

*Pacific  
Journal of  
Mathematics*

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 $SU_q(n)/SU_q(n-2)$**

PARTHA SARATHI CHAKRABORTY AND S. SUNDAR

Volume 252 No. 2

August 2011

## **$K$ -GROUPS OF THE QUANTUM HOMOGENEOUS SPACE $SU_q(n)/SU_q(n-2)$**

PARTHA SARATHI CHAKRABORTY AND S. SUNDAR

*Dedicated to Prof. K. R. Parthasarathy on his 75th birthday*

**Quantum Stiefel manifolds were introduced by Vainerman and Podkolzin, who classified the irreducible representations of the  $C^*$ -algebras underlying such manifolds. We compute the  $K$ -groups of the quantum homogeneous spaces  $SU_q(n)/SU_q(n-2)$  for  $n \geq 3$ . In the case  $n = 3$ , we show that  $K_1$  is a free  $\mathbb{Z}$ -module, and the fundamental unitary for quantum  $SU(3)$  is part of a basis for  $K_1$ .**

### **1. Introduction**

Quantization of mathematical theories is a major theme of research today. The theories of quantum groups and noncommutative geometry are two prime examples in this program. Both these programs started in the early 1980s. In the setting of operator algebras, the theory of quantum groups was initiated independently in [Woronowicz 1987] and [Vaksman and Soibelman 1988], for the case of quantum  $SU(2)$ . Later Woronowicz studied the family of compact quantum groups and obtained Tannaka-type duality theorems [Woronowicz 1988]. The notion of quantum subgroups and quantum homogeneous spaces soon followed [Podleś 1995].

The noncommutative differential geometry program of Alain Connes [1985] also started in the 1980s. In his interpretation, geometric data is encoded in elliptic operators or, more generally, in specific unbounded  $K$ -cycles, which he called spectral triples. It is natural to expect that, for compact quantum groups and their homogeneous spaces, there should be associated canonical spectral triples. Chakraborty and Pal [2003] showed that indeed that is the case for quantum  $SU(2)$ . In fact for odd-dimensional quantum spheres, one can construct finitely summable spectral triples that display Poincaré duality [Chakraborty and Pal 2010].

In this connection, a natural question is, are these examples somewhat singular or can one in general construct finitely summable spectral triples with further

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Chakraborty acknowledges financial support from Indian National Science Academy through its project "Noncommutative Geometry of Quantum Groups".

*MSC2000:* 46L80, 58B32.

*Keywords:*  $K$ -groups, quantum Stiefel manifolds, quantum groups, quantum homogeneous spaces.

properties like Poincaré duality, on quantum groups associated with Lie groups or their homogeneous spaces? Even though there are suggestions to construct such spectral triples [Neshveyev and Tuset 2010], their nontriviality as a  $K$ -cycle is not known. In fact, there are suggestions that, for quantum groups and their homogeneous spaces, one should look for a type-III formulation of noncommutative geometry. On this formulation also, there are currently two points of view, that of Alain Connes and Henri Moscovici [2008], and that of Carey–Phillips–Rennie [2010]. Therefore, to understand the true nature of the interplay between noncommutative geometry and quantum homogeneous spaces, it makes sense to take a closer look at these algebras.

The underlying  $C^*$ -algebras of these compact quantum groups were analyzed by Soibelman [1990] (also [Levendorskii and Soibelman 1991]) who described their irreducible representations. Exploiting their findings, Sheu went on to obtain composition sequences for these algebras. He initially obtained the results for  $SU_q(3)$  [Sheu 1991], and later extended them to the general  $SU_q(n)$  [Sheu 1997].

In this hierarchy of exploration, the next thing to look for would be  $K$ -groups; that is what we are looking for. But, instead of concentrating on quantum groups, we consider the quantum analogs of the Stiefel manifolds  $SU(n)/SU(n-m)$ , introduced by Podkolzin and Vainerman [1999]. Those authors have already described the structure of irreducible representations of the quantum Stiefel manifolds  $SU_q(n)/SU_q(n-m)$ . We take up the case of  $SU_q(n)/SU_q(n-2)$  when  $n \geq 3$ . We obtain the composition sequences for these algebras and then, utilizing them, we compute the  $K$ -groups. More importantly, as we remarked earlier, applications towards noncommutative geometry require an explicit understanding of generators for these  $K$ -groups; during our calculation we also achieve that. Specializing to the case  $n = 3$ , we get the  $K$ -groups of quantum  $SU(3)$ .

We should remark that these  $K$ -groups can be computed using the variant of  $KK$ -theory introduced by Nagy in [2000]. In fact, it is shown in [Nagy 1998] that  $SU_q(n)$  and  $SU(n)$  are  $KK$ -equivalent, but here we produce explicit generators, which is essential to test the nontriviality of  $K$ -cycles by computing the  $K$ -theory– $K$ -homology pairing. To our knowledge, there are not many instances of  $K$ -theory calculations for compact quantum groups. Other than the paper by Nagy, there is another related work by McClanahan [1992], where he computes the  $K$ -groups of the universal  $C^*$ -algebra generated by the elements of a unitary matrix, and shows that the associated  $K_1$  is generated by the defining unitary itself. This raises the question whether something similar holds for compact matrix quantum groups, namely, whether the defining unitary of a compact matrix quantum group is nontrivial in  $K_1$ . For quantum  $SU(2)$ , this was remarked by Connes [2004]. Here, we not only prove that the defining unitary of quantum  $SU(3)$  is nontrivial, the  $K_1$  is a free  $\mathbb{Z}$ -module, and the fundamental unitary for quantum  $SU(3)$  is part of a basis for  $K_1$ .

**2. The quantum Stiefel manifolds and their irreducible representations**

The quantum Stiefel manifold  $S_q^{n,m}$  was introduced in [Podkolzin and Vainerman 1999]. Throughout, we assume that  $q \in (0, 1)$ . Recall that the  $C^*$ -algebra  $C(SU_q(n))$  is the universal unital  $C^*$ -algebra generated by  $n^2$  elements  $u_{ij}$  satisfying the conditions

$$\sum_{k=1}^n u_{ik}u_{jk}^* = \delta_{ij}, \quad \sum_{k=1}^n u_{ki}^*u_{kj} = \delta_{ij},$$

$$\sum_{i_1=1}^n \sum_{i_2=1}^n \cdots \sum_{i_n=1}^n E_{i_1i_2\dots i_n} u_{j_1i_1} \cdots u_{j_ni_n} = E_{j_1j_2\dots j_n},$$

where

$$E_{i_1i_2\dots i_n} := \begin{cases} 0 & \text{if } i_1, i_2, \dots, i_n \text{ are not distinct,} \\ (-q)^{\ell(i_1, i_2, \dots, i_n)} & \text{otherwise,} \end{cases}$$

and where  $\ell(\sigma)$  denotes the length of a permutation  $\sigma$  on  $\{1, 2, \dots, n\}$ . The  $C^*$ -algebra  $C(SU_q(n))$  has a compact quantum group structure with comultiplication given by

$$\Delta(u_{ij}) := \sum_k u_{ik} \otimes u_{kj}.$$

Let  $1 \leq m \leq n - 1$ . Call  $v_{ij}$  the generators of  $SU_q(n - m)$ . The map  $\varphi : C(SU_q(n)) \rightarrow C(SU_q(n - m))$  defined by

$$(2-1) \quad \varphi(u_{ij}) := \begin{cases} v_{ij} & \text{if } 1 \leq i, j \leq n - m, \\ \delta_{ij} & \text{otherwise.} \end{cases}$$

is a surjective unital  $C^*$ -algebra homomorphism such that  $\Delta \circ \varphi = (\varphi \otimes \varphi)\Delta$ . In this way, the quantum group  $SU_q(n - m)$  is a subgroup of the quantum group  $SU_q(n)$ . The  $C^*$ -algebra of the quotient  $SU_q(n)/SU_q(n - m)$  is defined as

$$C(SU_q(n)/SU_q(n - m)) := \{a \in C(SU_q(n)) : (\varphi \otimes 1)\Delta(a) = 1 \otimes a\}.$$

We refer to [Podkolzin and Vainerman 1999] for the proof of the following:

**Proposition 2.1.** *The  $C^*$ -algebra  $C(SU_q(n)/SU_q(n - m))$  is generated by the last  $m$  rows of the matrix  $(u_{ij})$ , that is, by the set  $\{u_{ij} : n - m + 1 \leq i \leq n\}$ .*

In [Podkolzin and Vainerman 1999], the quotient space  $SU_q(n)/SU_q(n - m)$  is called a quantum Stiefel manifold and is denoted by  $S_q^{n,m}$ . We will use the same notation.

Before proceeding further, let us fix some notations. Let  $\mathbb{N}$  be the set of non-negative integers. Consider the number operator  $N$  and the left shift  $S$  on  $\ell^2(\mathbb{N})$

defined on the standard orthonormal basis  $\{e_n : n \geq 0\}$  by

$$S e_n := e_{n-1} \quad \text{and} \quad N e_n := n e_n.$$

Note that  $N$  is an unbounded selfadjoint operator. We denote by  $\tau$  the  $C^*$ -algebra generated by  $S$ . The  $C^*$ -algebra  $\tau$  is nothing but the Toeplitz algebra.

The irreducible representations of the  $C^*$ -algebra  $C(S_q^{n,m})$  was described in [Podkolzin and Vainerman 1999]. First, we recall the irreducible representations of  $C(\text{SU}_q(n))$  as in [Soibelman 1990]. The one-dimensional representations of  $C(\text{SU}_q(n))$  are parametrized by the torus  $\mathbb{T}^{n-1}$ . We consider  $\mathbb{T}^{n-1}$  as a subset of  $\mathbb{T}^n$  under the inclusion  $(t_1, t_2, \dots, t_{n-1}) \rightarrow (t_1, t_2, \dots, t_{n-1}, t_n)$ , where  $t_n := \prod_{i=1}^{n-1} \bar{t}_i$ . For  $t := (t_1, t_2, \dots, t_n) \in \mathbb{T}^{n-1}$ , let  $\tau_t : C(\text{SU}_q(n)) \rightarrow \mathbb{C}$  be defined as

$$\tau_t(u_{ij}) := t_{n-i+1} \delta_{ij}.$$

Then,  $\tau_t$  is a  $*$ -algebra homomorphism. The set  $\{\tau_t : t \in \mathbb{T}^{n-1}\}$  is a complete set of mutually inequivalent one-dimensional representations of  $C(\text{SU}_q(n))$ .

Denote the transposition  $(i, i + 1)$  by  $s_i$ . The map  $\pi_{s_i} : C(\text{SU}_q(n)) \rightarrow B(\ell^2(\mathbb{N}))$ , defined on the generators  $u_{rs}$  by

$$\pi_{s_i}(u_{rs}) := \begin{cases} \sqrt{1 - q^{2N+2}} S & \text{if } r = i, s = i, \\ -q^{N+1} & \text{if } r = i, s = i + 1, \\ q^N & \text{if } r = i + 1, s = i, \\ S^* \sqrt{1 - q^{2N+2}} & \text{if } r = i + 1, s = i + 1, \\ \delta_{ij} & \text{otherwise,} \end{cases}$$

is a  $*$ -algebra homomorphism. For any two representations  $\varphi$  and  $\xi$  of  $C(\text{SU}_q(n))$ , let  $\varphi * \xi := (\varphi \otimes \xi) \Delta$ . For  $\omega \in S_n$ , let  $\omega = s_{i_1} s_{i_2} \dots s_{i_k}$  be a reduced expression. Then, the representation  $\pi_\omega := \pi_{s_{i_1}} * \pi_{s_{i_2}} * \dots * \pi_{s_{i_k}}$  is an irreducible representation. Up to unitary equivalence, the representation  $\pi_\omega$  is independent of the reduced expression. For  $t \in \mathbb{T}^{n-1}$  and  $\omega \in S_n$  let  $\pi_{t,\omega} := \tau_t * \pi_\omega$ . We refer to [Soibelman 1990] for the proof of the following:

**Theorem 2.2.**  $\{\pi_{t,\omega} : t \in \mathbb{T}^{n-1}, \omega \in S_n\}$  is a complete set of mutually inequivalent irreducible representations of  $C(\text{SU}_q(n))$ .

The irreducible representations of  $C(S_q^{n,m})$  were studied in [Podkolzin and Vainerman 1999]. We recall them here. Embed  $\mathbb{T}^m$  into  $\mathbb{T}^{n-1}$  via the map

$$t = (t_1, t_2, \dots, t_m) \rightarrow (t_1, t_2, \dots, t_m, 1, 1, \dots, 1, t_n),$$

where  $t_n := \prod_{i=1}^m \bar{t}_i$ . For a permutation  $\omega \in S_n$ , let  $\omega^s$  be the permutation in the coset  $S_{n-m} \omega$  with the least possible length. We denote the restriction of the representation  $\pi_{t,\omega}$  to the subalgebra  $C(S_q^{n,m})$  by  $\pi_{t,\omega}$  itself.

**Theorem 2.3** [Podkolzin and Vainerman 1999]. *The set  $\{\pi_{t,\omega} : t \in \mathbb{T}^m, \omega \in S_n\}$  is a complete set of mutually inequivalent irreducible representations of  $C(S_q^{n,m})$ .*

### 3. Composition sequences

In this section, we derive certain exact sequences analogous to that of [Sheu 1997, Theorem 4]. We then apply the six-term sequence in  $K$ -theory to compute the  $K$ -groups of  $C(S_q^{n,2})$ .

**Lemma 3.1.** *Let  $t \in \mathbb{T}^m$  and  $\omega := s_{n-1}s_{n-2} \dots s_{n-k}$ . The image of  $C(S_q^{n,m})$  under the homomorphism  $\pi_{t,\omega}$  contains the algebra of compact operators  $\mathcal{K}(\ell^2(\mathbb{N}^k))$ .*

*Proof.* Since  $\pi_{t,\omega}(C(S_q^{n,m})) = \pi_\omega(C(S_q^{n,m}))$ , it is enough to show that  $\mathcal{K}(\ell^2(\mathbb{N}^k)) \subset \pi_\omega(C(S_q^{n,m}))$ . We prove this result by induction on  $n$ . Since

$$\pi_\omega(u_{nn}) := S^* \sqrt{1 - q^{2N+2}} \otimes 1,$$

it follows that  $S \otimes 1 \in \pi_\omega(C(S_q^{n,m}))$ . Hence,  $\mathcal{K}(\ell^2(\mathbb{N})) \otimes 1 \subset \pi_\omega(C(S_q^{n,m}))$ , and the result is true when  $n = 2$ .

Next, observe that  $(p \otimes 1) \pi_\omega(u_{n,i}) := p \otimes \pi_{\omega'}(v_{n-1,i})$  for  $1 \leq i \leq n - 1$ , where  $\omega' := s_{n-2}s_{n-3} \dots s_{n-k}$  and  $(v_{ij})$  denotes the generators of  $C(SU_q(n - 1))$ . Hence,  $\pi_\omega(C(S_q^{n,m}))$  contains the algebra  $p \otimes \pi_{\omega'}(C(S_q^{n-1,m}))$ . Now, by the induction hypothesis, it follows that  $\pi_\omega(C(S_q^{n,m}))$  contains  $p \otimes \mathcal{K}(\ell^2(\mathbb{N}^{k-1}))$ . Since  $\pi_\omega(C(S_q^{n,m}))$  contains both  $\mathcal{K}(\ell^2(\mathbb{N})) \otimes 1$  and  $p \otimes \mathcal{K}(\ell^2(\mathbb{N}^{k-1}))$ , it follows that  $\pi_\omega(C(S_q^{n,m}))$  contains the algebra of compact operators, which completes the proof.  $\square$

Let  $w$  be a word on  $s_1, s_2, \dots, s_n$ , say,  $w := s_{i_1}s_{i_2} \dots s_{i_n}$  (not necessarily a reduced expression). Define  $\psi_w := \pi_{s_{i_1}} * \pi_{s_{i_2}} * \dots * \pi_{s_{i_n}}$  and, for  $t \in \mathbb{T}^n$ , let  $\psi_{t,w} := \tau_t * \psi_w$ . Observe that the image of  $\psi_{t,w}$  is contained in  $\tau^{\otimes r}$ . We prove that, if  $w'$  is a subword of  $w$ , then  $\psi_{t,w'}$  factors through  $\psi_{t,w}$ .

**Proposition 3.2.** *Let  $w = w_1s_kw_2$  be a word on  $s_1, s_2, \dots, s_n$ . Denote the word  $w_1w_2$  by  $w'$  and let  $t \in \mathbb{T}^m$  be given. There exists a  $*$ -homomorphism*

$$\varepsilon : \psi_{t,w}(C(S_q^{n,m})) \rightarrow \psi_{t,w'}(C(S_q^{n,m}))$$

such that  $\psi_{t,w'} = \varepsilon \circ \psi_{t,w}$ .

*Proof.* If  $\ell(u)$  denotes the length of a word  $u$  on  $s_1, s_2, \dots, s_n$ , then  $\psi_{t,w}(C(S_q^{n,m}))$  is contained in  $\tau^{\otimes \ell(w_1)} \otimes \tau \otimes \tau^{\otimes \ell(w_2)}$ . Let  $\varepsilon$  denote the restriction of  $1 \otimes \sigma \otimes 1$  to  $\psi_{t,w}(C(S_q^{n,m}))$ , where  $\sigma : \tau \rightarrow \mathbb{C}$  is the homomorphism for which  $\sigma(S) = 1$ .

$$\psi_{t,w}(u_{rs}) = \sum_{j_1, j_2} \psi_{t,w_1}(u_{rj_1}) \otimes \pi_{s_k}(u_{j_1j_2}) \otimes \psi_{w_2}(u_{j_2s}).$$

Since  $\sigma(\pi_{s_k}(u_{j_1 j_2})) = \delta_{j_1 j_2}$ , it follows that

$$\varepsilon \circ \psi_{t,w}(u_{rs}) = \sum_j \psi_{t,w_1}(u_{rj}) \otimes \psi_{w_2}(u_{js}) = \psi_{t,w'}(u_{rs}).$$

This completes the proof. □

Let  $w$  be a word on  $s_1, s_2, \dots, s_n$ . Then, for  $n - m + 1 \leq i \leq n$  and  $1 \leq j \leq n$ , the map  $\mathbb{T}^m : t \rightarrow \psi_{t,w}(u_{ij}) \in \tau^{\otimes \ell(w)}$  is continuous. Thus, we get a homomorphism  $\chi_w : C(S_q^{n,m}) \rightarrow C(\mathbb{T}^m) \otimes \tau^{\otimes \ell(w)}$  such that  $\chi_w(a)(t) = \psi_{t,w}(a)$  for all  $a \in C(S_q^{n,m})$ .

**Remark 3.3.** Clearly, for a word  $w$  on  $s_1, s_2, \dots, s_n$ , the representations  $\psi_{t,w}$  factors through  $\chi_w$ . One can also prove, as in Proposition 3.2, that if  $w'$  is a subword of  $w$ , then  $\chi_{w'}$  factors through  $\chi_w$ .

Let us introduce some notation. Denote by  $\omega_{j,i}$  the permutation  $s_j s_{j-1} \dots s_i$  for  $j \geq i$ . If  $j < i$ , let  $\omega_{j,i} := 1$ . For  $1 \leq k \leq n$ , let  $\omega_k := \omega_{n-m,1} \omega_{n-m+1,1} \dots \omega_{n-1,n-k+1}$ .

**Theorem 3.4.** *The homomorphism  $\chi_{\omega_n} : C(S_q^{n,m}) \rightarrow C(\mathbb{T}^m) \otimes \tau^{\otimes \ell(\omega_n)}$  is faithful.*

*Proof.* If  $\omega_0 \in S_n$  then  $\omega_0^s$  (the representative in  $S_{n-m} \omega_0$  with the shortest length) is a subword of  $\omega_n$ . By Remark 3.3, it follows that every irreducible representation of  $C(S_q^{n,m})$  factors through  $\chi_{\omega_n}$ . Hence,  $\chi_{\omega_n}$  is faithful. This completes the proof. □

For  $1 \leq k \leq n$ , let  $C(S_q^{n,m,k}) := \chi_{\omega_k}(C(S_q^{n,m}))$ . Then,

$$C(S_q^{n,m,k}) \subset C(S_q^{n,m,1}) \otimes \tau^{\otimes(k-1)}.$$

For  $2 \leq k \leq n$ , let  $\sigma_k$  denote the restriction of  $(1 \otimes 1^{\otimes(k-2)} \otimes \sigma)$  to  $C(S_q^{n,m,k})$ . The image of  $\sigma_k$  is  $C(S_q^{n,m,k-1})$ . We determine the kernel of  $\sigma_k$  in the next proposition. We need the following two lemmas.

**Lemma 3.5.** *The algebra  $\chi_{\omega_{n-1,n-k}}(C(S_q^{n,1}))$  contains  $C^*(t_1) \otimes \mathcal{H}(\ell^2(\mathbb{N}^k))$ , which is isomorphic to  $C(\mathbb{T}) \otimes \mathcal{H}(\ell^2(\mathbb{N}^k))$ .*

*Proof.* Note that  $\chi_{\omega_{n-1,n-k}}(u_{nn}) = t_1 \otimes S^* \sqrt{1 - q^{2N+2}} \otimes 1$ . Hence it follows that the operator

$$1 \otimes \sqrt{1 - q^{2N+2}} \otimes 1 = \chi_{\omega_{n-1,n-k}}(u_{nn}^* u_{nn})$$

lies in the algebra  $\chi_{\omega_{n-1,n-k}}(C(S_q^{n,1}))$ . As  $\sqrt{1 - q^{2N+2}}$  is invertible,  $t_1 \otimes S^* \otimes 1 \in \chi_{\omega_{n-1,n-k}}(C(S_q^{n,1}))$ . Thus, the projection  $1 \otimes p \otimes 1$  is in the algebra  $C(S_q^{n,1,k+1})$ . Observe that, for  $1 \leq s \leq n - 1$ , one has

$$(3-1) \quad (1 \otimes p \otimes 1) \chi_{\omega_{n-1,n-k}}(u_{ns}) = t_1 \otimes p \otimes \pi_{\omega_{n-2,n-k}}(v_{n-1,s}),$$

where  $(v_{ij})$  are the generators of  $C(\text{SU}_q(n - 1))$ . If  $n = 2$ , then  $k = 1$ , and what we have shown is that  $C(S_q^{2,1,2})$  contains  $t_1 \otimes S^*$  and  $t_1 \otimes p$ . Hence,  $C^*(t_1) \otimes \mathcal{H}$  is contained in the algebra  $C(S_q^{2,1,2})$ .

We can now complete the proof by induction on  $n$ . Equation (3-1) shows that  $C^*(t_1) \otimes p \otimes \mathcal{H}^{\otimes(k-1)}$  is contained in the algebra  $C(S_q^{n,1,k+1})$ . Also,  $t_1 \otimes S^* \otimes 1 \in C(S_q^{n,1,k+1})$ . It follows that  $C^*(t_1) \otimes \mathcal{H}^{\otimes k}$  is contained in the algebra  $C(S_q^{n,1,k+1})$ . This completes the proof.  $\square$

**Lemma 3.6.** *Given  $1 \leq s \leq n$ , there exist compact operators  $x_s, y_s$  such that  $x_s \pi_{\omega_{n-1,n-k}}(u_{js}) y_s = \delta_{js}(p \otimes p \otimes \cdots \otimes p)$ , where  $p := 1 - S^* S$ .*

*Proof.* Let  $1 \leq s \leq n$  be given. Note that the operator

$$\omega_{n-1,n-k}(u_{ss}) = z_1 \otimes z_2 \otimes \cdots \otimes z_k,$$

where  $z_i \in \{1, \sqrt{1 - q^{2N+2}} S, S^* \sqrt{1 - q^{2N+2}}\}$ . Define  $x_i, y_i$  by

$$x_i := \begin{cases} p & \text{if } z_i = 1, \\ p & \text{if } z_i = \sqrt{1 - q^{2N+2}} S, \\ (1 - q^2)^{-\frac{1}{2}} p S & \text{if } z_i = S^* \sqrt{1 - q^{2N+2}}; \end{cases}$$

$$y_i := \begin{cases} p & \text{if } z_i = 1, \\ (1 - q^2)^{-\frac{1}{2}} S^* p & \text{if } z_i = \sqrt{1 - q^{2N+2}} S, \\ p & \text{if } z_i = S^* \sqrt{1 - q^{2N+2}}. \end{cases}$$

Then,  $x_i z_i y_i = p$  for  $1 \leq i \leq k$ . Now, let  $x_s := x_1 \otimes x_2 \otimes \cdots \otimes x_k$  and  $y_s := y_1 \otimes y_2 \otimes \cdots \otimes y_k$ . Then,

$$x_s \chi_{\omega_{n-1,n-k}}(u_{ss}) = \underbrace{p \otimes p \otimes \cdots \otimes p}_{k \text{ times}}.$$

Let  $j \neq s$  be given. Then,  $\chi_{\omega_{n-1,n-k}}(u_{js}) = a_1 \otimes a_2 \otimes \cdots \otimes a_k$  where

$$a_i \in \{1, \sqrt{1 - q^{2N+2}} S, S^* \sqrt{1 - q^{2N+2}}, -q^{N+1}, q^N\}.$$

Since  $j \neq s$ , there exists an  $i$  such that  $a_i \in \{q^N, -q^{N+1}\}$ . Let  $r$  be the largest integer for which  $a_r \in \{q^N, -q^{N+1}\}$ . Then,  $z_r \neq 1$  and hence  $x_r a_r y_r = 0$ . Thus,  $x_s \chi_{\omega_{n-1,n-k}}(u_{js}) y_s = 0$ , which completes the proof.  $\square$

**Proposition 3.7.** *Let  $2 \leq k \leq n$ . Then,  $C(S_q^{n,m,1}) \otimes \mathcal{H}(\ell^2(\mathbb{N}))^{\otimes(k-1)}$  is contained in the algebra  $C(S_q^{n,m,k})$ . Moreover, the kernel of the homomorphism  $\sigma_k$  is exactly  $C(S_q^{n,m,1}) \otimes \mathcal{H}(\ell^2(\mathbb{N}))^{\otimes(k-1)}$ . We have the exact sequence*

$$0 \longrightarrow C(S_q^{n,m,1}) \otimes \mathcal{H}^{\otimes(k-1)} \longrightarrow C(S_q^{n,m,k}) \xrightarrow{\sigma_k} C(S_q^{n,m,k-1}) \longrightarrow 0.$$

*Proof.* First, we prove that  $C(S_q^{n,m,1}) \otimes \mathcal{H}^{\otimes(k-1)}$  is contained in  $C(S_q^{n,m,k})$ . For  $a \in C(S_q^{n,1})$ , one has  $\chi_{\omega_k}(a) = 1 \otimes \chi_{\omega_{n-1,n-k+1}}(a)$ , and it follows from Lemma 3.5 that  $C(S_q^{n,m,k})$  contains  $1 \otimes \mathcal{H}(\ell^2(\mathbb{N}^{k-1}))$ . Let  $n - m + 1 \leq r \leq m$  and  $1 \leq s \leq n$  be



given. Note that

$$\chi_{\omega_k}(u_{rs}) = \sum_{j=1}^n \chi_{\omega_1}(u_{rj}) \otimes \pi_{\omega_{n-1, n-k+1}}(u_{js}).$$

By Lemma 3.6, there exist  $x_s, y_s \in C(S_q^{n,m,k})$  such that

$$x_s \chi_{\omega_k}(u_{rs}) y_s := \chi_{\omega_1}(u_{rs}) \otimes p^{\otimes(k-1)},$$

where  $p^{\otimes(k-1)} := p \otimes p \otimes \dots \otimes p$ . Thus, we have shown that  $C(S_q^{n,m,k})$  contains  $1 \otimes \mathcal{H}^{\otimes(k-1)}$  and  $C(S_q^{n,m,1}) \otimes p^{\otimes(k-1)}$ . Hence,  $C(S_q^{n,m,k})$  contains  $C(S_q^{n,m,1}) \otimes \mathcal{H}^{\otimes(k-1)}$ .

Clearly,  $\sigma_k$  vanishes on  $C(S_q^{n,m,1}) \otimes \mathcal{H}^{\otimes(k-1)}$ . Let  $\pi$  be an irreducible representation of  $C(S_q^{n,m,k})$  which vanishes on the ideal  $C(S_q^{n,m,1}) \otimes \mathcal{H}^{\otimes(k-1)}$ . Then,  $\pi \circ \chi_{\omega_k}$  is an irreducible representation of  $C(S_q^{n,m})$  and hence  $\pi \circ \chi_{\omega_k} = \pi_{t,\omega}$  for  $t \in \mathbb{T}^m$  and some  $\omega$  of the form  $\omega_{n-m, i_1} \omega_{n-m+1, i_2} \dots \omega_{n-1, i_{n-m}}$ . Since  $\pi \circ \chi_{\omega_k}(u_{n, n-k+1}) = 0$ , it follows that  $\pi_{t,\omega}(u_{n, n-k+1}) = 0$ . However,  $\pi_{t,\omega}(u_{n, n-k+1}) = t_n (1 \otimes \pi_{\omega_{n-1, i_{n-m}}}(u_{n, n-k+1}))$  and hence  $i_{n-m} > n - k + 1$ . In other words,  $\omega$  is a subword of  $\omega_{k-1}$ . Therefore,  $\pi \circ \chi_{\omega_k}$  factors through  $\chi_{\omega_{k-1}}$  and so there exists a representation  $\rho$  of  $C(S_q^{n,m,k-1})$  such that  $\pi \circ \chi_{\omega_k} = \rho \circ \chi_{\omega_{k-1}}$ . Since  $\chi_{\omega_{k-1}} = \sigma_k \circ \chi_{\omega_k}$ , it follows that  $\pi = \rho \circ \sigma_k$ .

We have shown that every irreducible representation of  $C(S_q^{n,m,k})$  which vanishes on the ideal  $C(S_q^{n,m,1}) \otimes \mathcal{H}^{\otimes(k-1)}$  factors through  $\sigma_k$ . Hence, the kernel of  $\sigma_k$  is exactly the ideal  $C(S_q^{n,m,1}) \otimes \mathcal{H}^{\otimes(k-1)}$ . This completes the proof.  $\square$

We apply the six-term exact sequence in  $K$ -theory to the exact sequence in Proposition 3.7 to compute the  $K$ -groups of  $C(S_q^{n,2,k})$  for  $1 \leq k \leq n$ . In the next section, we briefly recall the product operation in  $K$ -theory.

### 4. The operation $P$

The algebras that we consider will be nuclear. So, no problem arises with regard to tensor products. Let  $A$  and  $B$  be  $C^*$ -algebras. We have the product maps

$$\begin{aligned} K_0(A) \otimes K_0(B) &\rightarrow K_0(A \otimes B), & K_1(A) \otimes K_0(B) &\rightarrow K_1(A \otimes B), \\ K_0(A) \otimes K_1(B) &\rightarrow K_1(A \otimes B), & K_1(A) \otimes K_1(B) &\rightarrow K_0(A \otimes B). \end{aligned}$$

The first map is defined as  $[p] \otimes [q] \rightarrow [p \otimes q]$ ; the second one, as  $[u] \otimes [p] \rightarrow [u \otimes p + 1 - 1 \otimes p]$ ; and the third one likewise. The fourth map is defined using Bott periodicity and the first product; we describe it briefly, referring the reader to [Connes 1981, Appendix] for details.

Let  $h : \mathbb{T}^2 \rightarrow P_1(\mathbb{C}) := \{p \in \text{Proj}(M_2(\mathbb{C})) : \text{trace}(p) = 1\}$  be a degree-one map. Given unitaries  $u \in M_p(A)$  and  $v \in M_q(B)$ , the product  $[u] \otimes [v]$  is given by

$[h(u, v)] - [e_0]$ , where

$$e_0 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \in M_2(M_{pq}(A \otimes B))$$

and  $h(u, v)$  is the matrix with entries  $h_{ij}(u \otimes 1, 1 \otimes v)$ . We denote the image of  $[x] \otimes [y]$  by  $[x] \otimes [y]$  itself. Let  $A$  be a unital commutative  $C^*$ -algebra. Then, the multiplication  $m : A \otimes A \rightarrow A$  is a  $C^*$ -algebra homomorphism. Hence, we get a map at the  $K$ -theory level from  $K_1(A) \otimes K_1(A)$  to  $K_0(A)$ .

Suppose  $U$  and  $V$  are two commuting unitaries in a  $C^*$ -algebra  $B$ . If  $A := C^*(U, V)$ , then  $A$  is commutative. Define

$$P(U, V) := K_0(m)([U] \otimes [V]),$$

which is an element in  $K_0(A)$ . By composing with the inclusion map, we can think of it as an element in  $K_0(B)$ . From the formula of [Connes 1981] that we just recalled, the following properties are clear:

- (1) If  $U$  and  $V$  are commuting unitaries in  $A$ , and  $p$  is a rank-one projection in  $\mathcal{K}$ , then we have  $P(U \otimes p + 1 - 1 \otimes p, V \otimes p + 1 - 1 \otimes p) := P(U, V) \otimes p$ .
- (2) If  $U$  and  $V$  are commuting unitaries, and  $p$  is a projection that commutes with  $U$  and  $V$ , then  $P(U, Vp + 1 - p) = P(Up + 1 - p, Vp + 1 - p)$ .
- (3) If  $\varphi : A \rightarrow B$  is a unital homomorphism, and  $U$  and  $V$  are commuting unitaries in  $A$ , then  $K_0(\varphi)(P(U, V)) = P(\varphi(U), \varphi(V))$ .
- (4) If  $U$  is a unitary in  $A$ , then  $P(U, U) = 0$ . Since  $P_1(\mathbb{C})$  is simply connected, the matrix  $h(U, U)$  is path-connected to a rank-one projection in  $M_2(\mathbb{C})$ . Hence,  $P(U, U) = 0$ .

We need the following lemma in the six-term computation. Let  $z_1 \otimes 1$  and  $1 \otimes z_2$  be the generating unitaries of  $C(\mathbb{T}) \otimes C(\mathbb{T})$ . Then,  $K_0(C(\mathbb{T}^2))$  is isomorphic to  $\mathbb{Z}^2$  and is generated by  $1$  and  $P(z_1 \otimes 1, 1 \otimes z_2)$ .

**Lemma 4.1.** *Consider the exact sequence*

$$0 \longrightarrow C(\mathbb{T}) \otimes \mathcal{K} \longrightarrow C(\mathbb{T}) \otimes \tau \longrightarrow C(\mathbb{T}) \otimes C(\mathbb{T}) \longrightarrow 0$$

and the six-term sequence in  $K$ -theory

$$\begin{array}{ccccc} K_0(C(\mathbb{T}) \otimes \mathcal{K}) & \longrightarrow & K_0(C(\mathbb{T}) \otimes \tau) & \longrightarrow & K_0(C(\mathbb{T}) \otimes C(\mathbb{T})) \\ \uparrow \alpha & & & & \downarrow \delta \\ K_1(C(\mathbb{T}) \otimes C(\mathbb{T})) & \longleftarrow & K_1(C(\mathbb{T}) \otimes \tau) & \longleftarrow & K_1(C(\mathbb{T}) \otimes \mathcal{K}). \end{array}$$

The subgroup generated by  $\delta(P(z_1 \otimes 1, 1 \otimes z_2))$  coincides with the group generated by  $z_1 \otimes p + 1 - 1 \otimes p$ , which is  $K_1(C(\mathbb{T}) \otimes \mathcal{K}) \cong \mathbb{Z}$ .

*Proof.* The Toeplitz map  $\varepsilon : \tau \rightarrow C(\mathbb{T})$  induces an isomorphism at the  $K_0$ -level. Thus, by the Künneth theorem, it follows that the image of  $K_0(1 \otimes \varepsilon)$  is  $K_0(C(\mathbb{T})) \otimes K_0(C(\mathbb{T}))$ , which is the subgroup generated by  $[1]$ . The inclusion  $0 \rightarrow \mathcal{K} \rightarrow \tau$  induces the zero map at the  $K_0$  level and hence, again by the Künneth theorem, the inclusion  $0 \rightarrow C(\mathbb{T}) \otimes \mathcal{K} \rightarrow C(\mathbb{T}) \otimes \tau$  induces the zero map at the  $K_1$ -level. Thus, the image of  $\delta$  is  $K_1(C(\mathbb{T}) \otimes \mathcal{K})$ , which completes the proof.  $\square$

**Corollary 4.2.** *Let  $0 \rightarrow I \rightarrow A \xrightarrow{\varphi} B \rightarrow 0$  be a short exact sequence of  $C^*$ -algebras. Consider the six-term sequence in  $K$ -theory*

$$\begin{array}{ccccc} K_0(I) & \longrightarrow & K_0(A) & \xrightarrow{K_0(\varphi)} & K_0(B) \\ \vartheta \uparrow & & & & \delta \downarrow \\ K_1(B) & \xleftarrow{K_1(\varphi)} & K_1(A) & \xleftarrow{} & K_1(I). \end{array}$$

Suppose that  $U$  and  $V$  are two commuting unitaries in  $B$  such that there exists a unitary  $X$  and an isometry  $Y$  with  $\varphi(X) = U$  and  $\varphi(Y) = V$ . If  $X$  and  $Y$  commute, then the subgroup generated by  $\delta(P(U, V))$  coincides with the subgroup generated by the unitary  $X(1 - YY^*) + YY^*$  in  $K_1(I)$ .

*Proof.* Since  $C(\mathbb{T})$  is the universal  $C^*$ -algebra generated by a unitary, and  $\tau$  is the universal  $C^*$ -algebra generated by an isometry, there exists homomorphisms  $\Phi : C(\mathbb{T}) \otimes \tau \rightarrow A$  and  $\Psi : C(\mathbb{T}) \otimes C(\mathbb{T}) \rightarrow B$  such that

$$\Phi(z_1 \otimes 1) := X, \quad \Phi(1 \otimes S^*) := Y, \quad \Psi(z_1 \otimes 1) := U, \quad \Psi(1 \otimes z_2) := V.$$

We have the commutative diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & C(\mathbb{T}) \otimes \mathcal{K} & \longrightarrow & C(\mathbb{T}) \otimes \tau & \longrightarrow & C(\mathbb{T}) \otimes C(\mathbb{T}) \longrightarrow 0 \\ & & \downarrow \Phi & & \downarrow \Phi & & \downarrow \Psi \\ 0 & \longrightarrow & I & \longrightarrow & A & \xrightarrow{\varphi} & B \longrightarrow 0. \end{array}$$

By the functoriality of  $\delta$  and  $P$ , it follows that

$$\delta(P(U, V)) = K_1(\Phi)(\delta(P(z_1 \otimes 1, 1 \otimes z_2))).$$

By Lemma 4.1, it follows that the subgroup generated by  $\delta(P(U, V))$  is the subgroup generated by  $\Phi(z_1 \otimes p+1-1 \otimes p)$  in  $K_1(I)$ . Note that  $\Phi(z_1 \otimes p+1-1 \otimes p) = X(1 - YY^*) + YY^*$ . This completes the proof.  $\square$

### 5. $K$ -groups of $C(S_q^{n,2,k})$ for $k < n$

In this section, we compute the  $K$ -groups of  $C(S_q^{n,2,k})$  for  $1 \leq k < n$ , by applying the six-term sequence in  $K$ -theory to the exact sequence in Proposition 3.7. We

fix some notation. If  $q$  is a projection in  $\ell^2(\mathbb{N})$  then  $q_r$  denotes the projection  $q \otimes q \otimes \cdots \otimes q$  ( $r$  factors) in  $\ell^2(\mathbb{N}^r)$ . We define the unitaries  $U_k, V_k, u_k, v_k$  by

$$\begin{aligned} U_k &:= t_1 \otimes 1_{n-2} \otimes p_{k-1} + 1 - 1 \otimes 1_{n-2} \otimes p_{k-1}, \\ V_k &:= t_2 \otimes p_{n-2} \otimes 1_{k-1} + 1 - 1 \otimes p_{n-2} \otimes 1_{k-1}, \\ u_k &:= t_1 \otimes p_{n-2} \otimes p_{k-1} + 1 - 1 \otimes p_{n-2} \otimes p_{k-1}, \\ v_k &:= t_2 \otimes p_{n-2} \otimes p_{k-1} + 1 - 1 \otimes p_{n-2} \otimes p_{k-1}. \end{aligned}$$

Note that the operators  $U_k, V_k, u_k, v_k$  lies in the algebra  $C(S_q^{n,2,k})$ . Indeed,

$$\begin{aligned} U_k &= 1_{\{1\}}(u_{n,n-k+1}u_{n,n-k+1}^*)u_{n,n-k+1} + 1 - 1_{\{1\}}(u_{n,n-k+1}u_{n,n-k+1}^*), \\ V_k &= 1_{\{1\}}(u_{n-1,1}u_{n-1,1}^*)u_{n-1,1} + 1 - 1_{\{1\}}(u_{n-1,1}u_{n-1,1}^*), \\ u_k &= 1_{\{1\}}(u_{n,n-k+1}u_{n,n-k+1}^*u_{n-1,1}u_{n-1,1}^*)u_{n,n-k+1} + 1 \\ &\quad - 1_{\{1\}}(u_{n,n-k+1}u_{n,n-k+1}^*u_{n-1,1}u_{n-1,1}^*), \\ v_k &= 1_{\{1\}}(u_{n,n-k+1}u_{n,n-k+1}^*u_{n-1,1}u_{n-1,1}^*)u_{n-1,1} + 1 \\ &\quad - 1_{\{1\}}(u_{n,n-k+1}u_{n,n-k+1}^*u_{n-1,1}u_{n-1,1}^*). \end{aligned}$$

Note that the unitaries  $U_n, u_n$  and  $v_n$  lie in the algebra  $C(S_q^{n,2,n})$ . We start with the computation of the  $K$ -groups of  $C(S_q^{n,2,1})$ .

**Lemma 5.1.** *The  $K$ -groups  $K_0(C(S_q^{n,2,1}))$  and  $K_1(C(S_q^{n,2,1}))$  are both isomorphic to  $\mathbb{Z}^2$ . In fact,  $[U_1]$  and  $[V_1]$  form a  $\mathbb{Z}$ -basis for  $K_1(C(S_q^{n,2,1}))$ , while  $[1]$  and  $P(u_1, v_1)$  form a  $\mathbb{Z}$ -basis for  $K_0(C(S_q^{n,2,1}))$ .*

*Proof.* First, note that  $C(S_q^{n,2,1})$  is generated by  $t_1 \otimes 1_{n-2}$  and  $t_2 \otimes \pi_{\omega_{n-2,1}}(u_{n-1,j})$  for  $1 \leq j \leq n-1$ . The  $C^*$ -algebra generated by  $\{t_2 \otimes \pi_{\omega_{n-2,1}}(u_{n-1,j}) : 1 \leq j \leq n-1\}$  is isomorphic to  $C(S_q^{2n-3})$ . Hence,  $C(S_q^{n,2,1})$  is isomorphic to  $C(\mathbb{T}) \otimes C(S_q^{2n-3})$ . Also,  $K_0(C(S_q^{2n-3}))$  and  $K_1(C(S_q^{2n-3}))$  are both isomorphic to  $\mathbb{Z}$ , with  $[1]$  generating  $K_0(C(S_q^{2n-3}))$ , and  $[t_2 \otimes p_{n-2} + 1 - 1 \otimes p_{n-2}]$  generating  $K_1(C(S_q^{2n-3}))$ .

Now, by the Künneth theorem for the tensor product of  $C^*$ -algebras (see [Blackadar 1986]), it follows that  $C(S_q^{n,2,1})$  has both  $K_1$  and  $K_0$  isomorphic to  $\mathbb{Z}^2$ , with  $[U_1]$  and  $[V_1]$  generating  $K_1(C(S_q^{n,2,1}))$ , and  $[1]$  and

$$P(t_1 \otimes 1_{n-2}, t_2 \otimes p_{n-2} + 1 - 1 \otimes p_{n-2})$$

generating  $K_0(C(S_q^{n,2,1}))$ . The projection  $1 \otimes p_{n-2} = 1_{\{1\}}(\chi_{\omega_{n-2,1}}(u_{n-1,1}u_{n-1,1}^*))$  is in  $C(S_q^{n,2,1})$  and commutes with the unitaries  $t_1 \otimes 1_{n-2}$  and  $t_2 \otimes p_{n-2} + 1 - 1 \otimes p_{n-2}$ . Hence,

$$P(t_1 \otimes 1_{n-2}, t_2 \otimes p_{n-2} + 1 - 1 \otimes p_{n-2}) = P(u_1, v_1).$$

This completes the proof. □

**Proposition 5.2.** *Let  $1 \leq k < n$ . The  $K$ -groups  $K_0(C(S_q^{n,2,k}))$  and  $K_1(C(S_q^{n,2,k}))$  are both isomorphic to  $\mathbb{Z}^2$  and, in particular,  $[U_k]$  and  $[V_k]$  form a  $\mathbb{Z}$ -basis for  $K_1(C(S_q^{n,2,k}))$ , while  $[1]$  and  $P(u_k, v_k)$  form a  $\mathbb{Z}$ -basis for  $K_0(C(S_q^{n,2,k}))$ .*

*Proof.* We prove this result by induction on  $k$ . The case  $k = 1$  is just Lemma 5.1. Assume the result to be true for  $k$ . From Proposition 3.7, we have the short exact sequence

$$0 \longrightarrow C(S_q^{n,2,1}) \otimes \mathcal{H}^{\otimes(k)} \longrightarrow C(S_q^{n,2,k+1}) \xrightarrow{\sigma_{k+1}} C(S_q^{n,2,k}) \longrightarrow 0,$$

which gives rise to the following six-term sequence in  $K$ -theory:

$$\begin{array}{ccccc} K_0(C(S_q^{n,2,1}) \otimes \mathcal{H}^{\otimes k}) & \longrightarrow & K_0(C(S_q^{n,2,k+1})) & \xrightarrow{K_0(\sigma_{k+1})} & K_0(C(S_q^{n,2,k})) \\ \partial \uparrow & & & & \delta \downarrow \\ K_1(C(S_q^{n,2,k})) & \xleftarrow{K_1(\sigma_{k+1})} & K_1(C(S_q^{n,2,k+1})) & \xleftarrow{} & K_1(C(S_q^{n,2,1}) \otimes \mathcal{H}^{\otimes k}). \end{array}$$

To compute the six-term sequence, we determine  $\delta$  and  $\partial$ . Since  $\sigma_{k+1}(V_{k+1}) = V_k$ , it follows that  $\partial([V_k]) = 0$ . Since  $C(S_q^{n,2,k+1})$  contains the algebra  $C(S_q^{n,2,1}) \otimes \mathcal{H}^{\otimes k}$ , it follows that the operator

$$\tilde{X} := t_1 \otimes 1_{n-2} \otimes \underbrace{q^N \otimes q^N \otimes \dots \otimes q^N}_{(k-1) \text{ times}} \otimes S^*$$

is in the algebra  $C(S_q^{n,2,1})$ ; indeed, the difference  $X - \chi_{\omega_{k+1}}(u_{n,n-k+1})$  lies in the ideal  $C(S_q^{n,2,1}) \otimes \mathcal{H}^{\otimes k}$ . Let

$$X := 1_{\{1\}}(\tilde{X}^* \tilde{X}) \tilde{X} + 1 - 1_{\{1\}}(\tilde{X}^* \tilde{X}).$$

Then,  $X$  is an isometry such that  $\sigma_{k+1}(X) = U_k$  and hence

$$\partial([U_k]) = [1 - X^* X] - [1 - X X^*].$$

Thus,  $\partial([U_k]) = -[1 \otimes 1_{n-2} \otimes p_k]$ . The image of  $\partial$  is the subgroup of  $K_0(C(S_q^{n,2,1}) \otimes \mathcal{H}^{\otimes k})$  generated by  $[1 \otimes 1_{n-2} \otimes p_k]$ , while its kernel is  $[V_k]$ .

Next, we compute  $\delta$ . Since  $\sigma_{k+1}(1) = 1$ , it follows that  $\delta([1]) = 0$ . Let

$$Y := (1 \otimes p_{n-2} \otimes 1_k)(1 \otimes 1_{n-2} \otimes p_{k-1} \otimes 1) \tilde{X} + 1 - 1 \otimes p_{n-2} \otimes p_{k-1} \otimes 1.$$

Since  $1 \otimes p_{n-2} \otimes 1 = 1_{\{1\}}(\chi_{\omega_k}(u_{n-1,1}^* u_{n-1,1}))$  and  $1 \otimes 1_{n-2} \otimes p_{k-1} = 1_{\{1\}}(\tilde{X}^* \tilde{X})$ , it follows that  $Y \in C(S_q^{n,2,k+1})$ . Also,

$$Y = t_1 \otimes p_{n-2} \otimes p_{k-1} \otimes S^* + 1 - 1 \otimes p_{n-2} \otimes p_{k-1} \otimes 1.$$

Note that  $Y$  is an isometry such that  $\sigma_{k+1}(Y) = u_k$ . One has  $\sigma_{k+1}(v_{k+1}) = v_k$ . Observe that  $Y$  and  $v_{k+1}$  commute. By Lemma 4.1, it follows that the image of  $\delta$  is the subgroup generated by  $[v_{k+1}(1 - YY^*) + YY^*] = [V_1 \otimes p_k + 1 - 1 \otimes p_k]$ .

This computation with the six-term sequence implies that  $K_0(C(S_q^{n,2,k+1}))$  is isomorphic to  $\mathbb{Z}^2$  and is generated by  $P(u_1, v_1) \otimes p_k = P(u_k, v_k)$  and  $[1]$ . Also, the group  $K_1(C(S_q^{n,2,k+1}))$  is isomorphic to  $\mathbb{Z}^2$  and is generated by  $[V_{k+1}]$  and  $[U_1 \otimes p_k + 1 - 1 \otimes p_k] = [U_{k+1}]$ . This completes the proof.  $\square$

### 6. K-groups of $C(S_q^{n,2})$

In this section, we compute the  $K$ -groups of  $C(S_q^{n,2})$ . We start with a few observations.

**Lemma 6.1.** *In the permutation group  $S_n$ , one has  $\omega_{n-2,1}\omega_{n-1,1} = \omega_{n-1,1}\omega_{n-1,2}$ .*

*Proof.* First, note that  $s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}$ , and  $s_i s_j = s_j s_i$  if  $|i - j| \geq 2$ . Hence,  $\omega_{n-1,k}\omega_{n-1,1} = \omega_{n-1,k+1}\omega_{n-1,1}s_{k+1}$ . The result follows by induction on  $k$ .  $\square$

We denote the representation  $\chi_{\omega_{n-1,1}} * \pi_{\omega_{n-1,2}}$  by  $\tilde{\chi}_{\omega_n}$ . Since  $\omega_{n-1,1}\omega_{n-1,2}$  is a reduced expression for  $\omega_n$ , the representations  $\tilde{\chi}_{\omega_n}$  and  $\chi_{\omega_n}$  are equivalent. Let  $U$  be a unitary such that  $U\chi_{\omega_n}(\cdot)U^* = \tilde{\chi}_{\omega_n}(\cdot)$ . It is clear that

$$\tilde{\chi}_{\omega_n}(C(S_q^{n,2})) \subset C(\mathbb{T}^m) \otimes \tau \otimes \tau^{\otimes \ell(\omega_{n-1})}.$$

Let  $\tilde{\sigma}_n$  denote the restriction of  $1 \otimes \sigma \otimes 1^{\otimes (2(n-2))}$  to  $\tilde{\chi}_{\omega_n}(C(S_q^{n,2}))$ . Since

$$\tilde{\sigma}_n(\tilde{\chi}_{\omega_n}(u_{ij})) = \chi_{\omega_{n-1}}(u_{ij}),$$

we have the commutative diagram

$$\begin{array}{ccc} \chi_{\omega_n}(C(S_q^{n,2})) & \xrightarrow{U(\cdot)U^*} & \tilde{\chi}_{\omega_n}(C(S_q^{n,2})) \\ & \searrow \sigma_n & \swarrow \tilde{\sigma}_n \\ & C(S_q^{n,2,n-1}) & \end{array}$$

**Lemma 6.2.** *There exists a coisometry  $X \in \chi_{\omega_n}(C(S_q^{n,2}))$  such that  $\sigma_n(X) = V_{n-1}$  and  $X^*X = 1 - 1_{\{1\}}(\chi_{\omega_n}(u_{n1}^* u_{n1}))$ .*

*Proof.* From the commutative diagram above, it is enough to show that there exists a coisometry  $\tilde{X} \in \tilde{\chi}_{\omega_n}(C(S_q^{n,2}))$  such that

$$\tilde{\sigma}_n(X) = V_{n-1} \quad \text{and} \quad X^*X = 1 - 1_{\{1\}}(\tilde{\chi}_{\omega_n}(u_{n1}^* u_{n1})).$$

Note that

$$\tilde{\chi}_{\omega_n}(u_{n-1,1}^* u_{n-1,1} - q^2 u_{n1} u_{n1}) = 1 \otimes 1 \otimes \underbrace{q^{2N} \otimes q^{2N} \otimes \dots \otimes q^{2N}}_{(n-2) \text{ times}} \otimes 1_{n-2}.$$

Hence the projection  $1 \otimes 1 \otimes p_{n-2} \otimes 1_{n-2} = 1_{\{1\}}(\tilde{\chi}_{\omega_n}(u_{n-1,1}^* u_{n-1,1} - q^2 u_{n1}^* u_{n1}))$  is in the algebra  $\tilde{\chi}_{\omega_n}(C(S_q^{n,2}))$ . Now let  $Y := (1 \otimes 1 \otimes p_{n-2} \otimes 1_{n-2})\tilde{\chi}_{\omega_n}(u_{n-1,1})$ . Then,

$$Y := t_2 \otimes \sqrt{1 - q^{2N+2}} S \otimes p_{n-2} \otimes 1_{n-2}.$$

Hence, the operator  $Z := t_2 \otimes S \otimes p_{n-2} \otimes 1_{n-2}$  is in the algebra  $\tilde{\chi}_{\omega_n}(C(S_q^{n,2}))$ . Now, let  $\tilde{X} := Z + 1 - ZZ^*$ . Then,  $\tilde{X}$  is a coisometry such that  $\tilde{\sigma}_n(\tilde{X}) = V_{n-1}$  and  $\tilde{X}^* \tilde{X} = 1 - 1 \otimes p_{n-1} \otimes 1_{n-2}$ , which is  $1 - 1_{\{1\}}(\tilde{\chi}_{\omega_n}(u_{n1}^* u_{n1}))$ . This completes the proof. □

Observe that the operator

$$\tilde{Z} := t_1 \otimes 1_{n-1} \otimes \underbrace{q^N \otimes q^N \otimes \dots \otimes q^N}_{(n-2) \text{ times}} \otimes S^*$$

lies in the algebra  $C(S_q^{n,2,n})$ , since the difference  $\tilde{Z} - \chi_{\omega_n}(u_{n,2})$  lies in the ideal  $C(S_q^{n,2,1}) \otimes \mathcal{K}^{\otimes(n-1)}$ . Let  $Z := 1_{\{1\}}(\tilde{Z}^* \tilde{Z}) \tilde{Z}$  and  $Y_n := Z + 1 - Z^* Z$ . Then,

(6-1)  $Z_n = t_1 \otimes 1_{n-2} \otimes p_{n-2} \otimes S^*$ ,

(6-2)  $Y_n = t_1 \otimes 1_{n-2} \otimes p_{n-2} \otimes S^* + 1 - 1 \otimes 1_{n-2} \otimes p_{n-2} \otimes 1$ .

Hence,  $Y$  is an isometry and  $YY^* = 1 - 1_{\{1\}}(\chi_{\omega_n}(u_{n1}^* u_{n1}))$ . If  $X$  is a coisometry in  $C(S_q^{n,2,n})$  such that  $\sigma_n(X) = v_{n-1}$  and  $X^* X := 1 - 1_{\{1\}}(\chi_{\omega_n}(u_{n1}^* u_{n1}))$ , then  $XY$  is a unitary. (The existence of such an  $X$  was shown in Lemma 6.2.)

**Proposition 6.3.** *The  $K$ -groups  $K_0(C(S_q^{n,2}))$  and  $K_1(C(S_q^{n,2}))$  are both isomorphic to  $\mathbb{Z}^2$ . In particular,*

- (1) *the projections  $[1]$  and  $P(u_n, v_n)$  generate  $K_0(C(S_q^{n,2}))$ ;*
- (2) *the unitaries  $U_n$  and  $XY_n$  generate  $K_1(C(S_q^{n,2}))$ , where  $X$  is a coisometry in  $C(S_q^{n,2})$  such that  $\sigma_n(X) = V_{n-1}$  and  $X^* X = 1 - 1_{\{1\}}(u_{n1}^* u_{n1})$ , while  $Y_n$  is as in (6-2)*

*Proof.* By Proposition 3.7, we have the exact sequence

$$0 \longrightarrow C(S_q^{n,2,1}) \otimes \mathcal{K}^{\otimes(n-1)} \longrightarrow C(S_q^{n,2,n}) \xrightarrow{\sigma_n} C(S_q^{n,2,n-1}) \longrightarrow 0,$$

which gives rise to the six-term sequence in  $K$ -theory

$$\begin{CD} K_0(C(S_q^{n,2,1}) \otimes \mathcal{K}^{\otimes(n-1)}) @>>> K_0(C(S_q^{n,2,n})) @>{K_0(\sigma_n)}>> K_0(C(S_q^{n,2,k})) \\ @. @. @VV\delta V \\ K_1(C(S_q^{n,2,n-1})) @<<< K_1(C(S_q^{n,2,n})) @<<< K_1(C(S_q^{n,2,1}) \otimes \mathcal{K}^{\otimes(n-1)}) \\ @. @. @AA\partial A \end{CD}$$

Now, we evaluate  $\partial$  and  $\delta$  to compute the six-term sequence. Since  $[U_{n-1}]$  and  $[V_{n-1}]$  generate  $K_1(C(S_q^{n,2,n-1}))$ , it follows that  $[U_{n-1}]$  and  $[V_{n-1}U_{n-1}]$  generate  $K_1(C(S_q^{n,2,n-1}))$ . As  $XY_n$  is a unitary with  $\sigma_n(XY_n) = V_{n-1}U_{n-1}$ , it follows that  $\partial([V_{n-1}U_{n-1}]) = 0$ . Next,  $Y_n$  is an isometry with  $\sigma_n(Y_n) = U_{n-1}$ . Hence  $\partial([U_{n-1}]) = [1 - Y^*Y] - [1 - YY^*]$ . Thus,  $\partial([U_{n-1}]) = -[1 \otimes 1_{n-2} \otimes p_{n-1}]$ .

Now, we compute  $\delta$ . Since  $\sigma_n(1) = 1$ , it follows that  $\delta([1]) = 0$ . One observes that  $p_{n-2} \otimes S^*\pi_{\omega_{n-1,1}}(u_{j1}) = 0$  if  $j > 1$ . Hence,

$$Z_n \chi_{\omega_n}(u_{n-1,1}) = t_1 t_2 \otimes p_{n-2} \otimes p_{n-2} \otimes \sqrt{1 - q^{2N+2}},$$

where  $Z_n$  is as defined in (6-1). The operator  $R_n := t_1 t_2 \otimes p_{n-2} \otimes p_{n-2} \otimes 1$  lies in the algebra  $C(S_q^{n,2,n})$ , since the difference  $R_n - Z_n \chi_{\omega_n}(u_{n-1,1})$  lies in the ideal  $C(\mathbb{T}^2) \otimes \mathcal{H}^{\otimes(2n-3)}$ . Hence, the projection  $1 \otimes p_{n-2} \otimes p_{n-2} \otimes 1$  lies in the algebra  $C(S_q^{n,2,n})$ . Now, define

$$\begin{aligned} S_n &:= R_n + 1 - R_n R_n^*, \\ T_n &:= (1 \otimes p_{n-2} \otimes p_{n-2} \otimes 1)Z_n + 1 - 1 \otimes p_{n-2} \otimes p_{n-2} \otimes 1. \end{aligned}$$

Then,  $S_n$  is a unitary and  $T_n$  is an isometry such that  $\sigma_n(S_n) = u_{n-1}v_{n-1}$  and  $\sigma_n(T_n) = u_{n-1}$ . Moreover,  $S_n$  and  $T_n$  commute. Note that  $P(u_{n-1}, v_{n-1}) = P(u_{n-1}, u_{n-1}v_{n-1})$ . By Lemma 4.1, the image of  $\delta$  is the subgroup generated by  $S_n(1 - T_n T_n^*) + T_n T_n^*$  in  $K_1(C(S_q^{n,2,1}) \otimes \mathcal{H}^{\otimes(n-1)})$ . Now,

$$S_n(1 - T_n T_n^*) + T_n T_n^* = t_1 t_2 \otimes p_{n-2} \otimes p_{n-1} + 1 - 1 \otimes p_{n-2} \otimes p_{n-1}.$$

Since  $1 \otimes p_{n-2}$  is trivial in  $K_0(C(S_q^{2n-3}))$ , the unitary  $t_1 \otimes p_{n-2} + 1 - 1 \otimes p_{n-2}$  is trivial in  $K_1(C(S_q^{n,2,1})) = K_1(C(\mathbb{T}) \otimes C(S_q^{2n-3}))$ . One has  $[S_n(1 - T_n T_n^*) + T_n T_n^*] = [V_1 \otimes p_{n-1} + 1 - 1 \otimes p_{n-1}]$  in  $K_1(C(S_q^{n,2,1}) \otimes \mathcal{H}^{\otimes(n-1)})$ .

This computation, with the exactness of the six-term sequence, completes the proof. □

### 7. K-groups of quantum SU(3)

In this section, we show that when  $n = 3$  the unitary  $XY_n$  in Proposition 6.3 can be replaced by the fundamental  $3 \times 3$  matrix  $(u_{ij})$  of  $C(SU_q(3))$ . First, note that for  $n = 3$  we have  $C(S_q^{n,2}) = C(SU_q(3))$ , since  $C(SU_q(1)) = \mathbb{C}$ . The embedding  $SU_q(1) \subseteq SU_q(3)$  is given by the counit. The quotient  $C(SU_q(3)/SU_q(1))$  becomes isomorphic with  $C(SU_q(3))$ . In [Sheu 1997], the algebra  $C(S_q^{3,2,1})$  is denoted  $C(U_q(2))$ . Then,  $C(U_q(2)) = C(\mathbb{T}) \otimes C(SU_q(2))$ . Let  $\text{ev}_1 : C(\mathbb{T}) \rightarrow \mathbb{C}$  be the evaluation at the point 1. Then,  $\varphi = (\text{ev}_1 \otimes 1) \sigma_2 \sigma_3$ , where  $\varphi : C(SU_q(3)) \rightarrow C(SU_q(2))$  is the subgroup homomorphism defined in (2-1).

**Proposition 7.1.** *The K-group  $K_1(C(SU_q(3)))$  is isomorphic to  $\mathbb{Z}^2$ , generated by the unitary  $U_3 := t_1 \otimes p \otimes p + 1 - 1 \otimes p \otimes p$  and the fundamental unitary  $U = (u_{ij})$*



*Proof.* By Proposition 6.3, we know that  $K_1(C(\mathrm{SU}_q(3)))$  is isomorphic to  $\mathbb{Z}^2$  and generated by  $[U_3]$  and  $[XY_3]$ , where  $X$  is a coisometry such that  $\sigma_3(X) = V_2$  and  $X^*X = 1 - 1_{\{1\}}(\chi_{\omega_3}(u_{31}^*u_{31}))$ . Observe that  $\varphi(X) = t_2 \otimes p + 1 - 1 \otimes p$  and  $\varphi(Y_3) = 1$ . Hence,  $\varphi(XY_3) = t_2 \otimes p + 1 - 1 \otimes p$ . Also note that

$$\varphi(U_3) = 0 \quad \text{and} \quad \varphi(U) = \begin{bmatrix} u & 0 \\ 0 & 1 \end{bmatrix},$$

where  $u$  denote the fundamental unitary of  $C(\mathrm{SU}_q(2))$ . Since  $K_1(C(\mathrm{SU}_q(2)))$  is isomorphic to  $\mathbb{Z}$ , the proof is complete if we show that  $t_2 \otimes p + 1 - 1 \otimes p$  and  $[u]$  represent the same element in  $K_1(C(\mathrm{SU}_q(2)))$ ; we do this in the next lemma.  $\square$

Denote by  $u_q$  the  $2 \times 2$  fundamental unitary  $u = (u_{ij})$  of  $C(\mathrm{SU}_q(2))$ . Consider the representation  $\chi_{s_1} : C(\mathrm{SU}_q(2)) \rightarrow B(\ell^2(\mathbb{Z}) \otimes \ell^2(\mathbb{N}))$ . We let the unitary  $t$  act on  $\ell^2(\mathbb{Z})$  as the right shift, that is,  $te_n = e_{n+1}$ . Let  $\{e_{n,m} : n \in \mathbb{Z}, m \in \mathbb{N}\}$  be the standard orthonormal basis for the Hilbert space  $\ell^2(\mathbb{Z}) \otimes \ell^2(\mathbb{N})$ . For an integer  $k$ , denote by  $P_k$  the orthogonal projection onto the closed subspace spanned by  $\{e_{n,m} : n + m \leq k\}$ , and set  $F_k := 2P_k - 1$ . Note that  $F_k$  is a selfadjoint unitary.

**Proposition 7.2.** *For any integer  $k$ , the triple  $(\chi_{s_1}, \ell^2(\mathbb{Z}) \otimes \ell^2(\mathbb{N}), F_k)$  is an odd Fredholm module for  $C(\mathrm{SU}_q(2))$ , and we have the pairing*

- (1)  $\langle [u_q], F_k \rangle = -1$ ,
- (2)  $\langle t \otimes p + 1 - 1 \otimes p, F_k \rangle = -1$ , where  $p = 1 - S^*S$ .

*Proof.* It is not difficult to show that  $C(\mathrm{SU}_q(2))$  is generated by  $t \otimes S$  and  $t \otimes p$ . Now, one can see that  $[t \otimes S, P_k] = 0$  and  $[t \otimes p, P_k]$  is a finite-rank operator. Hence, the triple  $(\chi_{s_1}, \ell^2(\mathbb{Z}) \otimes \ell^2(\mathbb{N}), F_k)$  is an odd Fredholm module for  $C(\mathrm{SU}_q(2))$ . Since  $C(\mathrm{SU}_q(2))$  is generated by  $t \otimes S$  and  $t \otimes p$ , it follows that  $u_p \in C(\mathrm{SU}_q(2))$  for every  $p > 0$ . Also, as  $p \rightarrow 0$ ,  $u_p$  approaches  $u$  in norm, where  $u$  is given by

$$u := \begin{pmatrix} t \otimes S & 0 \\ \bar{t} \otimes p & \bar{t} \otimes S^* \end{pmatrix}.$$

Hence,  $[u_q] = [u]$  in  $K_1(C(\mathrm{SU}_q(2)))$ . It is now easy to check that  $\langle [u], F_k \rangle = -1$  and  $\langle [t \otimes p + 1 - 1 \otimes p], F_k \rangle = -1$ . This completes the proof.  $\square$

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Received July 23, 2010.

PARTHA SARATHI CHAKRABORTY  
THE INSTITUTE OF MATHEMATICAL SCIENCES  
CHENNAI 600113  
INDIA  
[parthac@imsc.res.in](mailto:parthac@imsc.res.in)

S. SUNDAR  
THE INSTITUTE OF MATHEMATICAL SCIENCES  
CHENNAI 600113  
INDIA  
[sundarsobers@gmail.com](mailto:sundarsobers@gmail.com)

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University of California  
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The subscription price for 2011 is US \$420/year for the electronic version, and \$485/year for print and electronic.

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The Pacific Journal of Mathematics (ISSN 0030-8730) at the University of California, c/o Department of Mathematics, 969 Evans Hall, Berkeley, CA 94720-3840, is published monthly except July and August. Periodical rate postage paid at Berkeley, CA 94704, and additional mailing offices. POSTMASTER: send address changes to Pacific Journal of Mathematics, P.O. Box 4163, Berkeley, CA 94704-0163.

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PJM peer review and production are managed by EditFLOW™ from Mathematical Sciences Publishers.

PUBLISHED BY PACIFIC JOURNAL OF MATHEMATICS

at the University of California, Berkeley 94720-3840

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Typeset in L<sup>A</sup>T<sub>E</sub>X

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Volume 252 No. 2 August 2011

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0030-8730(201108)252:2;1-9