Proc. Indian Acad. Sci. (Math. Sci.), Vol. 107, No. 3, August 1997, pp. 283–317.
© Printed in India

Failure of Palais–Smale condition and blow-up analysis for the critical exponent problem in $\mathbb{R}^2$

ADIMURTHI and S PRASHANTH

T.I.F.R. Centre, P.O. Box No. 1234, Bangalore 560012, India
e-mail: aditi@ns.tifrbng.res.in
pras@ns.tifrbng.res.in

MS received 17 July 1996; revised 3 February 1997

Abstract. Let Ω be a bounded smooth domain in $\mathbb{R}^2$. Let $f:\mathbb{R}\to\mathbb{R}$ be a smooth non-linearity behaving like $\exp\{s^2\}$ as $s\to\infty$. Let F denote the primitive of f. Consider the functional $J:H_0^1(\Omega)\to\mathbb{R}$ given by

$$J(u)=\frac{1}{2}\int_\Omega|\nabla u|^2\,dx-\int_\Omega F(u)\,dx.$$

It can be shown that J is the energy functional associated to the following nonlinear problem:

$$-\Delta u=f(u)\quad\text{in}\quad\Omega,$$
$$u=0\qquad\text{on}\quad\partial\Omega.$$

In this paper we consider the global compactness properties of J. We prove that J fails to satisfy the Palais–Smale condition at the energy levels $\{k/2\}$, k any positive integer. More interestingly, we show that J fails to satisfy the Palais–Smale condition at these energy levels along two Palais–Smale sequences. These two sequences exhibit different blow-up behaviours. This is in sharp contrast to the situation in higher dimensions where there is essentially one Palais–Smale sequence for the corresponding energy functional.

Keywords. Blow-up analysis; critical exponent problem in $\mathbb{R}^2$; Moser functions; Palais–Smale sequence; Palais–Smale condition.

1. Introduction

1.1 *Preliminaries*

Let Ω be a smooth bounded domain in $\mathbb{R}^n$, $n\geqslant 2$. For $n\geqslant 3$, let $\mathscr{A}_n$ denote the subset of $C^1(\bar{\mathbb{R}}_+,\bar{\mathbb{R}}_+)$ consisting of functions $g(s)$ which satisfy the following growth conditions:

$$\lim_{s\to\infty} g(s)\,s^{((n+2)/(n-2))}=\infty,$$

$$\lim_{s\to\infty} g(s)\,s^{-((n+2)/(n-2))}=0.$$

When $n=2$, let $\mathscr{B}$ denote the subset of $C^1(\bar{\mathbb{R}}_+,\bar{\mathbb{R}}_+)$ consisting of functions $h(s)$ which vanish only at $s=0$ and which satisfy the following growth conditions:

For every $\delta>0$,

$$\lim_{s\to\infty} h(s)\exp\{\delta s^2\}=\infty,$$

$$\lim_{s\to\infty} h(s)\exp\{-\delta s^2\}=0.$$