscillations, though such a model does not exhibit the usual, i.e., external-field-induced, Rabi oscillations. It is interesting to note that the correlation function [Eq. (3.3)] for the oscillator model is identical with that of Eq. (2.13) with $p_{nn} = \delta_{n0}$ for the two-level model. It is remarkable that the vacuum-field Rabi oscillations could occur in such a simple model of radiation–matter interaction.

4. EFFECT OF TRANSITION TO A NEIGHBORING LEVEL ON VACUUM-FIELD RABI OSCILLATIONS

In the previous sections, we have discussed the occurrence of vacuum-field Rabi oscillations under rather ideal conditions, such as a single atom, a two-level transition, and a single mode of the cavity. It is interesting to find out what happens to such oscillations if some of these conditions are not met. In this section, we examine the effect of transition to a neighboring level.[19] Let us assume that the ground state $|g\rangle$ is nearly degenerate with another energy level $|g'\rangle$. The Hamiltonian for the present three-level system interacting with a single mode of the radiation field is

$$\begin{aligned}H &= \hbar\omega a^+a + \hbar\omega_g|g\rangle\langle g| + \hbar\omega_{g'}|g'\rangle\langle g'| + \hbar\omega_e|e\rangle\langle e| \\ &\quad + (\hbar g_1 a|e\rangle\langle g| + \hbar g_2 a|e\rangle\langle g'| + \text{H.c.}).\end{aligned} \tag{4.1}$$

Here g_1 and g_1 are the coupling coefficients for the two transitions $|e\rangle \to |g\rangle, |e\rangle \to |g'\rangle$. Since our interest in this section is in vacuum-field Rabi oscillations, we take the initial state of the system as $|0, e\rangle$. From the structure of the Hamiltonian [Eq. (4.1)] it is clear that the state at time t will be

$$\begin{aligned}|\psi(t)\rangle &= \exp(-iHt/\hbar)|0,e\rangle \\ &= \alpha(t)|0,e\rangle + \beta(t)|1,g\rangle + \gamma(t)|1,g'\rangle,\end{aligned} \tag{4.2}$$

where α, β, and γ are given from the solution of

$$\frac{d}{dt}\begin{bmatrix}\alpha \\ \beta \\ \gamma\end{bmatrix} = -i\begin{bmatrix}\omega_e & g_1 & g_2 \\ g_1^* & \omega_g + \omega & 0 \\ g_2^* & 0 & \omega_{g'} + \omega\end{bmatrix}\begin{bmatrix}\alpha \\ \beta \\ \gamma\end{bmatrix}, \tag{4.3}$$

subject to the initial condition that $\alpha(0) = 1, \beta(0) = \gamma(0) = 0$.

A typical dipole correlation that determines spontaneous emission from such a system will be

$$\Gamma(t, \tau) = \langle 0, e| (|e\rangle\langle g|)_{t+\tau} (|g\rangle\langle e|)_t |0, e\rangle, \qquad (4.4)$$

which on simplification becomes

$$\begin{aligned}\Gamma(t, \tau) &= \langle \psi(t+\tau)|e\rangle\langle g|\exp(-iH\tau/\hbar)|g\rangle\langle e|\psi(t)\rangle \\ &= \alpha^*(t+\tau)\alpha(t)\langle 0, g|\exp(-iH\tau/\hbar)|0, g\rangle\end{aligned}$$

and hence

$$\Gamma(t, \tau) = \alpha^*(t+\tau)\alpha(t)\exp(-i\omega_g\tau). \qquad (4.5)$$

If λ_i's are the eigenvalues of the matrix in Eq. (4.3), then

$$\alpha(t) = \sum_{i=1}^{3} \alpha_i \exp(-i\lambda_i t). \qquad (4.6)$$

The structure of Eqs. (4.5) and (4.6) implies that the vacuum-field Rabi oscillations would now be related to frequencies $(\omega_g - \lambda_i)$. Thus the frequencies of such oscillations depend on the degeneracy of the level. For the special case when $\omega_g = \omega_{g'}$, $\omega_e - \omega_g = \omega$, the eigenvalues are found to be $\omega + \omega_g$, $\omega + \omega_g \pm (|g_1|^2 + |g_2|^2)^{1/2}$. Therefore the spontaneous-emission spectra will have a triplet structure with peaks at ω, $\omega \pm (|g_1|^2 + |g_2|^2)^{1/2}$. It may be noted that in a recent experiment[2] the degeneracy problem was avoided by suitable application of a dc magnetic field.

5. EFFECT OF COOPERATIVITY ON VACUUM-FIELD RABI OSCILLATIONS

In this section, we investigate the effect of the cooperativity on vacuum-field Rabi oscillations. In particular, we examine what happens to such oscillations if the atom in the cavity is in the company of several other excited atoms. The Hamiltonian for a collection of identical N two-level atoms interacting with a single mode of the field is

$$H = \hbar\omega_0 \sum_{i=1}^{N} S_i^z + \hbar\omega a^+a + \hbar \sum_i (gaS_i^+ + \text{H.c.}). \qquad (5.1)$$

For arbitrary number of atoms, no exact analytical[13] results for the Hamiltonian [Eq. (5.1)] are known, although numerically Eq. (5.1) can be solved for eigenfunctions,[12,14] etc.

In order to examine the simplest aspect of Eq. (5.1), let us look at the oscillator version of Eq. (5.1), i.e., $S_i^+ \to b_i^+$ with $[b_i, b_j^+] = \delta_{ij}$. The Heisenberg equations are now

$$\dot{a} = -i\omega a - ig \sum_i b_i, \qquad \dot{b}_i = -i\omega_0 b_i - iga. \qquad (5.2)$$

Assuming a vacuum state of the cavity mode and no correlations among the oscillators b_i, it can easily be shown from Eq. (5.2) that

$$\left\langle \sum_i b_i^+(t+\tau) \sum_j b_j(t) \right\rangle = e^{i\omega\tau} \cos g\sqrt{N}(t+\tau) \times \cos g\sqrt{N}t \left\langle \sum_{ij} b_i^+ b_j \right\rangle. \qquad (5.3)$$

Thus the effect of the cooperativity in the simple oscillator model[20] is to scale the vacuum-field Rabi frequency $g \to g\sqrt{N}$.

Since the full problem in Eq. (5.1) is not analytically tractable, we consider the effects arising owing to the presence of another atom. We thus solve the two-atom problem

$$H = \hbar\omega a^+a + \hbar\omega_0 S^z + \hbar(gaS^+ + \text{H.c.}), \qquad (5.4)$$

where $\mathbf{S}$ is now a spin-1 operator. We calculate the correlation function $\langle S^+(t+\tau)S^-(t)\rangle$, assuming that each atom initially is in the excited state and that the field is in the vacuum state. Let us denote the eigenstates of the operator S^z by $|f_1\rangle, |f_0\rangle, |f_{-1}\rangle$ instead of $|\pm 1\rangle, |0\rangle$ so that there is no confusion with the field eigenstates. From the structure of the Hamiltonian [Eq. (5.4)], the following relations are clearly true:

$$e^{-iHt/\hbar}|0, f_1\rangle = \alpha(t)|0, f_1\rangle + \beta(t)|1, f_0\rangle + \gamma(t)|2, f_{-1}\rangle, \qquad (5.5)$$

$$e^{-iHt/\hbar}|0, f_0\rangle = \mu(t)|0, f_0\rangle + \nu(t)|1, f_{-1}\rangle, \qquad (5.6)$$

$$e^{-iHt/\hbar}|1, f_{-1}\rangle = \varphi(t)|1, f_{-1}\rangle + \chi(t)|0, f_0\rangle. \qquad (5.7)$$

The coefficients that appear in Eqs. (5.5)–(5.7) are to be obtained from the solution of the Schrödinger equation, which leads to

$$\left[\frac{\mathrm{d}}{\mathrm{d}t} + i\begin{pmatrix} \omega_0 & g\sqrt{2} & 0 \\ g\sqrt{2} & \omega & 2g \\ 0 & 2g & 2\omega - \omega_0 \end{pmatrix}\right]\begin{bmatrix}\alpha(t) \\ \beta(t) \\ \gamma(t)\end{bmatrix} = 0,$$

$$\alpha(0) = 1, \qquad \beta(0) = \gamma(0) = 0, \qquad (5.8)$$

$$\left[\frac{\mathrm{d}}{\mathrm{d}t} + i\begin{pmatrix} 0 & g\sqrt{2} \\ g\sqrt{2} & +\Delta \end{pmatrix}\right]\begin{bmatrix}\mu(t) \\ \nu(t)\end{bmatrix} = 0,$$

$$\mu(0) = 1, \qquad \nu(0) = 0, \qquad \Delta = \omega_0 - \omega, \qquad (5.9)$$

$$\left[\frac{\mathrm{d}}{\mathrm{d}t} + i\begin{pmatrix} -\Delta & g\sqrt{2} \\ g\sqrt{2} & 0 \end{pmatrix}\right]\begin{bmatrix}\varphi(t) \\ \chi(t)\end{bmatrix} = 0,$$

$$\varphi(0) = 1, \qquad \chi(0) = 0. \qquad (5.10)$$

The dipole correlation function can be calculated in terms of the above coefficients:

$$\begin{aligned}\Gamma(t, \tau) &= \langle S^+(t+\tau)S^-(t)\rangle \\ &= 2[\alpha^*(t+\tau)\alpha(t)\mu(\tau) + \alpha(t)\beta^*(t+\tau)\nu(\tau) \\ &\quad + \alpha^*(t+\tau)\beta(t)\nu^*(-\tau) + \beta^*(t+\tau)\beta(t)\varphi(\tau)].\end{aligned} \qquad (5.11)$$

For finite detuning, Γ has a rather cumbersome form. Hence, in order to appreciate the effect of the presence of another atom, we present the result for the resonant case $\Delta = 0$:

$$\begin{aligned}\Gamma(t, \tau) = (2)e^{i\omega\tau}\Bigg(&\left\{\frac{4}{9} + \frac{2}{9}[\cos\sqrt{6}g\,(t+\tau) + \cos\sqrt{6}gt\,] \right.\\ &\left. + \frac{2}{9}[\cos\sqrt{6}g\tau - 1/2\cos\sqrt{6}g\,(2t+\tau)]\right\}\cos\sqrt{2}g\tau \\ &+ [\sin\sqrt{6}g\tau + 2\sin\sqrt{6}g\,(t+\tau) \\ &- 2\sin\sqrt{6}gt\,]\frac{\sin\sqrt{2}g\tau}{3\sqrt{3}}\Bigg).\end{aligned} \qquad (5.12)$$

A plot of the dipole correlation function [Eq. (5.12)] normalized to the number of atoms for two atoms is given in Fig. 1, where it is also compared with the corresponding result for the single-atom problem. The τ dependence of the above correlation suggests, for example, that the spontaneous-emission

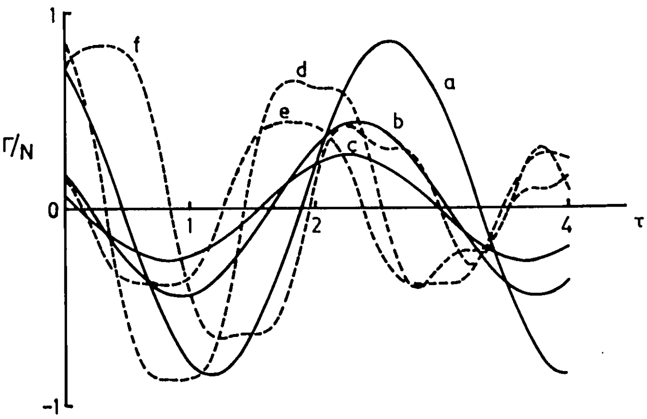

Fig. 1. The behavior of the dipole correlation function $\langle S^+(t+\tau)S^-(t)\rangle/N$ as a function of τ (in units of π) for fixed value of $t = \pi/4$ for a and d; $\pi/2$ for b and e; and 2π for c and f. Initially, the atoms are in the excited state. The full curves are for the single-atom case, whereas the dashed curves are for a cooperative system of two atoms.

spectrum will have many vacuum-field-induced Rabi splittings depending on the resolution

$$\omega_s - \omega = \pm g\sqrt{2}, \pm(\sqrt{3} \pm 1)\sqrt{2}g. \qquad (5.13)$$

Using the correlation function expressed in Eq. (5.12) and following the definition of Eberly and Wodkiewicz,[17] one can show that in the long time limit the spontaneous-emission spectrum is explicitly given by

$$S(\omega, T \to \infty) = \frac{2\Gamma/9}{\Gamma^2 + (\omega_s - \omega - \sqrt{3}g)^2} + (1/18)\left(1 - \frac{\sqrt{3}}{2}\right) \times \frac{\Gamma}{\Gamma^2 + [\omega_s - \omega - (\sqrt{6} + \sqrt{2})g]^2} + (1/18)\left(1 + \frac{\sqrt{3}}{2}\right) \times \frac{\Gamma}{\Gamma^2 + [\omega_s - \omega - (\sqrt{6} - \sqrt{2})g]^2} + g \to -g. \qquad (5.14)$$

Here Γ is the bandwidth of the detecting mechanism. It is thus clear that the presence of another excited atom in the vicinity of the atom can considerably complicate the spontaneous-emission spectra.

6. VACUUM-FIELD RABI OSCILLATIONS AND MULTIPHOTON TRANSITIONS

In this section, we show that the vacuum-field Rabi oscillations also occur in multiphoton transitions in atoms contained in cavities. For simplicity, we consider only two-photon transitions, which can be characterized by an effective Hamiltonian

$$H = \hbar\omega a^+ a + \hbar\omega_0 S^z + (\hbar g S^+ a^2 + \text{H.c.}) + \sum_{i=e,g} \hbar\beta_i a^+ a|i\rangle\langle i|, \qquad \omega_0 \approx 2\omega. \qquad (6.1)$$

Here, g is the two-photon matrix element and $\beta_i n$ gives the Stark shift of the ith level; the frequency difference ω_0 is also shifted owing to the spontaneous contributions to the Stark shift (Lamb shift of each state owing to the interaction of each level with a single mode of the cavity). The effective Hamiltonian [Eq. (6.1)] can be derived by using, say, canonical transformations,[21] keeping in view the quantized treatment of the radiation field.

The Hamiltonian [Eq. (6.1)] can be diagonalized in the same way as the Jaynes–Cummings Hamiltonian. The eigenfunctions and eigenvalues are found to be

$$|\psi_n^+\rangle = \cos\theta_n|n+2, g\rangle + \sin\theta_n|n, e\rangle,$$
$$|\psi_n^-\rangle = -\sin\theta_n|n+2, g\rangle + \cos\theta_n|n, e\rangle, \qquad (6.2)$$

$$\tan\theta_n = \frac{2g[(n+1)(n+2)]^{1/2}}{(\lambda_n^+ - \Delta - 2\beta_1 n)}, \qquad \Delta = (\omega_0 - 2\omega) \qquad (6.3)$$

$$E_n^\pm = \omega(n+1) + \lambda_n^\pm,$$
$$\lambda_n^\pm = \frac{\beta_1 n + \beta_2(n+2)}{2} \pm (1/2)\{4g^2(n+1)(n+2) + [\Delta + \beta_1 n - \beta_2(n+2)]^2\}^{1/2}. \qquad (6.4)$$

For this model, the probability of finding the field-atom system in the state $|n+2, g\rangle$ given that it was initially in the state $|n, e\rangle$ is found to be

$$p(t) = 4\cos^2\theta_n \sin^2\theta_n \sin^2\left(\frac{t\Omega_{n\Delta}}{2}\right), \qquad (6.5)$$

$$\Omega_{n\Delta} = \{4g^2(n+1)(n+2) + [\Delta + \beta_1 n - \beta_2(n+2)]^2\}^{1/2} \qquad (6.6)$$

i.e.,

$$\Omega_{n\Delta} = [4g^2(n+1)(n+2) + \Delta_n^2]^{1/2}. \qquad (6.7)$$

Here, Δ_n is the Stark-shift-dependent detuning parameter. The shift of each state depends on the number of photons, which is equal to $n(n+2)$ for the excited ground (state). For $n = 0$ (and $\Delta_n = 0$), we get the vacuum-field Rabi oscillations in the present model of two-photon transitions. These oscillations occur at the frequency $2\sqrt{2}g$. Note that in order to make the effect observable, one has to tune the system in such a way that there is resonant enhancement of the two-photon matrix element g.

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