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Ergodic theorems for transient one-dimensional diffusions

K.B. Athreya*[,1], A.P.N. Weerasinghe

Department of Mathematics, Iowa State University, Ames, IA 50011 USA

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Abstract

For one-dimensional diffusions X that drift off to $+\infty$ we give conditions on a set B and the drift and diffusion coefficients of X for $(1/t)\int_0^t I_B(X(u))\,\mathrm{d}u$ to converge w.p.l as $t \to \infty$.

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1. Introduction

In a recent paper Bingham and Rogers (1991) showed that if $X(t) = t + B(t)$, $t \geqslant 0$ where $B(\cdot)$ is standard Brownian motion then for any Borel set $A \subset [0, \infty)$,

$$\frac{1}{t}\int_0^t I_A(X(u))\,\mathrm{d}u - \frac{1}{t}\int_0^t I_A(u)\,\mathrm{d}u \to 0 \quad \text{a.s.} \tag{1}$$

The goal of this paper is to investigate similar phenomenon for a general diffusion on the line which drifts off to infinity. Clearly, if $X(t) \equiv \mu t + \sigma B(t)$ where $\mu > 0$, $\sigma > 0$, result (1) should hold. Hence, one expects that for any diffusion $\mathrm{d}X(t) = \mu(X(t))\,\mathrm{d}t + \sigma(X(t))\,\mathrm{d}B(t)$; (1) should hold if the functions $\mu(\cdot)$ and $\sigma(\cdot)$ are asymptotically constant. It is also tempting conjecture that if the diffusion term is not overwhelming then the diffusion trajectory and deterministic trajectory $\mathrm{d}\tilde{x} = \mu(\tilde{x})\,\mathrm{d}t$ spend asymptotically same proportion of time for many sets A.

In this paper we determine how far the above remarks are valid and find a set of reasonable sufficient conditions on the diffusion and drift coefficients $\mu(\cdot)$ and $\sigma(\cdot)$ of a general one-dimensional diffusion for the validity of a result similar to (1). It turns out that we are able to prove a ratio type theorem rather than the strong comparison result (1). The main result is Theorem 1 below. Corollaries 1 and 2 cover the cases

* Corresponding author.

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when μ and σ converge at ∞ and when μ and σ are periodic with a common period, respectively.

In higher dimensions a result similar to (1) or to our Theorem 1 is unlikely to hold even in the presence of a strong drift. For example if $(X_1(t), X_2(t))$ is a two-dimensional diffusion where there is a strong drift towards ∞ along the line $x_1 = x_2$, if the diffusion is nontrivial then once the path is away for $x_1 = x_2$ it could be subjected to a drift in a very different direction.

2. The main results

Let $\{X(t)\colon t \geqslant 0\}$ be a diffusion satisfying

$$\begin{aligned} \mathrm{d}X(t) &= \mu(X(t))\,\mathrm{d}t + \sigma(X(t))\,\mathrm{d}B(t), \quad t \geqslant 0, \\ X(0) &= x_0. \end{aligned} \tag{2}$$

We assume the following conditions on $\mu(\cdot)$ and $\sigma(\cdot)$:

(A.1) $\mu(\cdot)$ is Borel measurable and bounded in finite intervals and $\sigma(\cdot)$ is continuous,

(A.2) $\sigma^2(x) > 0$ for all x (and hence $\rho(u) = 2\mu(u)/\sigma^2(u)$ is locally integrable).

(A.3) $S(\cdot)$, the *scale function*, is defined by

$$S(x) \equiv \int_0^x \mathrm{e}^{-A(u)}\,\mathrm{d}u \quad \text{where } A(u) = \int_0^u \rho(u)\,\mathrm{d}u,$$

satisfies $S(-\infty) = -\infty$, $S(+\infty) < \infty$, where we use the convention that for $x < 0$, $\int_0^x f(u)\,\mathrm{d}u = -\int_x^0 f(u)\,\mathrm{d}u$.

(A.4) Any weak solution of (1) is nonexplosive in finite time.

It is known (see Karatzas and Shreve, 1988) that under (A.1) there is a weak solution to (2) and under (A.2) and (A.3) any such solution will satisfy $P_{x_0}(X(t) \to \infty$, as $t \to \infty) = 1$ for all x, where P_x is the probability distribution of the process X starting at $X(0) = x$. For sufficient conditions for (A.4) see Karatzas and Shreve (1988, p. 342).

In what follows $P_x(\cdot)$ denote the probability measure on the process corresponding to $X(0) = x$ and $E_x(\cdot)$ the expectation with respect to P_x.

Let

$$\tau_y = \inf\{t\colon t \geqslant 0,\ X(t) = y\}. \tag{3}$$

Then since $P_x(X(t) \to \infty) = 1$ for any x, $P_x(\tau_y < \infty) = 1$ for all $x < y$.

Theorem 1. *Assume, in addition to* (A.1)–(A.4), *that*

(i) $$\int_0^n \mathrm{e}^{-A(u)}\left(\int_{-\infty}^u \mathrm{e}^{A(r)}\frac{1}{\sigma^2(r)}\,\mathrm{d}r\right)\mathrm{d}u \sim c_1 n \quad \textit{as } n \to \infty,$$

(ii) $$\sup_n \int_n^{n+1} \mathrm{e}^{-A(u)}\left(\int_{-\infty}^u \mathrm{e}^{A(r)} E_r(\tau_{n+1})\frac{1}{\sigma^2(r)}\,\mathrm{d}r\right)\mathrm{d}u \leqslant K < \infty,$$

(iii) $$2\int_0^n e^{-A(u)}\left(\int_{-\infty}^u I_B(r)e^{A(r)}\frac{1}{\sigma^2(r)}\,dr\right)du \sim c_2 n \quad as\ n\to\infty,$$

(iv) $$\int_{-\infty}^0 e^{A(r)}\frac{1}{\sigma^2(r)}E_r(\tau_0)\,dr < \infty.$$

Then

$$\frac{1}{t}\int_0^t I_B(X(u))\,du \to \frac{c_2}{c_1} \quad as\ t\to\infty \ \ with\ probability\ one. \tag{4}$$

Some sufficient conditions for the validity of the assumptions of Theorem 1 will be given in Propositions 4 and 5 in the next section.

Corollary 1. *Let* $\mu(r)\to\mu$, $\sigma(r)\to\sigma$ *as* $r\to\infty$ *with* $0<\mu<\infty$, $0<\sigma<\infty$ *and conditions* (iii) *of Proposition* 4 *and* (ii) *of Proposition* 5 *hold and* $(1/t)\int_0^t I_B(r)\,dr \to C_B$, $0<C_B<\infty$. *Then* (4) *holds.*

Corollary 2. *Let* $\mu(\cdot)$ *and* $\sigma(\cdot)$ *be periodic with period one. Assume* $\int_0^1 \rho(u)\,du > 0$ *where* ρ *is as in* (A.2). *Let* $X(\cdot)$ *be a solution to* (2). *Then for a given Borel set* $B\subseteq[0,\infty)$, $(1/t)\int_0^t I_B(X(s))\,ds$ *converges w.p.1 as* $t\to\infty$ *if and only if*

$$\int_0^1\left(\frac{1}{n}\sum_{j=0}^{n-1} I_B(r+j)\right)\frac{1}{\sigma^2(r)}\psi(r)\,dr$$

converges as $n\to\infty$ *where*

$$\psi(r) = \int_0^\infty e^{-(A(r+s)-A(r))}$$

with $A(\cdot)$ *as in* (A.3).

In particular a sufficient condition for the above is that $T_n(r)\equiv(1/n)\sum_{j=0}^{n-1} I_B(r+j)$ *converges a.e. with respect to Lebesgue measure on* $[0,1]$.

A few remarks on the hypotheses of Theorem 1 and Corollaries 1 and 2 are in order. Condition (iii) of Theorem 1 is an asymptotic density condition on the set B and comes from estimating the mean value of the time spent in B by the process until it crosses level n. Similar considerations appear in Bingham and Goldie (1982). Condition (i) is a growth condition on $E_0(\tau_u)$. This needs the finiteness of $\int_{-\infty}^0 e^{A(r)}(1/\sigma^2(r)) \times E_r(\tau_0)\,dr < \infty$ which appears again as condition (iii) in Proposition 4 below. Conditions (ii) and (iv) are needed for $E_r\tau_u^2$ to be bounded.

3. Proof of the main results

Fix a Borel set B in $\mathbb{R}$ and set

$$\zeta_j = \int_{\tau_{j-1}}^{\tau_j} I_B(X(u))\,du, \quad \eta_j = (\tau_j - \tau_{j-1}) \qquad \text{for } j\geqslant 1. \tag{5}$$

Let $F_t = \sigma(X(u): u \leqslant t)$ and $\mathscr{F}_j = F_{\tau_j}$ be the stopped σ-algebra corresponding to τ_j. From Hall and Heyde (1962, Theorem 2.19, p. 36) we know that

$$\frac{1}{n}\sum_{j=1}^{n} (\zeta_j - E(\zeta_j \mid \mathscr{F}_{j-1})) \to 0, \tag{6a}$$

$$\frac{1}{n}\sum_{j=1}^{n} (\eta_j - E(\eta_j \mid \mathscr{F}_{j-1})) \to 0 \tag{6b}$$

a.s. if there exists a nonnegative random variable X and a constant C such that

$$\sup_j P(\eta_{j+1} \geqslant x \mid \mathscr{F}_j) \leqslant CP(X \geqslant x) \tag{7}$$

a.s. and $E|X| < \infty$. A sufficient condition for this is

$$E(\tau_{j+1} - \tau_j)^{1+\delta} \text{ is bounded in } j \text{ for some } \delta > 0. \tag{8}$$

If (6a) and (6b) hold then $(1/n)\sum_1^n \zeta_j$, $(1/n)\sum_{j=1}^n \eta_j$ converge w.p.1 iff $(1/n)\sum_{j=1}^n E(\zeta_j \mid \mathscr{F}_{j-1})$ and $(1/n)\sum_{j=1}^n E(\eta_j \mid \mathscr{F}_{j-1})$ converge as $n \to \infty$. By the continuity of sample paths of $X(t)$ and the strong Markov property $E(\cdot \mid \mathscr{F}_{j-1}) = E_{j-1}(\cdot)$ where E_x denotes expectation with respect to the process starting at $X(0) = x$

$$\begin{aligned}\sum_{j=1}^{n} E(\zeta_j \mid \mathscr{F}_{j-1}) &= \sum_{j=1}^{n} E_{j-1}\left(\int_{\tau_{j-1}}^{\tau_j} I_B(X(u))\,du\right) \\ &= \sum_{j=1}^{n} E\left(\int_{\tau_{j-1}}^{\tau_j} I_B(X(u))\,du\right) \quad \text{(by the strong Markov property)} \\ &= E\left(\int_0^{\tau_n} I_B(X(u))\,du\right) \quad (\text{since } \tau_0 = 0 \text{ under } X(0) = 0).\end{aligned}$$

Similarly

$$\sum_{j=1}^{n} E(\eta_j \mid \mathscr{F}_{j-1}) = E(\tau_n).$$

We need the following five propositions.

Proposition 1. (i) *For* $x < y$, $E_x(\tau_y) = 2\int_x^y e^{-A(u)}(\int_{-\infty}^u e^{A(r)}(1/\sigma^2(r))\,dr)\,du$.

(ii) *For* $k \geqslant 2$ *and* $x < y$, $E_x(\tau_y^k) = 2k\int_x^y e^{-A(u)}(\int_{-\infty}^u E_r(\tau_y^{k-1})e^{A(r)}(1/\sigma^2(r))\,dr)\,du$.

Next, to compute $E_x(\int_0^{\tau_y} I_B(X(u))\,du)$ we need to introduce the *local time* process L for the diffusion X. It is known that there exists a process $\{L(t,u,\omega): t \geqslant 0\}$ adapted to the filtration $\{\mathscr{F}_t\}$ such that a.s.

(i) $L(\cdot,\cdot,\omega)$ is jointly continuous in t and x,

(ii) for each x, $L(\cdot,x,\omega)$ is nondecreasing,

(iii) for any locally bounded Borel measurable $f: \mathbb{R} \to \mathbb{R}$

$$\int_0^t f(X_s)\sigma^2(X_s)\,ds = \int_{-\infty}^{+\infty} f(x)L(t,x,\omega)\,dx.$$

We refer to pp. 218–220 of Karatzas and Shreve (1988) for details.

The following result involving the expected value of L does not seem to be readily accessible in the literature.

Proposition 2. *Under the assumptions* (A.1)–(A.4),

$$E_x L(\tau_y, a, \omega) = \begin{cases} 2(S(y) - S(x))\mathrm{e}^{A(a)}, & a < x < y, \\ 2(S(y) - S(a))\mathrm{e}^{A(a)}, & x < a < y. \end{cases}$$

Proposition 3. *For* $x < y$

$$E_x\left(\int_0^{\tau_y} I_B(X_s)\,\mathrm{d}s\right) = 2\int_x^y \mathrm{e}^{-A(r)}\left(\int_{-\infty}^r I_B(u)\mathrm{e}^{A(u)}\frac{1}{\sigma^2(u)}\mathrm{d}u\right)\mathrm{d}r$$

provided the inner integral on the right-hand side is finite for all r.

Proposition 4. *Let there exist a* $\lambda \in (0, \infty)$ *such that* $\forall\, 0 < h < \infty$,

(i) $\quad F_r(h) \equiv \int_r^{r+h} \rho(u)\,\mathrm{d}u \to \lambda h \quad$ *as* $r \to \infty$,

(ii) $\quad k_0 \equiv \sup_{0<r} \sup_{0 \leqslant h \leqslant 1} |F_r(h)| < \infty$,

(iii) $\quad \int_{-\infty}^0 \mathrm{e}^{A(r)}\frac{1}{\sigma^2(r)}\mathrm{d}r < \infty$,

(iv) $\quad \frac{1}{t}\int_0^t I_D(r)\frac{1}{\sigma^2(r)}\mathrm{d}r \to C_D \quad$ *as* $t \to \infty$,

where $0 < C_D < \infty$ *for* $D = B$, *the given Borel set and for* $D = [0, \infty)$. *Then,* (i) *and* (iii) *of Theorem* 1 *hold.*

Proposition 5. *Let* (i)–(iii) *of Proposition* 4 *hold and in addition assume*

(i) $\quad \limsup_n \int_n^{n+1} \frac{1}{\sigma^2(r)}\mathrm{d}r < \infty.$

Then

(a) $\quad \sup_{n>0} E_n(\tau_{n+1}) < \infty.$

Suppose further that

(ii) $\quad \int_{-\infty}^0 \mathrm{e}^{A(r)}\frac{1}{\sigma^2(r)}E_r\tau_0\,\mathrm{d}r < \infty.$

Then,

(b) $\quad \sup_n E_n(\tau_{n+1}^2) < \infty,$

i.e. condition (ii) *of Theorem* 1 *holds.*

Thus the hypotheses of Propositions 4 and 5 are sufficient for the validity of conditions (i)–(iv) of Theorem 1.

Proposition 1 is not new. For example, formula (i) for $E_x(\tau_y)$ is derived in Bhattacharya and Waymire (1991) and (ii) is available in Dynkin (1965) and also in Athreya and Weerasinghe (1992).

Proof of Proposition 2. Consider the case $x < a < y$. Let M be a constant so that $M > \max\{|x|,|y|\}$. By Tanaka's formula (Karatzas and Shreve, 1988, p. 220).

$$|X(t) - a| = |x - a| + \int_0^t \operatorname{sign}(X(s) - a)\mu(X(s))\,\mathrm{d}s + \int_0^t \operatorname{sign}(X(s) - a)\sigma(X(s))\,\mathrm{d}B(s) + L(t,a,\omega). \tag{9}$$

Introduce $\tilde{\tau}_M = \inf\{t > 0\colon |X(t) \geqslant M\}$, and $\|\mu\|_M = \sup\{|\mu(x)|\colon |x| \leqslant M\}$.

Now replacing t by $t \wedge \tilde{\tau}_M$ in (9) and then taking expectations and using Doob's optional sampling theorem we conclude

$$E_x L(\tilde{\tau}_M, a) \leqslant \lambda(M,x) \equiv 4M + \|\mu\|_M E_x[\tau_M] \quad \text{for all } a \in [-M,M]. \tag{10}$$

We write $\tau = \tau_y \wedge \tilde{\tau}_M$ and then by (9), and the properties (ii) and (iii) of local time, we obtain

$$E_x|X(t\wedge\tau) - a| = |x - a| + \frac{1}{2}E_x\int_{-M}^{a} \operatorname{sign}(z - a)\rho(z)L(t\wedge\tau,z,\omega)\,\mathrm{d}z + E_x L(t\wedge\tau,a,\omega).$$

Since $\rho(\cdot)$ is locally integrable and $\lambda(M,x)$ in (10) is bounded for $x \in [-M,M]$, we observe $|\rho(z)|\,L(t\wedge\tau,z,\omega)$ is an integrable function of z and ω with respect to the product measure of Lebesgue measure on $(-M,a]$ and the probability measure P_x.

Consequently

$$E_x|X(t\wedge\tau) - a| = |x - a| + \int_{-M}^{a} \operatorname{sign}(z - a)\rho(z)E_x L(t\wedge\tau,z,\omega)\,\mathrm{d}z + E_x L(t\wedge\tau,a,\omega).$$

Now letting $t \to +\infty$ we get the integral equation for $-M < x < a < y$;

$$\phi_M(a) + \frac{1}{2}\int_{-M}^{a} \operatorname{sign}(z - a)\rho(z)\phi_M(z)\,\mathrm{d}z = E_x|X(\tau) - a| - |x - a|, \tag{11}$$

where $\phi_M(z) = E_x(L(\tau,z,\omega))$ which is finite for all z in $[-M,M]$, by (10).

For $-M < x < a < y$,

$$E_x|X(\tau) - a| = (y - a)\frac{S(x) - S(-M)}{S(y) - S(-M)} + (a + M)\cdot\frac{S(y) - S(x)}{S(y) - S(-M)}.$$

Since $\rho(\cdot)$ is locally integrable and the right-hand side of (11) is bounded in $[-M,M]$, the integral equation (11) has a unique solution in the class of functions that are bounded in $[-M, M]$ and vanishing at y. It can be verified that

$$\psi_M(\cdot) = 2\left(\frac{S(x) - S(-M)}{S(y) - S(-M)}\right)(S(y) - S(\cdot))\mathrm{e}^{A(\cdot)}$$

satisfies (11), and the boundary condition $\psi_M(y) = 0$ and is also bounded in $[-M, M]$. Hence by uniqueness, ψ_M coincides with ϕ_M. Now letting $M \to \infty$ and using the monotone convergence theorem we obain

$$E_x L(\tau_y, a, \omega) = 2(S(y) - S(a))\mathrm{e}^{A(a)} \quad \text{for } x < a < y.$$

Proof for the case $a < x < y$ is similar. □

Proof of Proposition 3. By the second part of Proposition 2

$$\begin{aligned}
E_x\left(\int_0^{\tau_y} I_B(X_s)\,\mathrm{d}s\right) &= E_x\left(\int_{-\infty}^{y} I_B(u)\frac{1}{\sigma^2(u)}L(\tau_y, u, \omega)\,\mathrm{d}u\right)\\
&= \int_{-\infty}^{y} I_B(u)\frac{1}{\sigma^2(u)}E_x(L(\tau_y, u, \omega))\,\mathrm{d}u\\
&= \int_{-\infty}^{x} I_B(u)\frac{1}{\sigma^2(u)}2(S(y) - S(x))\mathrm{e}^{A(u)}\,\mathrm{d}u\\
&\quad + \int_{x}^{y} I_B(u)\frac{1}{\sigma^2(u)}2(S(y) - S(u))\mathrm{e}^{A(u)}\,\mathrm{d}u\\
&= 2\left(\int_x^y \mathrm{e}^{-A(r)}\,\mathrm{d}r\right)\left(\int_{-\infty}^{x} I_B(u)\frac{1}{\sigma^2(u)}\mathrm{e}^{A(u)}\,\mathrm{d}u\right)\\
&\quad + 2\int_x^y \mathrm{e}^{-A(r)}\left(\int_x^r I_B(u)\frac{1}{\sigma^2(u)}\mathrm{e}^{A(u)}\,\mathrm{d}u\right)\mathrm{d}r\\
&= 2\int_x^y \mathrm{e}^{-A(r)}\left(\int_{-\infty}^{r} I_B(u)\frac{1}{\sigma^2(u)}\mathrm{e}^{A(u)}\,\mathrm{d}u\right)\mathrm{d}r.
\end{aligned}$$

This completes the proof of Proposition 3. □

Remark 1. (a) If we set $B = \mathbb{R}$ in the above we obtain $E_x\tau_y$ as in (i) of Proposition 1.

(b) Instead of using Proposition 3 above one could use Ito's formula to compute $E_x(\int_0^{\tau_y} f(X_s)\,\mathrm{d}s)$ for a bounded continuous f by using the solution to the differential equation

$$\tfrac{1}{2}\sigma^2(x)u''(x) + \mu(x)u'(x) = -f(x)$$

But it is not easy to generalize this method to f's that are not continuous but only measurable and bounded on finite intervals.

(c) It is possible to replace $I_B(\cdot)$ by a bounded Borel measurable $f(\cdot)$ in Proposition 3 provided the right side is well defined.

Proof of Theorem 1. Under (ii), (8) holds with $\delta = 1$ and hence (6a) and (6b) hold. Also by (i), (iii) and Proposition 1, $(1/n)E(\tau_n)$ and $(1/n)E(\int_0^{\tau_n} I_\beta(X(u))\,\mathrm{d}u)$ converge a.s. to c_1 and c_2 respectively. Thus, (a) and (b) follow.

Let $N(t) = \inf\{n\colon n \geqslant 1,\ \tau_{n+1} > t\}$. Then $\tau_{N(t)} \leqslant t < \tau_{N(t)+1}$ and (a) implies that

$$\frac{N(t)}{t} \to \frac{1}{c_1} \quad \text{a.s.}$$

Next,

$$\int_0^{\tau_{N(t)}} I_B(X(u))\,\mathrm{d}u \leqslant \int_0^t I_B(X(u))\,\mathrm{d}u \leqslant \int_0^{\tau_{N(t)+1}} I_B(X(u))\,\mathrm{d}u$$

and hence

$$\frac{c_2}{c_1} \leqslant \liminf_t \frac{1}{t}\int_0^t I_B(X(u))\,\mathrm{d}u \leqslant \limsup_t \frac{1}{t}\int_0^t I_B(X(u))\,\mathrm{d}u \leqslant \frac{c_2}{c_1}$$

yielding (c). □

Proof of Proposition 4. Fix a Borel set D in $\mathbb{R}$. Then

$$\begin{aligned}
&\int_0^t \mathrm{e}^{-A(u)}\left(\int_{-\infty}^u \mathrm{e}^{A(r)}\frac{1}{\sigma^2(r)}I_D(r)\,\mathrm{d}r\right)\mathrm{d}u \\
&\quad = \left(\int_{-\infty}^0 \mathrm{e}^{A(r)}\frac{1}{\sigma^2(r)}I_D(r)\,\mathrm{d}r\right)S(t) + \int_0^t \mathrm{e}^{-A(u)}\left(\int_0^u \mathrm{e}^{A(r)}\frac{1}{\sigma^2(r)}I_D(r)\,\mathrm{d}r\right)\mathrm{d}u \\
&\quad = L_1(t) + L_2(t) \quad \text{(say).}
\end{aligned}$$

Since $S(\infty) < \infty$ and $\int_{-\infty}^0 \mathrm{e}^{A(r)}(1/\sigma^2(r))\,\mathrm{d}r < \infty$, $L_1(\cdot)$ is bounded on $[0,\infty)$. We shall show that $L_2(t) \sim C_D \cdot t$ for $D = [0,\infty)$ and $D = B$, the given Borel set (for which (iv) holds).

By Fubini's theorem

$$\begin{aligned}
L_2(t) &= \int_0^t I_D(r)\frac{1}{\sigma^2(r)}\left(\int_r^t \mathrm{e}^{-(A(u)-A(r))}\,\mathrm{d}u\right)\mathrm{d}r \\
&= \int_0^t I_D(r)\frac{1}{\sigma^2(r)}\left(\int_0^{t-r} \mathrm{e}^{-F_r(v)}\,\mathrm{d}v\right)\mathrm{d}r.
\end{aligned}$$

By hypothesis (i)

$$k_r(v) = \mathrm{e}^{-F_r(v)} \to \mathrm{e}^{-\lambda v} \equiv k(v) \quad \text{as } r \to \infty \tag{12}$$

and also there is an r_0 such that for $r \geqslant r_0$, $F_r(1) \geqslant \lambda/2$.

Hence for $r \geqslant r_0$ and $n \leqslant v \leqslant n+1$

$$\begin{aligned}
k_r(v) &\leqslant k_n(v)\,\mathrm{e}^{-(A(r+v)-A(r+u))} \\
&\leqslant \mathrm{e}^{-v\lambda/2}\,\mathrm{e}^{k_0} \quad \text{(by hypothesis (ii))} \\
&= \tilde{k}(v) \quad \text{(say).}
\end{aligned}$$

Let

$$\tilde{L}_2(t) = \int_0^t \frac{1}{\sigma^2(r)} I_D(r) \left(\int_0^{t-r} k(v)\,\mathrm{d}v \right) \mathrm{d}r. \tag{13}$$

Then,

$$\frac{1}{t} |L_2(t) - \tilde{L}_2(t)| \leqslant \frac{1}{t} \int_0^t \frac{I_D(r)}{\sigma^2(r)} \left(\int_0^{t-r} |k_r(v) - k(v)|\,\mathrm{d}v \right) \mathrm{d}r$$
$$\leqslant \frac{1}{t} \int_0^t \frac{1}{\sigma^2(r)} \left(\int_0^{\infty} |k_r(v) - k(v)|\,\mathrm{d}v \right) \mathrm{d}r.$$

Since $k_r(v) \to k(v)$ as $r \to \infty$ and is dominated by $\tilde{k}(v)$ which is integrable,

$$\int_0^{\infty} |k_r(v) - k(v)|\,\mathrm{d}v \to 0 \quad \text{as } r \to \infty.$$

By hypotheses in (iv)

$$\frac{1}{t} \int_0^t \frac{1}{\sigma^2(r)} \mathrm{d}r \text{ is bounded in } t$$

and so we conclude that

$$\limsup_{t \to \infty} \frac{1}{t} |L_2(t) - \tilde{L}_2(t)| = 0.$$

Now for fixed $k > 0$ and $t > k$,

$$\tilde{L}_2(t) \geqslant \int_0^{t-k} \frac{I_D(r)}{\sigma^2(r)} \left(\int_0^k \mathrm{e}^{-\lambda v}\,\mathrm{d}v \right) \mathrm{d}r$$

yielding

$$\liminf_{t \to \infty} \frac{\tilde{L}_2(t)}{t} \geqslant C_D \int_0^k \mathrm{e}^{-\lambda v}\,\mathrm{d}v$$

and hence

$$\liminf_{t \to \infty} \frac{\tilde{L}_2(t)}{t} \geqslant C_D \lambda^{-1}.$$

Finally,

$$\tilde{L}_2(t) \leqslant \int_0^t \frac{I_D(r)}{\sigma^2(r)} \left(\int_0^{\infty} k(v)\,\mathrm{d}v \right) \mathrm{d}r$$

yielding

$$\limsup_{t \to \infty} \frac{\tilde{L}_2(t)}{t} \leqslant C_D \lambda^{-1}.$$

By taking $D = [0, \infty)$ and $D = B$, the given Borel set we get (i) and (iii) of Theorem 1 to hold. □

Proof of Proposition 5. By part (i) of Proposition 1

$$E_n(\tau_{n+1}) = 2\int_n^{n+1} e^{-A(u)}\left(\int_{-\infty}^{u} e^{A(r)}\frac{1}{\sigma^2(r)}\,dr\right)du.$$

By hypothesis (i) of Proposition 4, there exist r_0 such that

$$F_r(1) = A(r+1) - A(r) \geqslant \frac{\lambda}{2} \quad \text{for } r \geqslant r_0.$$

Thus for $n > r_0$

$$\begin{aligned} E_n(\tau_{n+1}) &= 2\int_n^{n+1} e^{-A(u)}\left(\int_{-\infty}^{r_0} e^{A(r)}\frac{1}{\sigma^2(r)}\,dr\right)du \\ &\quad + 2\int_n^{n+1} e^{-A(u)}\left(\int_{r_0}^{u} e^{A(r)}\frac{1}{\sigma^2(r)}\,dr\right)du \\ &= a_n + b_n \quad \text{(say)}. \end{aligned}$$

Now,

$$a_n \leqslant 2\left(\int_{-\infty}^{r_0} e^{A(r)}\frac{1}{\sigma^2(r)}\,dr\right)e^{-A(n)}e^{k_0},$$

where k_0 is as in hypothesis (ii) of Proposition 4. By hypothesis (i) of Proposition 4, $A(n) \to \infty$ and so $a_n \to 0$ and hence $\sup_n a_n < \infty$.

Next,

$$\begin{aligned} b_n &\leqslant 2\left(\int_{r_0}^{n+1} e^{A(r)}\frac{1}{\sigma^2(r)}\,dr\right)e^{-A(n)}e^{k_0} \\ &= 2e^{k_0}\sum_{k=r_0}^{n}\left(\int_k^{k+1} e^{(A(r)-A(n))}\frac{1}{\sigma^2(r)}\,dr\right) \\ &\leqslant 2e^{2k_0}\sum_{k=r_0}^{n}\left(\int_k^{k+1}\frac{1}{\sigma^2(r)}\,dr\right)e^{(A(k)-A(n))} \\ &\leqslant 2e^{2k_0}C\sum_{k=r_0}^{n} e^{-(n-k)\lambda/2} \quad \text{(by condition (i))} \\ &\leqslant 2e^{2k_0}C\sum_{0}^{\infty} e^{-\lambda/2 j}, \end{aligned}$$

where C is a generic constant. So $\sup_n b_n < \infty$ and hence $\sup_n (a_n + b_n) < \infty$ proving (a) of Proposition 5. Turning now to the proof of (b) we note from Proposition 1

$$\frac{1}{4}E_n(\tau_{n+1}^2) = \int_n^{n+1} e^{-A(u)}\left(\int_{-\infty}^{u}\frac{1}{\sigma^2(r)}e^{A(r)}E_r(\tau_{n+1})\,dr\right)du.$$

But for $r < 0 < n+1$, $E_r\tau_{n+1} = E_r\tau_0 + E_0\tau_{n+1}$ and hence

$$\frac{1}{4}E_n\tau_{n+1}^2 = \int_n^{n+1} e^{-A(u)}\left(\int_{-\infty}^0 \frac{e^{A(r)}}{\sigma^2(r)}E_r(\tau_0)\,dr\right)du$$

$$+\left(\int_n^{n+1} e^{-A(u)}\left(\int_{-\infty}^0 \frac{e^{A(r)}}{\sigma^2(r)}dr\right)du\right)E_0(\tau_{n+1})$$

$$+\int_n^{n+1} e^{-A(u)}\left(\int_0^u e^{A(r)}\frac{1}{\sigma^2(r)}E_r(\tau_{n+1})\,dr\right)du$$

$$= \tilde{a}_n + \tilde{b}_n + \tilde{c}_n \quad \text{(say)}.$$

Now

$$\tilde{a}_n \leqslant e^{-A(n)}e^{k_0}\left(\int_{-\infty}^0 e^{A(r)}\frac{1}{\sigma^2(r)}E_r(\tau_0)\,dr\right) \to 0,$$

as $n \to \infty$ and so $\sup_n|\tilde{a}_n| < \infty$.

Next,

$$\tilde{b}_n \leqslant e^{k_0}\left(\int_{-\infty}^0 e^{A(r)}\frac{1}{\sigma^2(r)}\,dr\right)e^{-\lambda n/2}\,C_n$$

for all n large, since $\sup_j E_j\tau_{j+1} = c < \infty$, and $A(n)/n \to \lambda$ as $n \to \infty$. Thus $\tilde{b}_n \to 0$ and $\sup_n|\tilde{b}_n| < \infty$. Finally,

$$\tilde{c}_n \leqslant e^{k_0}\sum_{k=0}^{n}\int_k^{k+1} e^{(A(k)-A(n+1))}E_r(\tau_{n+1})\,dr$$

$$\leqslant e^{2k_0}\sum_{k=0}^{n} e^{(A(k)-A(n+1))}E_k(\tau_{n+1})$$

$$\leqslant e^{2k_0}C\left(\sum_{k=0}^{r_0} e^{A(k)}(n+1-k)\right)e^{-A(n+1)} \quad (C \text{ is a generic constant})$$

$$+\, e^{2k_0}C\sum_{k=r_0}^{n}(n+1-k)e^{-(\lambda/2)(n+1-k)}$$

Thus

$$\tilde{c}_n \leqslant e^{2k_0}(n+1)e^{-A(n+1)}\left(\sum_{k=0}^{k_0} e^{A(k)}\right) + e^{2k_0}C\sum_{j=0}^{\infty} je^{-\lambda j/2}$$

The first term goes to zero and so

$$\limsup_n(\tilde{a}_n + \tilde{b}_n + \tilde{c}_n) \leqslant e^{2k_0}C\sum_0^{\infty} je^{-j\lambda/2} < \infty. \qquad \square$$

Remark 2. A set of sufficient conditions for the validity of Propositions 4 and 5 is the following:

(1) there exists $0 < \lambda \leqslant \infty$ such that $\liminf_r \int_r^{r+h} \rho(u)\,\mathrm{d}u \geqslant \lambda h$ for all $h > 0$,

(2) $\sup_r \sup_{0 \leqslant h \leqslant 1} |\int_r^{r+h} \rho(u)\,\mathrm{d}u| < \infty$,

(3) $\int_{-\infty}^{0} \mathrm{e}^{A(r)}(1/\sigma^2(r))\,\mathrm{d}r < \infty$,

(4) $\limsup \int_n^{n+1} (1/\sigma^2(r))\,\mathrm{d}r < \infty$,

(5) $\int_{-\infty}^{0} \mathrm{e}^{A(r)}(1/\sigma^2(r))\,\mathrm{d}r < \infty$.

Proof of Corollary 2. Since μ and σ are periodic with period one the same is true of $\rho(\cdot)$. Further the assumption $\int_0^1 \rho(u)\,\mathrm{d}u > 0$ implies that $S(+\infty) < \infty$ and $S(-\infty) = -\infty$ where $S(\cdot)$ is as in (A.3). Thus the process X defined in (2) goes to ∞ w.p.1. Also, by periodicity, $E(\tau_{n+1} - \tau_n)^k = E_0\tau_1^k$, $k = 1, 2$, which can be shown to be finite using periodicity. Following the discussion in section 2 and the proof of Theorem 1, we see that $(1/t)\int_0^t I_B(X(u))\,\mathrm{d}u$ is convergent w.p.1 if and only if $(1/n)E_0(\int_0^{\tau_n} I_B(X(u))\,\mathrm{d}u)$ is convergent. By Proposition 3, this last quantity equals

$$\frac{2}{n}\int_0^n \mathrm{e}^{-A(r)}\left(\int_{-\infty}^{r} I_B(u)\,\mathrm{e}^{A(u)}\frac{1}{\sigma^2(u)}\,\mathrm{d}u\right)\mathrm{d}r,$$

which converges if and only if $(2/n)\int_0^n \mathrm{e}^{-A(r)}(\int_0^r I_B(u)\mathrm{e}^{A(u)}(1/\sigma^2(u))\,\mathrm{d}u)\,\mathrm{d}r$ converges, since $\int_{-\infty}^{0} I_B(u)\mathrm{e}^{A(u)}(1/\sigma^2(u))\,\mathrm{d}u < \infty$ and $\int_0^\infty \mathrm{e}^{-A(r)}\,\mathrm{d}r = S(+\infty)$. Now

$$\begin{aligned}
&\frac{2}{n}\int_0^n \mathrm{e}^{-A(r)}\left(\int_0^r I_B(u)\,\mathrm{e}^{A(u)}\frac{1}{\sigma^2(u)}\,\mathrm{d}u\right)\mathrm{d}r \\
&\quad = \frac{2}{n}\int_0^n I_B(u)\frac{1}{\sigma^2(u)}\left(\int_0^{n-u} \mathrm{e}^{(A(u+s)-A(u))}\,\mathrm{d}s\right)\mathrm{d}u \\
&\quad = \frac{2}{n}\int_0^n I_B(u)\frac{1}{\sigma^2(u)}\psi(u)\,\mathrm{d}u - \frac{2}{n}\int_0^n I_B(u)\frac{1}{\sigma^2(u)}\left(\int_{n-u}^{\infty} \mathrm{e}^{-(A(u+s)-A(u))}\,\mathrm{d}s\right)\mathrm{d}u,
\end{aligned}$$

where $\psi(u)$ is as in Corollary 2. By periodicity, there are constants c_1 and c_2 such that $\alpha s + c_1 \leqslant A(u+s) - A(u) \leqslant \alpha s + c_2$ for all u and s where $\alpha = \int_0^1 \rho(u)\,\mathrm{d}u$. Therefore the second term is dominated by

$$\frac{C}{n}\int_0^n \int_{n-u}^{\infty} \mathrm{e}^{-\alpha s}\,\mathrm{d}s = \frac{C}{n\alpha}\int_0^n \mathrm{e}^{-\alpha(n-u)}\,\mathrm{d}u \leqslant \frac{C}{n\alpha},$$

where C is a generic constant. (We have used (A.1) and the periodicity to conclude that $\inf \sigma(\cdot) > 0$). Finally, by periodicity of μ, σ and hence of ψ,

$$\frac{2}{n}\int_0^n I_B(u)\frac{1}{\sigma^2(u)}\psi(u)\,\mathrm{d}u = \int_0^n \left(\frac{2}{r}\sum_0^{n-1} I_B(r+j)\right)\frac{1}{\sigma^2(r)}\psi(r)\,\mathrm{d}r.$$

This completes the proof of Corollary 2. □

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