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Maps into projective spaces

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Abstract. We compute the cohomology of the Picard bundle on the desingularization $\tilde{J}^d(Y)$ of the compactified Jacobian of an irreducible nodal curve Y. We use it to compute the cohomology classes of the Brill–Noether loci in $\tilde{J}^d(Y)$.

We show that the moduli space M of morphisms of a fixed degree from Y to a projective space has a smooth compactification. As another application of the cohomology of the Picard bundle, we compute a top intersection number for the moduli space M confirming the Vafa–Intriligator formulae in the nodal case.

Keywords. Nodal curves; torsionfree sheaves; Picard bundle.

1. Introduction

Let Y be an integral nodal curve of arithmetic genus g, with m (ordinary) nodes as only singularities, defined over an algebraically closed field of characteristic 0. Let $\bar{J}^d(Y)$ denote the compactified Jacobian of Y i.e., the space of torsion-free sheaves of rank 1 and degree d on Y. The generalized Jacobian $J^d(Y) \subset \bar{J}^d(Y)$, the subset consisting of locally free sheaves, is the set of nonsingular points of $\bar{J}^d(Y)$. There is a natural desingularization of $\bar{J}^d(Y)$ (Proposition 12.1, p. 64 of [9])

$$h : \tilde{J}^d(Y) \to \bar{J}^d(Y).$$

Let $\tilde{\theta}$ denote the pullback of the theta divisor (or the theta line bundle) on $\bar{J}^d(Y)$ to $\tilde{J}^d(Y)$. Let $\tilde{\mathcal{P}}$ be the pullback to $\tilde{J}^d(Y) \times Y$ of the Poincaré sheaf $\mathcal{P}$ on $\bar{J}^d(Y) \times Y$. For $d \geq 2g - 1$, the direct image E_d of the Poincaré sheaf $\tilde{\mathcal{P}}$ is a vector bundle on $\tilde{J}^d(Y)$ called the degree d Picard bundle. Unlike in the case of a nonsingular curve, E_d is neither $\tilde{\theta}$-stable nor ample [6]. However, as the following theorem shows, the Chern classes of this bundle are given by a formula exactly the same as that in the smooth case.

Theorem 1.1. *The total Segre class $s(E_d)$ of the Picard bundle E_d is*

$$s(E_{\mathrm{d}}) = \mathrm{e}^{\tilde{\theta}} \text{ and hence } c(E_{\mathrm{d}}) = \mathrm{e}^{-\tilde{\theta}}.$$

We give a few applications of this theorem. The Brill–Noether scheme $B_Y(1, d, r) \subset \bar{J}^d(Y)$ is the scheme whose underlying set is the set of torsion-free sheaves of rank 1 and degree d on Y with at least r independent sections. The expected dimension of $B_Y(1,d,r)$ is given by the Brill–Noether number

$$\beta_Y(1,d,r) = g - r(r-d-1+g)\,.$$

Let $\tilde{B}_Y(1,d,r) := h^{-1}B_Y(1,d,r) \subset \tilde{J}^d(Y)$ be the Brill–Noether locus in $\tilde{J}^d(Y)$. Since h is a finite surjective map, $B_Y(1,d,r)$ is nonempty if and only if $\tilde{B}_Y(1,d,r)$ is nonempty. Using Theorem 1.1, we compute the fundamental class of $\tilde{B}_Y(1,d,r)$ and use it to give an effective proof of the nonemptiness of $\tilde{B}_Y(1,d,r)$ for $\beta_Y(1,d,r) \geq 0$.

Theorem 1.2. *If $\tilde{B}_Y(1,d,r)$ is empty or if $\tilde{B}_Y(1,d,r)$ has the expected dimension $\beta_Y(1,d,r)$, then the fundamental class $\tilde{b}_{1,d,r}$ of $\tilde{B}_Y(1,d,r)$ coincides with*

$$b_{1,d,r} = \prod_{\alpha=0}^{r-1} \frac{\alpha!}{g-d+r-1+\alpha}\,\tilde{\theta}^{r(g-d+r-1)}\,.$$

COROLLARY 1.3

$B_Y(1,d,r)$ and $\tilde{B}_Y(1,d,r)$ are nonempty for $\beta_Y(1,d,r) \geq 0$.

For a fixed positive integer r, consider the direct sum of r copies of E_d,

$$\mathcal{E} = \oplus_r E_d\,.$$

For $\tilde{L} \in \tilde{J}^d(Y)$ with $h(\tilde{L}) = L$, the fibre of $\mathbb{P}(\mathcal{E})$ is isomorphic to $\mathbb{P}(\oplus_r\ H^0(Y,L))$. A point in the fibre may be written as a class

$$(\tilde{L},\bar{\phi}) = (\tilde{L},\phi_1,\ldots,\phi_r),$$

where

$$\tilde{L} \in \tilde{J}^d(Y),\ \phi_i \in H^0(Y,L) \quad \text{and} \quad (\phi_1,\ldots,\phi_r) \neq (0,\ldots,0).$$

Let V_ϕ be the subspace of $H^0(Y,L)$ generated by $\phi_1,\ldots,\phi_r$. Define

$$M := \{(\tilde{L},\bar{\phi}) \in \mathbb{P}(\mathcal{E}) \mid L \text{ locally free, } V_\phi \text{ generates } L\}. \tag{1.1}$$

We show that M can be regarded as the moduli space of morphisms

$$Y \to \mathbb{P}^{r-1}$$

of degree d for $d \geq 2g-1, d \geq 0, r \geq 2$.

Theorem 1.4.

(1) *There exists a morphism F_M from $M \times Y$ to a $\mathbb{P}^{r-1}$-bundle over $M \times Y$ such that for any element $a \in M$, $F_M\,|_{a\times Y}$ determines a morphism $f_a : Y \to \mathbb{P}^{r-1}$ of degree d.*
(2) *Given a scheme S and a morphism $F_S : S \times Y \to \mathbb{P}^{r-1}$ such that for any $s \in S$, the morphism $F_s : Y \to \mathbb{P}^{r-1}$ is of degree d, there is a morphism*

$$\alpha_S : S \to M$$

such that the base change of F_M by $\alpha_S \times id$ gives F_S.

Thus $\bar{M} := \mathbb{P}(\mathcal{E})$ can be regarded as a compactification of the moduli space of morphisms $Y \to \mathbb{P}^{r-1}$ of degree d.

Fixing a nonsingular point $t \in Y$, we may assume that the Poincaré bundle is normalized so that $\tilde{\mathcal{P}} \mid \tilde{J}^d(Y) \times t$ is the trivial line bundle on $\tilde{J}^d(Y)$. Then we see that the restriction of F_M to $M \times t$ gives

$$F_t : M \to \mathbb{P}^{r-1}$$

defined by

$$F_t(\tilde{L}, \bar{\phi}) = (\phi_1(t), \dots, \phi_r(t))\,.$$

Fixing a hyperplane H of $\mathbb{P}^{r-1}$, we get a Cartier divisor on M with the underlying set

$$X = X_H := \{(\tilde{L}, \bar{\phi}) \in M \mid F_t((\tilde{L}, \bar{\phi})) \in H\}.$$

One shows that there exists a variety $Z \subset \bar{M}$ such that $c_1(\mathcal{O}_{\bar{M}}(1)) = Z$ and $Z \cap M = X$. We define the top intersection number $\langle X^n \rangle$ of X in M as the top intersection number $Z^n[\bar{M}]$ of Z, n being the dimension of M.

Theorem 1.5.

$$\langle X^n \rangle \;=\; r^g\,.$$

In case Y is smooth, a formula for the top intersection number was given by Vafa [10] and Intriligator (eq. (5.5) of [8]). The formula was verified to be true by Bertram and others (Theorem 5.11 of [3]). Theorem 1.5 shows that the intersection number is the same in the nodal case.

2. Cohomology of the Picard bundle

2.1 *Notation*

Let Y be an integral nodal curve of arithmetic genus g with m (ordinary) nodes defined over an algebraically closed field of characteristic 0. Let $y_1, \dots, y_m$ be the nodes of Y. We denote by X_k the curve with k nodes obtained by blowing up the nodes $y_{k+1}, \dots, y_m$, thus $Y = X_m$ and X_0 is the normalization of Y. Denote the normalization map by

$$p : X_0 \to Y.$$

Let $p^{-1}(y_k) = \{x_k, z_k\} \in X_0$ be the inverse image of the nodal point y_k in Y. By abuse of notation, we denote by the same x_k and z_k, the image of these points in X_j, for all $j < k$. Let $p_k : X_{k-1} \to X_k$ be the natural morphism obtained by identifying x_k and z_k to the single node y_k.

Let us denote by Y^o and X_0^o the smooth irreducible open subsets $Y - \{\cup_{j=1}^m y_j\}$ and $X_0 - \cup_{j=1}^m \{x_j, z_j\}$ respectively. Then one has $Y^o \cong X_0^o$. Note that X_0^o maps isomorphically onto an open subset X_k^o of X_k for each k. For $x \in X_0^o$, we use the same notation for x and its image in X_k^o for all k (if no confusion is possible).

Once and for all, fix a (sufficiently general) point $t \in Y^o$.

2.2 *The cycles $\tilde{W}_d$ and the nonsingular variety Bl^d*

Let $\bar{J}^d(Y)$ denote the compactified Jacobian i.e., the space of torsion-free sheaves of rank 1 and degree d on Y. It is a seminormal variety. The generalized Jacobian $J^d(Y) \subset \bar{J}^d(Y)$, the subset consisting of locally free sheaves, is the set of nonsingular points of $\bar{J}^d(Y)$. The compactified Jacobian $\bar{J}^d(Y)$ has a natural desingularization

$$h : \tilde{J}^d(Y) \to \bar{J}^d(Y).$$

It is a $\mathbb{P}^1 \times \cdots \times \mathbb{P}^1$-bundle ($m$-fold product) over $J^d(X_0)$ (Prop. 12.1, p. 64 of [9], [4]). Since h is an isomorphism over $J^d(Y)$, the Jacobian $J^d(Y)$ is canonically embedded in $\tilde{J}^d(Y)$.

We have the Abel–Jacobi map

$$Y \to \bar{J}^1(Y)$$

which is an embedding [1]. However, it does not extend to a morphism $S^d(Y) \to \bar{J}^d(Y)$, where $S^d(Y)$ is the symmetric d-th power of Y. The problem being that, unlike in the smooth case, the tensor products of non-locally free sheaves on Y have torsion.

Certainly the restriction of the Abel–Jacobi map to Y^o extends to $S^d(Y^o)$ giving a morphism

$$f'_d : S^d(Y^o) \to J^d(Y)$$

defined by

$$[x_1, \ldots, x_d] \mapsto \mathcal{O}_Y(x_1 + \cdots + x_d) \in J^d(Y).$$

Define cycles $\tilde{W}_d \subset J^d(Y) \subset \tilde{J}^d(Y)$ to be the closure of the image of f'_d in $\tilde{J}^d(Y)$ with the reduced scheme structure. In particular, $\tilde{W}_{g-1}$ is a divisor. Define the theta divisor $\tilde{\theta}$ on $\tilde{J}^{g-1}(Y)$ as $\tilde{W}_{g-1}$. We identify $\tilde{W}_d$ with its isomorphic image in $\tilde{J}^0(Y)$ under translation by $\mathcal{O}_Y(-dt)$.

In Theorem 3.1 of [5], we have proved the following generalization of the Poincaré formula. For $1 \leq d \leq g$, one has

$$\tilde{W}_{g-d} = \frac{\tilde{\theta}^d}{d!} \tag{2.1}$$

as cycles in $\tilde{J}^0(Y)$ modulo numerical equivalence.

We also identified $\tilde{W}_d$ with the Brill–Noether locus $\tilde{B}_Y(1, d, 1)$ whose underlying set is

$$\tilde{B}_Y(1, d, 1) := \{N \in \tilde{J}^d(Y) \mid h^0(Y, h(N)) \geq 1\}.$$

We constructed in § 4.1 of [5] a nonsingular variety Bl^d and a morphism

$$\psi : Bl^d \to \tilde{J}^d(Y)$$

with image $\tilde{W}_d$. The morphism ψ is analogous to the natural morphism

$$\psi_0 : S^d(X_0) \to W_d(X_0) \subseteq J^d(X_0),$$

where $W_d(X_0) = B_{X_0}(1, d, 1)$. We define $Bl_0^d = S^d(X_0)$. The variety $Bl^d := Bl_m^d$ was constructed from Bl_0^d by induction on the number of nodes. For $k = 1, \ldots, m$, we have divisors $\Sigma_{x_k} \cong S^{d-1}(X_0) \times x_k \subset S^d(X_0)$ and $\Sigma_{z_k} \cong S^{d-1}(X_0) \times z_k \subset S^d(X_0)$. Bl_1^d is obtained by blowing up $\Sigma_{x_1} \cap \Sigma_{z_1}$ in Bl_0^d. Inductively Bl_k^d is obtained by blowing up $D(x_k) \cap D(z_k)$ in Bl_{k-1}^d where $D(x_k)$ and $D(z_k)$ are respectively the proper transforms of Σ_{x_k} and Σ_{z_k}.

For $x \in Y^o$, let $\Sigma_x \subset S^d(X_0)$ be the divisor isomorphic to $S^{d-1}(X_0) \times x$ and D_x its proper transform in Bl^d. Let $[D_x]$ denote the class of D_x. Note that for $d \leq g$, $\psi : Bl^d \to \tilde{W}_d$ is a surjective birational morphism. Therefore, for the cycle $[D_x]^i$ of codimension i in Bl^d, the cycle $\psi_*[D_x]^i$ is of codimension i in $\tilde{W}_d$. Since $\tilde{W}_d$ is of codimension $g - d$ in $\tilde{J}^d(Y)$, it follows that $\psi_*[D_x]^i$ is of codimension $g - d + i$ in $\tilde{J}^d(Y)$. In fact, we have the following explicit description of the latter cycle.

PROPOSITION 2.1

$$\psi_*[D_x]^i = \frac{\tilde{\theta}^{g-d+i}}{(g-d+i)!}\,.$$

Proof. We have a commutative diagram

$$\begin{array}{ccc} Bl^d & \xrightarrow{\psi} & \tilde{J}^d(Y) \\ \pi\big\downarrow & & \big\downarrow p' \\ Bl_0^d & \xrightarrow{\psi_0} & J^d(X_0)\,. \end{array}$$

The fibre of ψ_0 over $L_0 \in J^d(X_0)$ is $F_{\psi_0} \cong \mathbb{P}(H^0(X_0, L_0))$, the space of 1-dimensional subspaces of $H^0(X_0, L_0)$. A point $\tilde{L} \in \tilde{J}^d(Y)$ corresponds to a tuple $(L_0, Q_1, \ldots, Q_m)$ where $L_0 \in J^d(X_0)$ and Q_j are 1-dimensional quotients of $(L_0)_{x_j} \oplus (L_0)_{z_j}$. One has $h(\tilde{L}) \subset p_* L_0$ and hence $H^0(Y, h(\tilde{L})) \subset H^0(Y, p_* L_0) \cong H^0(X_0, L_0)$. As in the proof of Proposition 4.3 of [5], it follows that the map $Bl^d \to Bl_0^d$ induces an injection of fibres $F_\psi \to F_{\psi_0}$. The fibre F_ψ of ψ over $\tilde{L} \in \tilde{J}^d(Y)$ is isomorphic to $\mathbb{P}(H^0(Y, h(\tilde{L}))$ (Proposition 4.3 of [5]) and the injection $F_\psi \to F_{\psi_0}$ is the canonical injection $H^0(Y, h(\tilde{L})) \subset H^0(X_0, L_0)$.

The elements of $S^d(X_0)$ can be identified with divisors on X_0. For $x \in X_0^o$,

$$\Sigma_x = \{D \in S^d(X_0) \mid D = x + D', D' \in S^{d-1}(X_0)\}.$$

Equivalently, $\Sigma_x = \{D \in S^d(X_0) \mid D - x \geq 0\}$. Thus

$$\psi_0(\Sigma_x) = \{L_0 \in J^d(X_0) \mid L_0(-x) \in W_{d-1}(X_0)\}$$

is a translate of $W_{d-1}(X_0)$.

One has $F_{\psi_0} \cap \Sigma_x \cong H^0(X_0, L_0(-x))$ [3]. Hence

$$D_x \cap F_\psi = H^0(X_0, L_0(-x)) \cap H^0(Y, h(\tilde{L})) = H^0(Y, h(\tilde{L})(-x)).$$

It follows that $\psi(D_x)$ is an x-translate of $\tilde{W}_{d-1} \cong \tilde{B}_Y(1, d, 1)$. More generally, if $x_1, \ldots x_i$ are general elements of Y^o, then one has $\psi(D_{x_1} \cap \cdots \cap D_{x_i})$ is an $(\sum_{j=1}^{i} x_j)$-translate of $\tilde{W}_{d-i}$. By generalized Poincaré formula on Y (equation (2.1), Theorem 3.8 of [5]), we have

$$[\tilde{W}_{d-i}] = \frac{\tilde{\theta}^{g-d+i}}{(g-d+i)!}\,.$$

Thus $\psi_*[D_{x_1} \cap \cdots \cap D_{x_i}] = \tilde{\theta}^{g-d+i}/(g-d+i)!$ and hence

$$\psi_*[D_x]^i = \frac{\tilde{\theta}^{g-d+i}}{(g-d+i)!}$$

for all $x \in Y^o$. □

2.3 *The Picard bundle*

Recall that we have fixed a point $t \in Y^o$. There exists a Poincaré sheaf $\mathcal{P} \to \bar{J}^d(Y) \times Y$ normalized by the condition that $\mathcal{P} \mid \bar{J}^d(Y) \times t$ is the trivial line bundle on $\bar{J}^d(Y)$ (see [7]). Let

$$\tilde{\mathcal{P}} \to \tilde{J}^d(Y) \times Y$$

be the pullback of $\mathcal{P}$ to $\tilde{J}^d(Y) \times Y$. Then $\tilde{\mathcal{P}}$ is a family of torsion-free sheaves of rank 1 and degree d on Y parametrized by $\tilde{J}^d(Y)$ and $\tilde{\mathcal{P}} \mid \tilde{J}^d(Y) \times t$ is the trivial line bundle on $\tilde{J}^d(Y)$. Let ν (respectively, p_Y) denote the projections from $\tilde{J}^d(Y) \times Y$ to $\tilde{J}^d(Y)$ (respectively, Y).

For $d \geq 2g-1$, the direct image E_d of the Poincaré sheaf $\tilde{P}$ on $\tilde{J}^d(Y) \times Y$ is a vector bundle on $\tilde{J}^d(Y)$ called the degree d Picard bundle. It is a vector bundle of rank $d+1-g$.

PROPOSITION 2.2

For $d \geq 2g$, Bl^d is isomorphic to the projective bundle $\mathbb{P}(E_d)$.

Proof. We prove the result by induction on the number k of nodes. Recall that X_k denotes the curve with k nodes $y_1, \ldots, y_k$. Let $g(X_k)$ be the genus of X_k. For each k and $d \in \mathbb{Z}$, let $J^d(X_k)$ be the Jacobian and $\bar{J}^d(X_k)$ the compactified Jacobian of degree d on X_k. We have a $\mathbb{P}^1$-bundle

$$\pi_k : \tilde{J}^d(X_k) \to \tilde{J}^d(X_{k-1})\,.$$

We identify $\bar{J}^0(X_k)$ with $\bar{J}^d(X_k)$ by the morphism $L \mapsto L(dt)$ for $L \in \bar{J}^0(X_k)$ for all k. This also gives an identification of $\tilde{J}^0(X_k)$ with $\tilde{J}^d(X_k)$. Let $E_{d,k}$ be the Picard bundle on $\tilde{J}^d(X_k)$. Let Bl_k^d be the variety Bl^d corresponding to X_k.

For $k = 0$, set $Bl_0^d = S^d(X_0)$ and it is well-known that this symmetric product is isomorphic to $\mathbb{P}(E_{d,0})$ for $d \geq 2g(X_0)$. Now, by induction, we may assume that for $k \geq 1$, we have $\mathbb{P}(E_{d,k-1}) \cong Bl_{k-1}^d$. Hence

$$\pi_k^* \mathbb{P}(E_{d,k-1}) \cong Bl_{k-1}^d \times_{\tilde{J}^d(X_{k-1})} \tilde{J}^d(X_k) \subset Bl_{k-1}^d \times \tilde{J}^d(X_k).$$

There is an injective morphism $i_k : E_{d,k} \to \pi_k^* E_{d,k-1}$ (Proposition 5.1 of [6]) so that

$$\mathbb{P}(i_k E_{d,k}) \subset Bl^d_{k-1} \times \tilde{J}^d(X_k)\,.$$

On the other hand, by the construction of Bl^d_k,

$$Bl^d_k \subset Bl^d_{k-1} \times \tilde{J}^d(X_k).$$

In fact it is the closure of the graph of a rational map $\psi'_k : Bl^d_{k-1} \to \tilde{J}^d(X_k)$. We recall the definition of ψ'_k. There exists an open set $U_{k-1} \subset S^d(X^o_0)$ embedded in Bl^d_{k-1} (i.e. isomorphic to an open subset of Bl^d_{k-1}) such that ψ'_k is well-defined on U_{k-1} and is defined as follows: For $\sum_i p_i \in U_{k-1}$, one has $\psi'_k(\sum_i p_i) = (j, Q) \in \tilde{J}^d(X_k)$ where j corresponds to the line bundle $L = \mathcal{O}_{X_{k-1}}(\sum p_i)$, L has a unique (up to a scalar) section s with zero scheme $\sum_i p_i$ and Q is the quotient of $L_{x_k} \oplus L_{z_k}$ by the 1-dimensional subspace generated by $s(x_k)+s(z_k)$. The pair (j, Q) determines $j' = h(j, Q)$ and s gives a section s' of the line bundle L' corresponding to j'.

Recall that by the definition of the direct image, the elements of $E_{d,k}$ correspond to all the pairs (j', s'), $j' \in \tilde{J}^d(X_k)$, $s' \in H^0(X_k, L')$ where L' is the torsionfree sheaf corresponding to $h(j')$. Let $j = h(\pi_k(j)')$, $j \in \bar{J}_{X_{k-1}}$. If L corresponds to j, then the injection $(i_k)_{j'}$ corresponds to the inclusion $H^0(X_k, L') \subset H^0(X_{k-1}, L)$. It follows that $\mathbb{P}(i_k(E_{d,k})) \subset Bl^d_{k-1} \times \tilde{J}^d(X_k)$ contains the graph of ψ'_k and hence its closure Bl^d_k. Since Bl^d_k and $\mathbb{P}(i_k(E_{d,k}))$ are irreducible and of the same dimension, it follows that they coincide. Since both Bl^d_k and $\mathbb{P}(i_k(E_{d,k}))$ are nonsingular, the injective homomorphism i_k induces an isomorphism from $\mathbb{P}(E_{d,k})$ onto Bl^d_k. □

Theorem 2.3 (Theorem 1.1). *The total Segre class $s(E_d)$ of the Picard bundle is*

$$s(E_d) = \mathrm{e}^{\tilde{\theta}} \quad \text{and} \quad c(E_d) = \mathrm{e}^{-\tilde{\theta}}\,.$$

Proof. Since $\tilde{\mathcal{P}}\mid_{\tilde{J}^d(Y)\times t} \cong \mathcal{O}_{\tilde{J}^d(Y)}$, the restriction of the evaluation map $ev : \nu^*\nu_*\tilde{\mathcal{P}} \to \tilde{\mathcal{P}}$ to $\tilde{J}^d(Y) \times t$ gives a surjective homomorphism

$$ev_t : \nu_*\tilde{\mathcal{P}} = E_d \to \mathcal{O}_{\tilde{J}^d(Y)}\,.$$

This defines a section s_t of $\mathcal{O}_{\mathbb{P}(E_d)}(1)$ whose zero set is the divisor

$$D_t = \{(\tilde{L}, \phi) \in \mathbb{P}(E_d) \mid \phi(t) = 0\}\,.$$

Thus we have

$$[D_t] = c_1(\mathcal{O}_{\mathbb{P}(E_d)}(1))\,. \tag{2.2}$$

By Proposition 2.1,

$$\psi_*[D_t]^{d-g+\ell} = \frac{\tilde{\theta}^\ell}{\ell!}\,. \tag{2.3}$$

By VII(4.3), p. 318 of [2], if $c(-E_d) := \frac{1}{c(E_d)}$ denotes the Segre class of E_d, then

$$c_l(-E_d) = \psi_*[D_t]^{d+1-g-1+l} = \psi_*[D_t]^{d-g+l}\,.$$

Hence by eq. (2.3), one has $c_l(-E_d) = \frac{\tilde{\theta}^l}{l!}$ and hence $c(-E_d) = \mathrm{e}^{\tilde{\theta}}$. Thus

$$c(E_d) = \mathrm{e}^{-\tilde{\theta}}\,.$$

□

3. Brill–Noether loci

The Brill–Noether scheme $B_Y(1,d,r) \subset \bar{J}^d(Y)$ is the scheme whose underlying set is the set of torsion-free sheaves of rank 1 and degree d on Y with at least r independent sections. The expected dimension of $B_Y(1,d,r)$ is given by the Brill–Noether number

$$\beta_Y(1,d,r) = g - r(r-d-1+g)\,.$$

Let $\tilde{B}_Y(1,d,r) := h^{-1}B_Y(1,d,r) \subset \tilde{J}^d(Y)$ be the Brill–Noether locus in $\tilde{J}^d(Y)$. Since h is a surjective map, $\tilde{J}^d(Y)$ is nonempty if and only if $\tilde{B}_Y(1,d,r)$ is nonempty. In this section, we compute the fundamental class of $\tilde{B}_Y(1,d,r) \in \tilde{J}^d(Y)$ using the Porteous' formula. An application of this computation is an effective proof of nonemptiness of $\tilde{B}_Y(1,d,r)$ for $\beta_Y(1,d,r) \geq 0$ following VII, Theorem 4.4 of [2]. We note that since $\tilde{J}^d(Y)$ is a smooth algebraic variety, the Porteous' formula is valid in the Chow ring of $\tilde{J}^d(Y)$ (II(4.2) of [2]). We recall the formula.

For a vector bundle V on a variety Z, one has $c_t(V) = 1 + c_1(V)t + c_2(V)t^2 + \cdots$ and c_t extends to a homomorphism from the Grothendieck group $K(Z)$ to the multiplicative group of the invertible elements in the power series ring $H^*(Z)[[t]]$. Let $-V$ denote the negative of the class of V in $K(Z)$ and $V_1 - V_0$ the difference of the classes of V_1 and V_0 in $K(Z)$.

For a formal power series $a(i) = \sum_{-\infty}^{\infty} a_i t^i$, set

$$\Delta_{p,q}(a) = \det A,$$

where A is the matrix

$$\begin{pmatrix} a_p & \cdots & a_{p+q-1} \\ . & \cdots & . \\ . & \cdots & . \\ . & \cdots & . \\ a_{p-q+1} & \cdots & a_p \end{pmatrix}.$$

3.1 *Porteous' formula*

Let V_0 and V_1 be holomorphic vector bundles of respective ranks n and m over a complex manifold Z and $\Psi : V_0 \to V_1$ a holomorphic mapping. Let $Z_k(\Psi)$ be the k-th degeneracy locus associated to Ψ. It is supported on the set

$$Z_k(\Psi) = \{z \in Z \mid \text{ rank } \Psi_z \leq k\}\,.$$

Then if $Z_k(\Psi)$ is empty or has the expected dimension dim $Z-(n-k)(m-k)$, the fundamental class z_k of $Z_k(\Psi)$ coincides with

$$\Delta_{m-k,n-k}(c_t(V_1-V_0))=(-1)^{(m-k)(n-k)}\Delta_{n-k,m-k}(c_t(V_0-V_1)).$$

Theorem 3.1 (Theorem 1.2). *If $\tilde{B}_Y(1,d,r)$ is empty or if $\tilde{B}_Y(1,d,r)$ has the expected dimension $\beta_Y(1,d,r)$, then the fundamental class $\tilde{b}_{1,d,r}$ of $\tilde{B}_Y(1,d,r)$ coincides with*

$$b_{1,d,r}=\prod_{\alpha=0}^{r-1}\frac{\alpha!}{g-d+r-1+\alpha}\;\tilde{\theta}^{r(g-d+r-1)}.$$

Proof. Recall that $\tilde{\mathcal{P}}$ denotes the Poincaré sheaf on $\tilde{J}^d(Y)\times Y$ and $\nu:\tilde{J}^d(Y)\times Y\to\tilde{J}^d(Y)$, $p_Y:\tilde{J}^d(Y)\times Y\to Y$ denote the projections.

Fix a Cartier divisor E on Y of degree $m\geq 2g-d-1$ and let $n=m+d-g+1$. Then (as seen in section 5.1 of [5]) $\tilde{B}_Y(1,d,r)\subset\tilde{J}^d(Y)$ is the $(n-r)$-th degenaracy locus of the morphism

$$\Psi:\tilde{V_0}\to\tilde{V_1},$$

where

$$\tilde{V_0}:=\nu_*(\tilde{\mathcal{P}}\otimes\tilde{p}_Y^*\mathcal{O}_Y(E))\quad\text{and}\quad\tilde{V_1}:=\tilde{\nu}_*(\tilde{\mathcal{P}}\otimes\tilde{p}_Y^*\mathcal{O}_Y(E)\mid_{\tilde{p}_Y^{-1}(E)}).$$

The sheaves $\tilde{V_0}$ and $\tilde{V_1}$ are locally free sheaves of rank n and m respectively. The vector bundle $\tilde{V_1}$ is a direct sum of line bundles with the first Chern class 0 and so it has a trivial (total) Chern class. Hence by Porteous' formula, one has

$$\tilde{b}_{1,d,r}=\Delta_{g-d+r-1,r}(c_t(-V_0)).$$

By Theorem 1.1, $c(-V_0)=\mathrm{e}^{\tilde{\theta}}$. Then

$$\tilde{b}_{1,d,r}=\Delta_{g-d+r-1,r}(\mathrm{e}^{t\tilde{\theta}}).$$

By calculations exactly the same as those on page 320 of [2] (in the proof of VII, Theorem (4.4) of [2]), we finally have

$$\tilde{b}_{1,d,r}=\prod_{\alpha=0}^{r-1}\frac{\alpha!}{g-d+r-1+\alpha}\;\tilde{\theta}^{r(g-d+r-1)}.$$

□

COROLLARY 3.2 (Corollary 1.3)

$B_Y(1,d,r)$ and $\tilde{B}_Y(1,d,r)$ are nonempty for $\beta_Y(1,d,r)\geq 0$.

Proof. Note that $\beta_Y(1,d,r)=g-r(g-d+r-1)\geq 0$ if and only if $g\geq r(g-d+r-1)$ so that $b_{1,d,r}$ is nonzero if $\beta_Y(1,d,r)\geq 0$. The fundamental class $\tilde{b}_{1,d,r}$ of $\tilde{B}_Y(1,d,r)$ coincides with $b_{1,d,r}$ (by Theorem 1.2) and hence is nonzero for $\beta_Y(1,d,r)\geq 0$. Hence $\tilde{B}_Y(1,d,r)$ is nonempty for $\beta_Y(1,d,r)\geq 0$. It follows that $B_Y(1,d,r)$ is nonempty for $\beta_Y(1,d,r)\geq 0$. □

4. Maps from Y to $\mathbb{P}^{r-1}$

Assume that $d \geq 2g-1, d \geq 0, r \geq 2$. Let $\mathcal{E} = \oplus_r E_d$ where E_d is the Picard bundle on $\tilde{J}^d(Y)$ defined in § 2.3. Let

$$u : \mathbb{P}(\mathcal{E}) \to \tilde{J}^d(Y)$$

be the projection map. For $\tilde{L} \in \tilde{J}^d(Y)$ with $h(\tilde{L}) = L$, the fibre of $\bar{M}$ over $\tilde{L}$ is isomorphic to $\mathbb{P}(\oplus_r H^0(Y, L))$. A point in the fibre may be written as a class $(\tilde{L}, \bar{\phi}) = (\tilde{L}, \phi_1, \ldots, \phi_r)$ with $\tilde{L} \in \tilde{J}^d(Y)$, $\phi_i \in H^0(Y, L)$ and $(\phi_1, \ldots, \phi_r) \neq (0, \ldots, 0)$. Let V_ϕ be the subspace of $H^0(Y, L)$ generated by $\phi_1, \ldots, \phi_r$. Let

$$M = \{(\tilde{L}, \bar{\phi}) \in \mathbb{P}(\mathcal{E}) \mid L \text{ locally free, } V_\phi \text{ generates } L\}.$$

The following theorem shows that M can be regarded as the moduli space of morphisms

$$Y \to \mathbb{P}^{r-1}$$

of degree d.

Theorem 4.1 (Theorem 1.4).

(1) *There exists a morphism F_M from $M \times Y$ to a projective bundle on $M \times Y$ such that for any element $a \in M$, $F_M |_{a \times Y}$ gives a morphism $f_a : Y \to \mathbb{P}^{r-1}$ of degree d.*
(2) *Given a scheme S and a morphism $F_S : S \times Y \to \mathbb{P}^{r-1}$ such that for any $s \in S$, the morphism $F_s = F_S |_{s \times Y}: Y \to \mathbb{P}^{r-1}$ is of degree d, there is a morphism*

$$\alpha_S : S \to M$$

such that the base change of F_M by α_S gives F_S.

Thus $\bar{M} := \mathbb{P}(\mathcal{E})$ may be regarded as a compactification of the moduli space of morphisms $Y \to \mathbb{P}^{r-1}$ of degree d.

Proof.

(1) Over $\tilde{J}^d(Y) \times Y$, we have the evaluation map $ev_r : \nu^* \nu_*(\oplus_r \tilde{\mathcal{P}}) \to \oplus_r \tilde{\mathcal{P}}$. Pulling back to $\bar{M} \times Y$ by $u' = u \times Id_Y$ gives the map

$$u'^* e_{v_r} : u'^* \nu^* \nu_*(\oplus_r \tilde{\mathcal{P}}) \to u'^*(\oplus_r \tilde{\mathcal{P}})\,.$$

Its restriction to $M \times Y$ induces a map

$$e_M : \mathbb{P}(u'^* \nu^* \nu_*(\oplus_r \tilde{\mathcal{P}})) \to \mathbb{P}(u'^*(\oplus_r \tilde{\mathcal{P}}))\,.$$

Note that the fibre of the bundle $u'^* \nu^* \nu_*(\oplus_r \tilde{\mathcal{P}})$ over $(\tilde{L}, \bar{\phi}, y)$ is $\oplus_r H^0(Y, L)$. By definition, $M = \mathbb{P}(\nu_*(\oplus_r \tilde{\mathcal{P}}))$. Hence $\mathbb{P}(u'^* \nu^* \nu_*(\oplus_r \tilde{\mathcal{P}})) \to M \times Y$ has a canonical section σ defined by $\sigma(\tilde{L}, \bar{\phi}, y) = (\phi_1, \ldots, \phi_r)$. Then

$$F_M := e_M \circ \sigma : M \times Y \to \mathbb{P}(u'^*(\oplus_r \tilde{\mathcal{P}})) \cong \mathbb{P}((h \circ u) \times id_Y)^*(\oplus_r \mathcal{P})) \quad (4.1)$$

is the required morphism. To see this, note that the restriction of this composite morphism to $(\tilde{L}, \bar{\phi}) \times Y$ gives $(\phi_1, \ldots, \phi_r) \in \mathbb{P}(H^0(Y, L))$ and hence determines the morphism

$$f_{\tilde{L},\bar{\phi}} : Y \to \mathbb{P}^{r-1}$$

defined by

$$f_{\tilde{L},\bar{\phi}}(y) = (\phi_1(y), \ldots, \phi_r(y)) .$$

We remark that in case Y is a smooth curve, this map is the same as the pointwise map defined in Proposition 2.7 of [3].

(2) Let $F_S : S \times Y \to \mathbb{P}^{r-1}$ be a morphism such that for any $s \in S$, the morphism $F_s = F_S \mid_{s \times Y}: Y \to \mathbb{P}^{r-1}$ is of degree d. Let

$$N := F_S^*(\mathcal{O}_{\mathbb{P}^{r-1}}(1)) .$$

Note that for all $s \in S$, $N_s = N \mid_{s \times Y}$ is a line bundle of degree d generated by global sections. The coordinate functions z_i, $i = 1, \ldots, r$, on $\mathbb{C}^r$ define sections z_i of $\mathcal{O}_{\mathbb{P}^{r-1}}(1)$. F_S gives sections $\Phi_i = F_S^*(z_i)$ of N such that $\Phi_i \mid_{s \times Y}$, $i = 1, \ldots, r$, generate N_s of all s. Define

$$F'_S : S \times Y \to \mathbb{P}(\oplus_r N),$$
$$F'_S(s, y) := (\Phi_1(s, y), \ldots, \Phi_r(s, y)) \in \mathbb{P}(\oplus_r N_{s,y}) .$$

Since $N := F_S^*(\mathcal{O}_{\mathbb{P}^{r-1}}(1))$, we have a map $\beta_r : \mathbb{P}(\oplus_r N) \to \mathbb{P}(\oplus_r \mathcal{O}_{\mathbb{P}^{r-1}}(1))$ lying over F_S. One has $\beta_r((\Phi_i(s, y))_i) = (z_i(F_S(s, y))_i)$. Hence there is a commutative diagram

$$\begin{array}{ccc} \mathbb{P}(\oplus_r N) & \overset{\beta_r}{\to} & \mathbb{P}(\oplus_r \mathcal{O}_{\mathbb{P}^{r-1}}(1)) \\ \uparrow F'_S & & \downarrow \pi \\ S \times Y & \overset{F_S}{\to} & \mathbb{P}^{r-1} \end{array}$$

showing that F_S can be recovered from F'_S.

Let $\mathcal{P}' = \tilde{\mathcal{P}} \mid_{J^d(Y)}$. By the universal property of the Jacobian, the line bundle $N \to S \times Y$ defines a morphism $\alpha : S \to J^d(Y) \subset \tilde{J}^d(Y)$. One has $(\alpha \times id)^*\mathcal{P}' \cong N \otimes p_S^* N_1$, where N_1 is a line bundle on S. Thus

$$(\alpha \times id)^*(\oplus_r \mathcal{P}') \cong (\oplus_r N) \otimes p_S^* N_1 . \tag{4.2}$$

By the projection formula, we have

$$\begin{aligned} \alpha^*(\nu_* \oplus_r \mathcal{P}') &\cong p_{S*}(\alpha \times id)^*(\oplus_r \mathcal{P}') \cong p_{S*}((\oplus_r N) \otimes p_S^* N_1) \\ &\cong (p_{S*}(\oplus_r N)) \otimes N_1 . \end{aligned}$$

Thus $\alpha^*(\mathcal{E}) \cong (p_{S*}(\oplus_r N)) \otimes N_1$. We have

$$\alpha^*(\bar{M}) = \alpha^*(\mathbb{P}\mathcal{E}) = \mathbb{P}(\alpha^*\mathcal{E})$$

and hence

$$\alpha^*(\bar{M}\mid_{J^d(Y)}) \cong \mathbb{P}(p_{S*}(\oplus_r N))\,.$$

This gives the cartesian diagram

$$\begin{array}{ccc} \mathbb{P}(p_{S*}(\oplus_r N)) & \stackrel{\bar{\alpha}}{\to} & \bar{M}\mid_{J^d(Y)} \\ \downarrow & & \downarrow^{u} \\ S & \stackrel{\alpha}{\to} & J^d(Y)\,. \end{array}$$

The sections $\Phi_i \in H^0(S \times Y, N)$ give $\Phi \in \oplus_r H^0(S \times Y, N)) \cong H^0(S, p_{S*}(\oplus_r N)$. Since N is generated by Φ_i's, this gives a section $\bar{\phi}_S$ of $\mathbb{P}(p_{S*}(\oplus_r N))$ over S such that if

$$\alpha_S = \bar{\alpha} \circ \bar{\phi}_S,\ \text{then}\ \alpha_S(S) \subset M\,.$$

Then $u \circ \alpha_S = \alpha$ and the isomorphism (4.2) implies that

$$(\alpha_S \times id)^*(u'^* \oplus_r \mathcal{P}') = (\alpha \times id)^*(\oplus_r \mathcal{P}') \cong (\oplus_r N) \otimes p_S^*(N_1)$$

so that

$$(\alpha_S \times id)^*(\mathbb{P}(u'^* \oplus_r \mathcal{P}')) = \mathbb{P}(\oplus_r N)\,.$$

It follows that the family $F'_S : S \times Y \to \mathbb{P}(\oplus_r N)$ is the base change of the family $F_M : M \times Y \to \mathbb{P}(u'^* \oplus_r \mathcal{P}')$ by $\alpha_S \times id$. As explained in the beginning, F'_S gives F_S.

This completes the proof of the theorem. □

We remark that the proof of Theorem 1.4 is valid for any integral curve Y with its (compactified) Jacobian irreducible.

4.1 *Top intersection number*

Recall that $u : \bar{M} \to \tilde{J}^d(Y)$ is a projective bundle so that

$$n = \dim \bar{M} = r(d + 1 - g) + \dim \tilde{J}^d(Y) - 1 = r(d + 1 - g) + g - 1\,.$$

Let $\mathcal{O}_{\bar{M}}(1) = \mathcal{O}_{\mathbb{P}(\oplus_r E_d)}(1)$ be the relative ample line bundle.

The restriction of F_M to $M \times t$ followed by the projection to $\mathbb{P}^{r-1}$ gives

$$F_t : M \to \mathbb{P}(\oplus_r \mathcal{O}_M) = M \times \mathbb{P}^{r-1} \to \mathbb{P}^{r-1}$$

defined by

$$F_t(\tilde{L}, \bar{\phi}) = (\phi_1(t), \ldots, \phi_r(t))\,.$$

Fix a section $s \in H^0(\mathbb{P}^{r-1}, \mathcal{O}_{\mathbb{P}^{r-1}}(1))$. It determines a hyperplane H of $\mathbb{P}^{r-1}$. Then $F_t^* s \in H^0(M, F_t^* \mathcal{O}_{\mathbb{P}^{r-1}}(1))$ defines a Cartier divisor on M. The underlying set of the Cartier divisor is given by

$$X = X_H := \{(\tilde{L}, \bar{\phi}) \in M \mid F_t((\tilde{L}, \bar{\phi})) \in H\}.$$

Lemma 4.2. *There exists a variety $Z \subset \bar{M}$ such that $c_1(\mathcal{O}_{\bar{M}}(1)) = Z$ and $Z \cap M = X$.*

Proof. Restricting the evaluation map $ev : \nu^* \nu_* \tilde{\mathcal{P}} \to \tilde{\mathcal{P}}$ to $\tilde{J}^d(Y) \times t$ we get

$$ev_t : \nu_* \tilde{\mathcal{P}} \to (\tilde{\mathcal{P}})_t = \mathcal{O}_{\tilde{J}^d(Y)}.$$

Composing a projection (say 1st) $\nu_*(\oplus_r \tilde{\mathcal{P}}) \to \nu_* \tilde{\mathcal{P}}$ with this map ev_t gives the surjective homomorphism $\nu_*(\oplus_r \tilde{\mathcal{P}}) = \mathcal{E} \to \mathcal{O}_{\tilde{J}^d(Y)}$. This defines a section s_t of $\mathcal{O}_{\bar{M}}(1)$ whose zero set is

$$Z := Z(s_t) = \{(\tilde{L}, \bar{\phi}) \mid \phi_1(t) = 0\}.$$

Then $Z \cap M = X_{H_1}$ where $H_1 = \{(z_1, \ldots, z_r) \in \mathbb{P}^{r-1} \mid z_1 = 0\}$. □

DEFINITION 4.3

We define the top intersection number $\langle X^n \rangle$ of X in M as the intersection number

$$\langle X^n \rangle \ := \ Z^n[\bar{M}] \ = \ c_1(\mathcal{O}_{\bar{M}}(1))^n[\bar{M}].$$

Theorem 4.4. (Theorem 1.5).

$$\langle X^n \rangle \ = \ r^g.$$

Proof. By Lemma 4.2 and VII(4.3), p. 318 of [2] applied to the vector bundle $\mathcal{E} = \oplus_r E_d$, we have (for $u_* : H^n(\mathbb{P}\mathcal{E}) \to H^g(\tilde{J}^d(Y))$)

$$u_* Z^n = c_g(-\mathcal{E}) = s_g(\mathcal{E}),$$

where $s_g(\mathcal{E})$ is the g-th Segre class of $\mathcal{E}$. By Theorem 1.1, $s(\mathcal{E}) = e^{r\tilde{\theta}}$ so that

$$s_g(\mathcal{E}) = \frac{r^g \tilde{\theta}^g}{g!}.$$

By the generalized Poincaré formula (see eq. (2.1)), $\tilde{\theta}^g[\tilde{J}^d(Y)] = g!$ so that

$$s_g(\mathcal{E})[\tilde{J}^d(Y)] = r^g,$$

proving the theorem. □

4.2 *The formulas of Vafa and Intriligator*

Let X be a smooth curve (a compact Riemann surface). The formula for the top intersection number for the space of maps from X to projective spaces (and more generally for intersection numbers for the space of maps from X to Grassmannians) was given by Vafa and worked out in detail by Intriligator (eq. (5.5) of [8], [10]). The formula was verified to be true by Bertram *et al* (Theorem 5.11 of [3]) by showing that the top intersection number is r^g. Our Theorem 1.5 generalizes this to maps from nodal curves to projective spaces and shows that the top intersection number has the same value.

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