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Domains in complex surfaces with non-compact automorphism groups ☆

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Abstract

Let X be an arbitrary complex surface and D a domain in X that has a non-compact group of holomorphic automorphisms. A characterization of those domains D that admit a smooth, weakly pseudoconvex, finite type boundary orbit accumulation point is obtained.

Keywords: Non-compact automorphism group; Complex surface; Weakly pseudoconvex; Orbit accumulation point

1. Introduction

Let D be a bounded domain in $\mathbf{C}^n$, $n \geqslant 1$ and let $\mathrm{Aut}(D)$ denote the group of holomorphic automorphisms of D. Suppose that the orbit of some point in D under the natural action of $\mathrm{Aut}(D)$ accumulates at $\zeta_0 \in \partial D$. In this situation, it has been demonstrated (see [3–5,8,14,22,24,27]) that the nature of ∂D near ζ_0 provides global information about D. The point ζ_0 is referred to as an orbit accumulation point. The question of investigating this phenomenon when D is a domain in a complex manifold was raised in [12] and [18] and it was shown in the latter article that the Wong–Rosay theorem does remain valid when D is a domain in an arbitrary complex manifold with $\zeta_0 \in \partial D$ a strongly pseudoconvex orbit accumulation point. Indeed, it was shown that D is biholomorphically equivalent to $\mathbf{B}^n$, the unit ball in $\mathbf{C}^n$. The purpose of this article is to study this phenomenon for domains $D \subset X$, X an arbitrary complex surface and $\zeta_0 \in \partial D$ is a smooth (henceforth meaning C^∞), weakly pseudoconvex, finite type, orbit accumulation point. It must be mentioned that [3] and [8] have already studied the case when $D \subset \mathbf{C}^2$ and a complete list of all possible model domains can be found there.

The boundary of D is smooth, weakly pseudoconvex and of finite type near ζ_0 if there is a coordinate chart (U_α, ϕ_α) such that $\phi_\alpha(\zeta_0) = 0 \in \mathbf{C}^2$ and the smooth real hypersurface $\phi_\alpha(U_\alpha \cap \partial D)$ near the origin is given by a smooth defining function of the form

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$$\rho_\alpha(z_1, z_2) = 2\Re z_2 + H_\alpha(z_1, \bar{z}_1) + O\big(|z_1|^{2m}, \Im z_2\big),$$

where $H_\alpha(z_1, \bar{z}_1)$ is a homogeneous, subharmonic polynomial of degree $2m$ (the 1-type of ζ_0 being $2m$ for some positive integer m) and without harmonic terms. If (U_β, ϕ_β) is another chart centered at ζ_0 with the property that the smooth real hypersurface $\phi_\beta(U_\beta \cap \partial D)$ near the origin has the form

$$\rho_\beta(z_1, z_2) = 2\Re z_2 + H_\beta(z_1, \bar{z}_1) + O\big(|z_1|^{2m}, \Im z_2\big)$$

then the mapping $\phi_\beta \circ \phi_\alpha^{-1}$ which is defined in a neighborhood of the origin and fixes it, maps the real hypersurface $\{\rho_\alpha = 0\}$ biholomorphically onto $\{\rho_\beta = 0\}$. Thus there exists a smooth, non-vanishing function r near the origin such that

$$\rho_\beta \circ \big(\phi_\beta \circ \phi_\alpha^{-1}\big) = r \cdot \rho_\alpha.$$

Comparing the terms of order $2m$ on both sides shows that $H_\beta(z_1, \bar{z}_1) = cH_\alpha(z_1, \bar{z}_1)$ for some $c \in \mathbf{C}^*$. Hence, modulo a transformation of the form $(z_1, z_2) \mapsto (c^{1/2m} z_1, z_2)$ it suffices to fix a chart (U, ϕ) centered at ζ_0 so that the smooth, real hypersurface $\phi(U \cap \partial D)$ is defined by

$$\rho(z) = 2\Re z_2 + H(z_1, \bar{z}_1) + O\big(|z_1|^{2m}, \Im z_2\big) \tag{1.1}$$

in the neighborhood $\phi(U) \subset \mathbf{C}^2$, where as before $H(z_1, \bar{z}_1)$ is a homogeneous, subharmonic polynomial of degree $2m$ without harmonic terms. Henceforth the general situation considered is the following: X is an arbitrary complex surface, $D \subset X$ a domain which is smooth, weakly pseudoconvex and of finite type near $\zeta_0 \in \partial D$, i.e., in a fixed local coordinate system the defining function is of the form (1.1). The neighborhood U is chosen small enough so that $U \cap D$ is connected. Also, there exists a point $p \in D$ and a sequence $\psi_j \in \mathrm{Aut}(D)$ (which is equipped with the topology of uniform convergence on compact subsets of D) such that $\psi_j(p) \to \zeta_0$.

Theorem 1.1. *The domain D is biholomorphically equivalent to a domain of the form $\Omega = \{(z_1, z_2) \in \mathbf{C}^2\colon 2\Re z_2 + H(z_1, \bar{z}_1) < 0\}$ where $H(z_1, \bar{z}_1)$ is as in* (1.1).

The strategy employed is that of [8]. After an initial scaling, D is shown to be equivalent to an algebraic domain of the form $\{(z_1, z_2) \in \mathbf{C}^2\colon 2\Re z_2 + P(z_1, \bar{z}_1) < 0\}$, where $P(z_1, \bar{z}_1)$ is a real valued subharmonic polynomial of degree at most $2m$ that does not have harmonic terms. The orbit $\{\psi_j(p)\}$ is then studied in the algebraic realization of D and three cases (cf. Section 3) arise depending on the nature of this orbit and this leads to the determination of $H(z_1, \bar{z}_1)$ in some cases. The arguments of [8] and [3] need to be suitably localized to ensure that we are in the affine situation and this is achieved by systematically using the attraction property of analytic discs. Moreover, in the parabolic case, the vanishing of the tangential vector field $\mathcal{X}$ (see Proposition 3.3) at the orbit accumulation point can be achieved without first proving the parabolicity of the action of g_t on D. This was also done in [5] assuming global boundary hypotheses. However, this also holds under local hypotheses as Proposition 3.3 shows. Thus the main theorem of [8] can be recovered completely even for domains in complex surfaces.

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2. Scaling the domain D

As a prelude to the initial scaling of D, the following observation will be needed. This was proved in [17] (see also [8]) for domains in $\mathbf{C}^n$ and then extended to the case of strongly pseudoconvex domains in complex manifolds in [18]. The infinitesimal Kobayashi metric will be denoted by $F_D^K(p, v)$ for $p \in D$ and $v \in T_p D$, while the associated Kobayashi distance between $p, q \in D$ is $d_D^K(p, q)$. The Kobayashi ball centered at $p \in D$ of radius R is denoted by $B_D^K(p, R)$. Finally X is equipped with a length function arising from a fixed hermitian metric on TX.

Proposition 2.1. *For every open neighborhood V of $\zeta_0 \in \partial D$ in X and every relatively compact subset K in D, there exists $j_0 > 0$ such that $\psi_j(K) \subset V \cap D$ for $j \geqslant j_0$. Moreover D is complete Kobayashi hyperbolic.*

Proof. Since ∂D is smooth, weakly pseudoconvex and of finite type near ζ_0, there exists a local holomorphic peak function f at ζ_0 by[15]. Choose a neighborhood W of ζ_0 which is relatively compact in V so that $f(\zeta_0) = 1$ and

$1/2 < |f(z)| < 1$ on $(W \cap \overline{D}) \setminus \{\zeta_0\}$. Then $u(z) = \log|f(z)|$ defined on $W \cap \overline{D}$ is a negative plurisubharmonic function that vanishes exactly at ζ_0 and which is strictly negative at all other points of $W \cap \overline{D}$. Choose $\eta > 0$ small enough so that

$$\Omega_{3\eta} = \{z \in W \cap \overline{D}\colon\ u > -3\eta\}$$

is relatively compact in W. The function

$$\tilde{u} = \begin{cases} u & \text{if } z \in \Omega_\eta \cap \overline{D}, \\ -\eta & \text{if } z \in \overline{D} \setminus \Omega_\eta \end{cases}$$

is then a negative plurisubharmonic function on D that peaks at ζ_0. Let $g : \bar{\Delta}(0,1) \to D$ be an analytic disc. The function $\tilde{u} \circ g$ is subharmonic and the sub-mean value property shows that for every negative α such that $-\alpha < \eta/2$ and $\tilde{u} \circ g(0) > \alpha$, the Lebesgue measure of the set $E_\alpha = \{\theta \in [0, 2\pi]\colon\ \tilde{u} \circ g(e^{\imath\theta}) \geqslant 2\alpha\}$, denoted by $|E_\alpha|$ satisfies

$$|E_\alpha| \geqslant \pi. \tag{2.1}$$

Now choose $\epsilon > 0$ small enough so that

$$\inf_{z \in \partial\Omega_\eta \cap \overline{D}} \{\tilde{u}(z) + \epsilon \log|\phi(z)|\} = -c_1 > -c_2 = \sup_{z \in \partial\Omega_{2\eta} \cap \overline{D}} \{\tilde{u}(z) + \epsilon \log|\phi(z)|\}.$$

The function $\rho(z)$ defined on $\overline{D}$ by

$$\rho(z) = \begin{cases} \tilde{u}(z) + \epsilon \log|\phi(z)| & \text{if } z \in \Omega_\eta \cap \overline{D}, \\ \max\{\tilde{u}(z) + \epsilon \log|\phi(z)|, -(c_1 + c_2)/2\} & \text{if } z \in (\Omega_{2\eta} \setminus \Omega_\eta) \cap \overline{D}, \\ -(c_1 + c_2)/2 & \text{if } z \in \overline{D} \setminus \Omega_{2\eta} \end{cases}$$

is then a negative plurisubharmonic function with $\rho^{-1}(-\infty) = \{\zeta_0\}$. By the Poisson integral formula it follows that for all $0 < r < 1$ there exists $C = C(r) > 0$ such that

$$(\rho \circ g)(\zeta) \leqslant C \int_0^{2\pi} (\rho \circ g)(e^{\imath\theta})\, d\theta \tag{2.2}$$

for all $\zeta \in \Delta(0, r)$. Moreover, since $\tilde{u}$ is a plurisubharmonic peak function at ζ_0 and $\rho(\zeta_0) = -\infty$, there exists for each $L > 0$, an arbitrarily small negative constant α such that $\tilde{u}(z) \geqslant 2\alpha$ implies $\rho(z) < -L$.

Let $g : \bar{\Delta} \to D$ be an analytic disc with $\tilde{u} \circ g(0) > \alpha$. Then from (2.1), we have $|E_\alpha| \geqslant \pi$ and hence $\rho \circ g(e^{\imath\theta}) < -L$ for $\theta \in E_\alpha$. Using (2.2) we see that for $\zeta \in \Delta(0, r)$, where $0 < r < 1$ is fixed,

$$(\rho \circ g)(\zeta) \leqslant C \int_0^{2\pi} (\rho \circ g)(e^{\imath\theta})\, d\theta = C\left(\int_{E_\alpha} (-L)\, d\theta + \int_{[0,2\pi]\setminus E_\alpha} (\rho \circ g)(e^{\imath\theta})\, d\theta\right).$$

The second term above is negative since $\rho \leqslant 0$ and hence

$$(\rho \circ g)(\zeta) \leqslant -LC\pi \tag{2.3}$$

for $\zeta \in \Delta(0, r)$. Now the family $G_n = \{z \in \overline{D}\colon\ \rho(z) < -C\pi n\}$, $n \geqslant 1$ is a neighborhood basis of $\zeta_0 \in \overline{D}$ since $\rho^{-1}(-\infty) = \zeta_0$. For each $n \geqslant 1$, choose α_n such that $\tilde{u}(z) \geqslant 2\alpha_n$ implies $\rho(z) < -n$. Let $G'_n = \{z \in \overline{D}\colon\ \tilde{u}(z) > \alpha_n\}$. Then from (2.3) it follows that for each n and each fixed $r \in (0, 1)$

$$g(0) \in G'_n \quad \Rightarrow \quad g(\Delta(0, r)) \subset G_n. \tag{2.4}$$

Henceforth we will work with a fixed large n chosen so that G'_n, G_n are both relatively compact in W. Also shrink G'_n to make it relatively compact in G_n. Dropping the subscripts for brevity, (2.4) says that if the center of an analytic disc is close to the peak point, then an a priori fixed portion of the disc cannot wander too far away from it. This implies that

$$F_D^K(z, v) \geqslant r F_{G \cap D}^K(z, v) \tag{2.5}$$

for $z \in G'$ and $v \in T_z D$.

Choose $z' \in \partial G \cap D$ and let $\gamma : [0,1] \to D$ be a differentiable path with $\gamma(0) = z'$ and $\gamma(1) = z$. Let t_0 be such that $\gamma(t_0) \in \partial G' \cap D$ and $\gamma((t_0, 1]) \subset G' \cap D$. Then

$$\int_0^1 F_D^K\big(\gamma(t), d\gamma(t)(1)\big)\, dt \geqslant r \int_{t_0}^1 F_{G\cap D}^K\big(\gamma(t), d\gamma(t)(1)\big)\, dt.$$

Since $f : W \to \Delta(0,1)$ peaks at ζ_0 and $\sup |f| = c < 1$ on $\partial G' \cap D$, it follows that

$$r \int_{t_0}^1 F_{G\cap D}^K\big(\gamma(t), d\gamma(t)(1)\big)\, dt \geqslant r \int_{t_0}^1 F_{\Delta(0,1)}^K\big(f \circ \gamma(t), (f \circ \gamma)'(t)\big)\, dt \geqslant d_{\Delta(0,1)}^K\big(f \circ \gamma(t_0), f(z)\big) \to \infty$$

as $z \to \zeta_0$. It follows that for all $z' \in \partial G \cap D$

$$d_D^K(z, z') \to \infty \tag{2.6}$$

as $z \to \zeta_0$. The first claim made in the proposition follows. Indeed, let K be a compact, connected sub-domain of D that contains z_0 and V a fixed but arbitrarily small neighborhood of $\zeta_0 \in \partial D$ in X. Suppose that $p_{j_n} \in \psi_{j_n}(K) \cap \partial V$ for some subsequence $j_n \to \infty$. Then

$$d_D^K\big(z_0, \psi_{j_n}^{-1}(p_{j_n})\big) = d_D^K\big(\psi_{j_n}(z_0), p_{j_n}\big)$$

and since $\psi_{j_n}(z_0) \to \zeta_0$, (2.6) shows that the right side above becomes arbitrarily large. On the other hand the left side is uniformly bounded since z_0 and $\psi_{j_n}^{-1}(p_{j_n}) \in K$ for all j_n. This is a contradiction. Thus $\psi_j(K) \subset V \cap D$ for all large j. This also shows that $V \cap D$ is hyperbolic. Indeed, suppose that $F_D^K(p, v) = 0$ for some $p \in V$ and a non-zero $v \in T_p D$. The upper semi-continuity of F_D^K shows that there exists $p_j \to p$, and $v_j \in T_{p_j} D$ converging to v such that $F_D^K(p_j, v_j) < 1/j$ for all $j \geqslant 1$. Consequently there are holomorphic mappings $g_j : \Delta(0, r_j) \to D$, $r_j \to \infty$ with $g_j(0) = p_j$ and $|dg_j(0)| = |v_j| \to |v| > 0$. By Brody's theorem there exists a non-constant mapping $g : \mathbf{C} \to D$ with $g(0) = p$. But then $\tilde{u} \circ g$ is a non-constant, negative subharmonic function on $\mathbf{C}$ which is a contradiction. Thus $V \cap D$ is hyperbolic and since each compact subset of D is mapped into $V \cap D$ by some ψ_j, it follows that D is hyperbolic as well.

To establish completeness, choose the neighborhood V of ζ_0 small enough so that each $\zeta \in V \cap \partial D$ has a holomorphic function f_ζ defined in $V \cap D$ that peaks at ζ. Suppose that $B_D^K(p, R)$ for some fixed $p \in D$ and $R > 0$ has points $p_j \to \partial D$. Then $\psi_J(B_D^K(p, R)) \subset V \cap D$ for some large fixed J. Also note that $\{\psi_J(p_j)\}$ can cluster only at $V \cap \partial D$, say at $\tilde{\zeta} \in V \cap \partial D$. It follows that

$$d_D^K(p, p_j) = d_D^K\big(\psi_J(p), \psi_J(p_j)\big) \geqslant d_{\Delta(0,1)}^K\big((f_{\tilde{\zeta}} \circ \psi_J)(p), (f_{\tilde{\zeta}} \circ \psi_J)(p_j)\big)$$

which is a contradiction since $(f_{\tilde{\zeta}} \circ \psi_J)(p_j) \to 1$ and thus the right side above becomes arbitrarily large while $d_D^K(p, p_j) < R$ for all j. $\square$

Let $p \in D$ be an arbitrary point. Then $\psi_j(p) \to \zeta_0$. It will be useful to describe the scaling of D and the corresponding model domain in terms of the given point p and the sequence $\{\psi_j(p)\}$. Recall the fixed coordinate system (U, ϕ) centered at ζ_0 as in (1.1). Then $\phi \circ \psi_j(p) \subset \phi(U)$ for large j. Choose $\zeta_j \in \phi(\partial D \cap U)$ defined by

$$\zeta_j = \phi \circ \psi_j(p) + (0, \epsilon_j), \quad \epsilon_j > 0.$$

From [11] it follows by shrinking U if necessary that there is a sequence of biholomorphisms h_j defined on $\phi(U)$ such that $h_j(\zeta_j) = (0,0)$, $h_j(\phi \circ \psi_j(p)) = (0, -\epsilon_j)$, and the defining equation for $h_j(\phi(\partial D \cap U))$ around the origin is

$$2\Re z_2 + \sum_{l=2}^{2m} P_{l,j}(z_1, \bar{z}_1) + R_j(\Im z_2, z_1) = 0. \tag{2.7}$$

Here $P_{l,j}(z_1, \bar z_1)$ are real valued homogeneous polynomials of degree l without harmonic terms. Also, $P_{l,j}(z_1, \bar z_1) \to 0$ and $P_{2m,j}(z_1, \bar z_1) \to H(z_1, \bar z_1)$ as $j \to \infty$. Let $\|\cdot\|$ be a fixed norm on the finite dimensional space of all real valued polynomials on the complex plane with degree at most $2m$ that do not contain harmonic terms. Define $\tau_j > 0$ by

$$\left\| \sum_{l=2}^{2m} \epsilon_j^{-1} P_{l,j}(\tau_j z_1, \tau_j \bar z_1) \right\| = 1. \tag{2.8}$$

Since $P_{2m,j} \to H$ which is a non-zero polynomial, it follows that $\sup_j \{\epsilon_j^{-1} \tau_j^{2m}\} < \infty$. Let $T_j(z_1, z_2)$ be the sequence of dilations defined by $(z_1, z_2) \mapsto (z_1/\tau_j, z_2/\epsilon_j)$. It follows from (2.7) and (2.8) that the domains $T_j \circ h_j(\phi(U \cap D))$ converge to

$$\Omega_p = \left\{(z_1, z_2) \in \mathbf{C}^2\colon\ 2\Re z_2 + P_p(z_1, \bar z_1) < 0\right\}$$

where P_p is a real valued subharmonic polynomial of degree at most $2m$ without harmonic terms and $\|P_p\| = 1$.

Proposition 2.2. *The domain D is biholomorphic to Ω_p. Moreover, there exist points p arbitrarily close to ζ_0 for which the polynomial P_p that occurs in the defining equation for Ω_p is of the form*

$$P_p(z_1, \bar z_1) = H(z_1, \bar z_1) + R(z_1, \bar z_1)$$

where $H(z_1, \bar z_1)$ is as in (1.1) *and the degree of $R(z_1, \bar z_1)$ is strictly less than $2m$. In particular, for these choices of p, the degree of P_p is $2m$.*

Proof. Let K be a compact sub-domain of D that contains p. Then $\psi_j(K) \subset U \cap D$ for j large and thus the holomorphic mappings $\phi_\circ \psi_j$ are well defined on K. Note that $(T_j \circ h_j \circ \phi \circ \psi_j)(p) = (0, -1)$ for all large j. By [9] it follows that $\{T_j \circ h_j \circ \phi \circ \psi_j\}$ is a normal family on K and by exhausting D with an increasing family of such K, we get a holomorphic mapping $\Phi_p : D \to \bar\Omega_p$ with $\Phi_p(p) = (0, -1)$. The maximum principle shows that $\Phi_p : D \to \Omega_p$. To show that Φ_p is biholomorphic onto Ω_p, let $\tilde K$ be relatively compact in Ω_p with $(0, -1) \in \tilde K$. Since $T_j \circ h_j(\phi(U \cap D)) \to \Omega_p$ it follows that $\tilde K$ is relatively compact in $T_j \circ h_j(\phi(U \cap D))$ for all j large and hence the holomorphic mappings

$$(T_j \circ h_j \circ \phi \circ \psi_j)^{-1} : \tilde K \to D$$

are well defined and map $(0, -1)$ to $p \in D$. Since D is complete hyperbolic, some subsequence converges to a well defined holomorphic mapping that maps $\tilde K$ into D. Exhausting Ω_p by such $\tilde K$, we get a holomorphic mapping $\Psi_p : \Omega_p \to D$. It follows that $\Phi_p \circ \Psi_p = \Psi_p \circ \Phi_p = \mathrm{id}$.

The points $p_j = (0, -1/j)$, $j \geqslant 1$ are in $\phi(U \cap D)$ and lie on the inward real normal to $\phi(U \cap \partial D)$ at the origin. Let $\Phi_j : D \to \Omega_j$ be biholomorphic mappings such that $\Phi_j(\phi^{-1}(p_j)) = (0, -1)$ where $\Omega_j = \{(z_1, z_2) \in \mathbf{C}^2\colon\ 2\Re z_2 + P_j(z_1, \bar z_1) < 0\}$ is a model domain corresponding to the point $\phi(p_j)$. Compose Φ_j with a mapping of the form $(z_1, z_2) \mapsto (\lambda_j z_1, z_2)$, $\lambda_j > 0$, if the need be, to ensure that $\|P_j\| = 1$ for all j. Then P_j converges to a real valued subharmonic, polynomial of degree at most $2m$ without harmonic terms. Call this limit P_∞ and note that $\|P_\infty\| = 1$. Thus $\Omega_j \to \Omega_\infty = \{(z_1, z_2) \in \mathbf{C}^2\colon\ 2\Re z_2 + P_\infty(z_1, \bar z_1) < 0\}$. Consider the dilations $\Lambda_j(z_1, z_2) = (j^{1/2m} z_1, j z_2)$ and the domains $G_j = \Lambda(\phi(U \cap D))$ and $D_j = \Phi_j(U \cap D) \subset \Omega_j$. Thus we have a sequence of biholomorphisms

$$\tau_j = \Phi_j \circ \phi^{-1} \circ \Lambda_j^{-1} : G_j \to D_j$$

such that $\tau_j(0, -1) = (0, -1)$. It is evident that $G_j \to G_\infty = \{(z_1, z_2) \in \mathbf{C}^2\colon\ 2\Re z_2 + H(z_1, \bar z_1) < 0\}$. On the other hand $D_j \to \Omega_\infty$. Indeed, if K is a relatively compact in Ω_∞ and contains $(0, -1)$, then K is eventually relatively compact in Ω_j and the proof of Proposition 2.1 shows that $\Phi_j^{-1} \to \zeta_0$ uniformly on K. This means that $\Phi_j^{-1}(K) \subset U \cap D$ and hence that K is relatively compact in D_j for all j large. Conversely, when $K \subset \mathbf{C}^2 \setminus \Omega_\infty$, then $K \subset \mathbf{C}^2 \setminus \Omega_j \subset \mathbf{C}^2 \setminus D_j$ for j large. The convergence of τ_j follows from Lemma 2.3 of [8] while that of τ_j^{-1} from [9]. Applying the maximum principle, it follows from [19] that there exists a biholomorphism $\tau_\infty : G_\infty \to \Omega_\infty$ with $\tau_\infty(0, -1) = (0, -1)$.

To conclude from this that the degree of $P_\infty = 2m$, it suffices to show that there are points on ∂G_∞, excluding the point at infinity, whose cluster sets under τ_∞ contain at least one finite point on $\partial \Omega_\infty$. This can be done using the holomorphic peak function at infinity for Ω_∞ as in Lemma 3.2 of [8] or using the argument of Lemma 3.1 in [13] (which does not require the existence of holomorphic peak functions at infinity). Thus the degree of P_j equals the degree of $P_\infty = 2m$ for all large j. Finally note that

$$P_\infty = \lim_{j\to\infty} \sum_{l=2}^{2m} P_{l,j}(z_1, \bar{z}_1)\epsilon_j^{-1}\tau_j^l$$

where $P_{2m,j}(z_1, \bar{z}_1) \to H(z_1, \bar{z}_1)$ and $P_{l,j} \to 0$ for $l < 2m$. But since the degree of P_∞ is $2m$, it follows that $\tau_j^{2m} \approx \epsilon_j$ and hence the homogeneous part of P_∞ of degree $2m$ is a scalar multiple of $H(z_1, \bar{z}_1)$. This multiple can be absorbed in $H(z_1, \bar{z}_1)$ by a global change of coordinates. □

3. Model domains for D

It will now be convenient to fix $p \in D$ sufficiently close to ζ_0 so that Ω_p and P_p are as in the previous proposition. Having done so we drop the subscripts on Ω_p, P_p and Φ_p for brevity and simply write Ω, P and Φ. The symbols P_α, $P_{\alpha,\bar{\beta}}$ denote the derivatives $\partial^\alpha P/\partial z_1^\alpha$, $\partial^{\alpha+\beta} P/\partial z_1^\alpha \bar{\partial} z_1^\beta$ respectively. The sequence $\psi_j(p) \to \zeta_0$ and there are three possibilities for the image sequence $\Phi \circ \psi_j(p) \in \Omega$. To describe them let $\Phi \circ \psi_j(p) = (a_j, b_j)$ and define $\eta_j = 2\Re b_j + P(a_j, \bar{a}_j)$. Then exactly one of the following holds:

(i) $|a_j|$ is bounded after passing to a subsequence and $|\eta_j| \to 0$, or
(ii) $|a_j| \to \infty$ and $|\eta_j| \to 0$, or
(iii) $|\eta_j| > c > 0$ for some uniform c, after possibly passing to a subsequence.

Proposition 3.1. *In case* (i)*, there exists* $a \in \mathbf{C}$ *such that*

$$P(z_1, \bar{z}_1) = H(z_1 - a) + 2\Re\left(\sum_{\alpha=0}^{2m} \frac{P_\alpha(a)}{\alpha!}(z_1 - a)^\alpha\right)$$

and D is biholomorphic to $\{(z_1, z_2) \in \mathbf{C}^2\colon\ 2\Re z_2 + H(z_1, \bar{z}_1) < 0\}$.

In case (ii), $P(z_1, \bar{z}_1) = H(z_1, \bar{z}_1) = \lambda((2\Re e^{\imath\theta} z)^{2m} - 2\Re(e^{\imath\theta} z)^{2m})$ *for some* $\lambda > 0$ *and* $0 \leqslant \theta < 2\pi$*. In particular, D is biholomorphic to* $\{(z_1, z_2) \in \mathbf{C}^2\colon\ 2\Re z_2 + H(z_1, \bar{z}_1) < 0\}$.

Proof. Define a sequence of subharmonic polynomials of degree at most $2m$ by

$$P^j(z_1, \bar{z}_1) = \sum_{\alpha,\beta>0} \frac{P_{\alpha,\bar{\beta}}(a_j)}{(\alpha+\beta)!}\eta_j^{-1}\tau_j^{\alpha+\beta} z_1^\alpha \bar{z}_1^\beta$$

where $\tau_j > 0$ is chosen so that $\|P^j\| = 1$ for all j. Passing to a subsequence if necessary, $P^j \to P^\infty$, which is a real valued subharmonic polynomial of degree at most $2m$ and without harmonic terms. The mapping $\sigma^j \in \mathrm{Aut}(\mathbf{C}^2)$ whose components are

$$\sigma_1^j(z) = \tau_j^{-1}(z_1 - a_j), \qquad \sigma_2^j(z) = \eta_j^{-1}\left(z_2 - b_j - \eta_j + 2\sum_{\alpha=1}^{2m} \frac{P_\alpha(a_j)}{\alpha!}(z_1 - a_j)^\alpha\right)$$

maps Ω biholomorphically onto $\Omega^j = \{(z_1, z_2) \in \mathbf{C}^2\colon\ 2\Re z_2 + P^j(z_1, \bar{z}_1) < 0\}$ and $\sigma^j(a_j, b_j) = (0, -1) \in \Omega^j$. It is evident that $\Omega^j \to \Omega^\infty = \{(z_1, z_2) \in \mathbf{C}^2\colon\ 2\Re z_2 + P^\infty(z_1, \bar{z}_1) < 0\}$. The family of mappings $\sigma^j \circ \Phi \circ \psi_j : D \to \Omega^j$ and map p to $(0, -1)$. By Lemma 2.3 of [8] this is a normal family and thus $\sigma^j \circ \Phi \circ \psi_j \to \sigma$ uniformly on compact subsets of D and $\sigma : D \to \Omega^\infty$ by the maximum principle. On the other hand, the inverses $(\sigma^j \circ \Phi \circ \psi_j)^{-1}$ are well defined on any compact $K \subset \Omega^\infty$ since $\Omega^j \to \Omega^\infty$. As the inverses map each such K into D which is complete hyperbolic and $(\sigma^j \circ \Phi \circ \psi_j)^{-1}((0, -1)) = p \in D$, there is a convergent subsequence on K. Hence there is a holomorphic mapping $\omega : \Omega^\infty \to D$ and we note that σ and ω are inverses. This shows that D is biholomorphic to Ω^∞ and the proof of Proposition 2.2 now shows that if p was chosen sufficiently close to $\zeta_0 \in \partial D$, the degree of P^∞ equals $2m$. But the definition of $P^j(z_1, \bar{z}_1)$ shows that this happens precisely when $\tau_j^{2m} \approx \eta_j$ and $P_{\alpha,\bar{\beta}}(a_j) \to 0$

as $j \to \infty$ for all $\alpha, \beta > 0$ and $\alpha + \beta < 2m$. In case (i), $(a_j, b_j) \to (a, b) \in \partial\Omega$ and we may pass to the limit to get $P_{\alpha,\bar{\beta}}(a) = 0$ for all $\alpha, \beta > 0$ and $\alpha + \beta < 2m$. The description of $P(z_1, \bar{z}_1)$ as claimed follows since the highest degree summand is $H(z_1, \bar{z}_1)$. It remains to see that Ω is equivalent to the domain $\{(z_1, \zeta_2) \in \mathbf{C}^2\colon\ 2\Re z_2 + H(z_1, \bar{z}_1) < 0\}$ by the mapping

$$(z_1, z_2) \mapsto \left(z_1 - a, z_2 + \sum_{\alpha=0}^{2m} \frac{P_\alpha(a)}{\alpha!}(z_1 - a)^\alpha\right).$$

In case (ii), we have that $|a_j| \to \infty$ and the arguments used in Lemma 4.2 in [8] can be applied without any changes as they do not involve any considerations on D, and it follows that $P(z_1, \bar{z}_1) = H(z_1, \bar{z}_1)$ must be of the form as claimed. □

It remains to analyze case (iii). In this case note that (a_j, b_j) converge to the point at infinity in $\partial\Omega$. The defining function for $\partial\Omega$ shows that $L_t(z_1, z_2) = (z_1, z_2 + \imath t)$ for $t \in \mathbf{R}$ is a one-parameter group of automorphisms of Ω. The corresponding holomorphic vector field is

$$\left.\frac{dL_t(z)}{dt}\right|_{t=0} = (0, \imath) = \imath \frac{\partial}{\partial z_2} = \frac{\imath}{2}\frac{\partial}{\partial x_2} + \frac{1}{2}\frac{\partial}{\partial y_2} \tag{3.1}$$

where as usual $z_j = x_j + \imath y_j$ for $j = 1, 2$. The pull-back $\mathcal{X} = \Phi^*(\imath\partial/\partial z_2)$ is a holomorphic vector field on D and its real part $\Re\mathcal{X}$ generates the one-parameter group $g_t = \Phi^{-1} \circ L_t \circ \Phi = \exp(t\Re\mathcal{X}) \in \mathrm{Aut}(D)$.

Proposition 3.2. *The group (g_t) induces a local one-parameter group of smooth diffeomorphisms on ∂D in a neighborhood of ζ_0. In particular $\mathcal{X}$ extends smoothly up to ∂D near ζ_0.*

Proof. The coordinate neighborhood U centered at ζ_0 is assumed to be small enough to ensure the following: first, for all $\zeta \in U \cap \partial D$, there exists (see [15]) a plurisubharmonic function u_ζ in $U \cap D$ that extends continuously up to $U \cap \overline{D}$ and satisfies

$$-M|z - \zeta| \leqslant u_\zeta(z) \leqslant -|z - \zeta|^{2m} \tag{3.2}$$

for all $z \in U \cap \overline{D}$, the constant $M > 0$ being uniform for all $\zeta \in U \cap \partial D$. Second, for all $z \in U \cap D$, the nearest point to z on ∂D, denoted by $\pi(z)$ is well defined and $\pi(z) \in U$. Here z is a local coordinate near the origin and all calculations in the ensuing proof will be done in these coordinates without explicitly mentioning the mapping ϕ. Extend u_ζ to D (see [26]) as a negative, continuous, plurisubharmonic function by composing it with a suitable smooth, convex function and then comparing it with a constant.

Then there exists $\delta > 0$ and an arbitrarily small neighborhood V of ζ_0 so that if for any $t \in (-\delta, \delta)$ and $z \in V \cap D$ it happens that $g_t(z) \in V \cap D$, then

$$\mathrm{dist}\big(g_t(z), \partial D\big) \leqslant C\, \mathrm{dist}(z, \partial D)$$

with $C > 0$ being uniform for all $t \in (-\delta, \delta)$. Indeed, let V be any small neighborhood of ζ_0 and fix a compact $K \cap V \cap D$ with non-empty interior. Since D is complete hyperbolic, the self map $g \mapsto g^{-1}$ of $\mathrm{Aut}(D)$ is continuous and it follows that there exists $\delta > 0$ such that

$$\tilde{K} = \bigcup_{t \in (-\delta,\delta)} g_{-t}(K)$$

is relatively compact in $V \cap D$. For each $t \in (-\delta, \delta)$ and $z \in V \cap D$, the function

$$v_{z,t}(\tau) = u_{\pi(z)} \circ g_{-t}(\tau)$$

is a negative, continuous plurisubharmonic function of $\tau \in D$ and satisfies

$$\begin{aligned}
\max\big\{v_{z,t}(\tau)\colon\ \tau \in K\big\} &= \max\big\{u_{\pi(z)}(\tau)\colon\ \tau \in g_{-t}(K)\big\} \\
&\leqslant -\min\big\{|\tau - \zeta|^{2m}\colon\ \tau \in g_{-t}(K),\ \zeta \in V \cap \partial D\big\} \\
&= \big(\mathrm{dist}\big(g_{-t}(K), V \cap \partial D\big)\big)^{2m} \leqslant -L < 0.
\end{aligned} \tag{3.3}$$

The constant $L > 0$ is independent of $t \in (-\delta, \delta)$ and $z \in V \cap D$ since $\tilde{K}$ is compact in $V \cap D$. By the version of the Hopf lemma proved in [23] (which carries over to domains in complex manifolds as well), it follows that

$$\left|v_{z,t}(\tau)\right| \geqslant C\,\mathrm{dist}(\tau, \partial D)$$

where $C > 0$ is uniform for all $t \in (-\delta, \delta)$ and $z \in V \cap D$. From (3.2) it follows that

$$C\,\mathrm{dist}\big(g_t(z), \partial D\big) \leqslant \left|v_{z,t}\big(g_t(z)\big)\right| = \left|u_{\pi(z)}(z)\right| \leqslant M\,\mathrm{dist}(z, \partial D) \tag{3.4}$$

as was claimed.

This implies that for every sufficiently small neighborhood V_2 of ζ_0, there exists another neighborhood V_1 of ζ_0 with $V_1 \subset V_2$ and $\delta > 0$ such that $g_t(V_1) \subset V_2$ for all $t \in (-\delta, \delta)$. Indeed, if not, then for a given V_2 there exists $\delta_j \to 0$ and a sequence of neighborhoods W_j shrinking to ζ_0 and points $p_j, q_j \in W_j$ such that $g_{\delta_j}(p_j) \in V_2$ but $g_{\delta_j}(q_j) \in \partial V_2 \cap D$. Using the boundary behavior of F_D^K near ζ_0 that follows from (2.4), (2.5) and (3.2) and its invariance under g_t for all t, we get

$$C^{-1}\frac{|dg_t(z)v|}{\mathrm{dist}(g_t(z), \partial D)^{1/2m}} \leqslant F_D^K\big(g_t(z), dg_t(z)v\big) = F_D^K(z, v) \leqslant C\frac{|v|}{\mathrm{dist}(z, \partial D)}$$

for some uniform $C > 0$ and for all t such that both $z, g_t(z)$ are near ζ_0 and $v \in T_z D$. Using (3.4), this gives

$$\left|dg_t(z)v\right| \leqslant C|v|\big(\mathrm{dist}(z, \partial D)\big)^{-1+1/2m}.$$

Integrating this estimate along a polygonal path consisting of three line segments that joins p_j, q_j, we get

$$\left|g_{\delta_j}(p_j) - g_{\delta_j}(q_j)\right| \leqslant C|p_j - q_j|^{1/2m} \tag{3.5}$$

and this is a contradiction. Thus each g_t for $t \in (-\delta, \delta)$ induces a local homeomorphism on ∂D near ζ_0 and by [6] this extension is a smooth diffeomorphism. In other words, the action $g : \mathbf{R} \times D \to D$ where $g(t, z) = g_t(z)$ extends smoothly to $\overline{D}$ near ζ_0 for each fixed $t \in (-\delta, \delta)$ for some $\delta > 0$. Fix a small neighborhood V of ζ_0 of diameter $\eta > 0$ so that all previous considerations hold on $V \cap \overline{D}$. The choice of η will be determined later. To prove that g is smooth on $(-\delta, \delta) \times (V \cap \overline{D})$, it suffices to establish its joint continuity there by the Bochner–Montgomery theorem (cf. Theorem 4 in [10]). For this let $(t_j, \zeta_j) \to (t, \zeta) \in (-\delta, \delta) \times (V \cap \overline{D})$ and fix $a \in V \cap D$. Then

$$\left|g_{t_j}(\zeta_j) - g_t(\zeta)\right| \leqslant \left|g_{t_j}(\zeta_j) - g_{t_j}(a)\right| + \left|g_{t_j}(a) - g_t(a)\right| + \left|g_t(a) - g_t(\zeta)\right|.$$

For a given $\epsilon > 0$, the second term is less than $\epsilon/3$ for $t_j \approx t$, while the first and third terms are dominated by a constant times $|a - \zeta|^{1/2m}$ which can be made less than $\epsilon/3$ for a suitable choice of η. □

Proposition 3.3. *The vector field $\mathcal{X}$ vanishes at ζ_0. Moreover the order of vanishing at ζ_0 is finite.*

Proof. Extend $\mathcal{X}$ to a neighborhood of ζ_0 in X (see [25] and also the references therein) as a smooth vector field. For $\tau \in \mathbf{C}$, note that $(a_j, b_j + |\eta_j|\tau) \subset \Omega$ provided that $\Re\tau < 1$. With $\mathcal{H} = \{\tau \in \mathbf{C}\colon \Re\tau < 1\}$, the analytic disc $f_j : \mathcal{H} \to D$ where

$$f_j(\tau) = \Phi^{-1}\big(a_j, b_j + |\eta_j|\tau\big)$$

is well defined. Note that $f_j(0) = \Phi^{-1}(a_j, b_j) = \psi_j(p) \to \zeta_0$. The integral curves of $\imath\partial/\partial z_2$ passing through (a_j, b_j) are of the form $\gamma_j(t) = (a_j, b_j + \imath t)$, $t \in \mathbf{R}$ and hence $\Phi^{-1} \circ \gamma_j(t)$ defines the integral curves for $\mathcal{X}$ through $\psi_j(p)$. For j fixed, as τ varies on the imaginary axis in $\mathcal{H}$, $(a_j, b_j + |\eta_j|\tau)$ sweeps out the integral curves of $\imath\partial/\partial z_2$ through (a_j, b_j). Moreover, since $|\eta_j| > c > 0$, the image of the interval $(-M, M)$, for any $M > 0$, under the map $t \mapsto (a_j, b_j + |\eta_j|\imath t)$ contains the line segment from $(a_j, b_j - \imath cM)$ to $(a_j, b_j + \imath cM)$ which equals $\gamma_j((-cM, cM))$ for all j.

The family $f_j : \mathcal{H} \to D$ is normal since D is complete hyperbolic and since $f_j(0) \to \zeta_0$, it follows that some subsequence of f_j converges uniformly on compact subsets of $\mathcal{H}$ to a holomorphic mapping $f : \mathcal{H} \to \partial D$ with $f(0) = \zeta_0$. However ζ_0 is a local plurisubharmonic peak point and this forces $f(\tau) \equiv \zeta_0$ for all $\tau \in \mathcal{H}$. For an arbitrarily small neighborhood V of ζ_0 and the compact interval $[-\imath M, \imath M] \in \mathcal{H}$, this means that

$$\Phi^{-1} \circ \gamma\big((-cM, cM)\big) \subset f_j\big([-\imath M, \imath M]\big) \subset V$$

for all large j which implies that the integral curve of $\mathcal{X}$ through $\psi_j(p)$ is contained in V for all $t \in (-cM, cM)$. Since f_j is normal, $\Phi^{-1} \circ \gamma_j$ converges uniformly on $[-cM, cM]$ to a path $\gamma_\infty : [-cM, cM] \to V \cap \overline{D}$ with $\gamma_\infty(0) = \zeta_0$. Evidently γ_∞ is the integral curve of $\mathcal{X}$ through ζ_0. Since V and M are arbitrary, it follows that $\mathcal{X}(\zeta_0) = 0$.

Since (g_t) induces a smooth local one-parameter group of diffeomorphisms on ∂D near ζ_0, it follows from the Hopf lemma in [1] (see also [20]) that g_t vanishes to finite order at ζ_0 and hence so does $\mathcal{X}$. □

Remark. In [3] and [8] the parabolicity of the action of (g_t) was used to deduce that $\mathcal{X}$ must vanish at ζ_0. A different argument was used in [5] to show that $\mathcal{X}$ vanishes at ζ_0 without knowing the parabolicity of (g_t). This however used the global real analyticity of the boundary of the domain. Proposition 3.3 is also in the same spirit and it shows that the vanishing of $\mathcal{X}$ can be deduced under local hypotheses as well.

Observe that g_t preserves ∂D near ζ_0 for all small t and this shows that $\mathcal{X}$ is tangent to ∂D, that is $\mathcal{X}(\zeta) \in T_\zeta(\partial D)$, for ζ near ζ_0. Working in local coordinates near ζ_0, it follows from Lemma 3.4 in [3] that $\mathcal{X} = \mathcal{X}_1 \partial/\partial z_1 + \mathcal{X}_2 \partial/\partial z_2$ where $\mathcal{X}_1, \mathcal{X}_2$ are holomorphic in D and extend smoothly up to $\overline{D}$ near ζ_0 and there are only two possibilities for $\mathcal{X}_1$ and $\mathcal{X}_2$. Assigning weights $1/2m$, 1 to z_1, z_2 respectively and $-1/2m$, -1 to $\partial/\partial z_1$, $\partial/\partial z_2$ respectively, the possibilities are:

$$\begin{aligned} &\mathcal{X}_1(z) = (\alpha + \imath\beta) z_1 + \cdots, \\ &\mathcal{X}_2(z) = 2m\alpha z_2 + \cdots, \end{aligned} \tag{3.6}$$

where $\alpha, \beta \in \mathbf{R}$ are both not zero and if $\beta \neq 0$, then $H(z_1, \bar{z}_1) = \lambda |z_1|^{2m}$ for some $\lambda > 0$.

$$\begin{aligned} &\mathcal{X}_1(z) = \imath\alpha z_1 z_2 + \cdots, \\ &\mathcal{X}_2(z) = \imath m\alpha z_2^2 + \cdots, \end{aligned} \tag{3.7}$$

where α is non-zero real and $H(z_1, \bar{z}_1) = \lambda |z_1|^{2m}$, $\lambda > 0$. The lower dots in the equations above indicate terms of higher weight.

Proposition 3.4. *The one parameter group* $(g_t) \in \operatorname{Aut}(D)$ *is parabolic in the sense that*

$$\lim_{t \to \infty} \Phi^{-1}(z_1, z_2 \pm \imath t) = \zeta_0$$

for all $(z_1, z_2) \in \Omega$.

Proof. According to [2] there exists $\tilde{h} \in \mathcal{O}(\Omega) \cap C^0(\bar{\Omega})$ that satisfies

$$\bigl|\tilde{h}(z_1, z_2)\bigr| \approx \bigl(|z_1|^{2m} + |z_2|^2\bigr)^{1/N}$$

for some integer N and $\arg \tilde{h}(z) \in [-\pi/4, \pi/4]$ when $|z|$ is very large. For $\alpha > 0$ small enough, the function $h(z) = (\alpha\tilde{h}(z) - 1)/(\alpha\tilde{h}(z) + 1)$ is then a holomorphic peak function for Ω at infinity. Let $B(0, A)$ be the euclidean ball of radius $A > 0$ around the origin and define the negative, plurisubharmonic function u on Ω as follows:

$$u = \begin{cases} \max(|h|^2 - 1, -a), & \Omega \setminus \bar{B}(0, A), \\ -a, & \text{elsewhere} \end{cases}$$

for suitable constants $A, a > 0$. Evidently $|u|$ grows no faster than a constant times $(|z_1|^{2m} + |z_2|^2)^{-1/N}$ near infinity. Now $u \circ \Phi \circ \phi^{-1}$ is a negative, plurisubharmonic function on $\phi(U \cap D) \subset \mathbf{C}^2$ and by the Hopf lemma, there exists a neighborhood V of the origin such that

$$\operatorname{dist}\bigl(\phi \circ \Phi^{-1}(z_1, z_2), \phi(U \cap \partial D)\bigr) \leqslant C\bigl(|z_1|^{2m} + |z_2|^2\bigr)^{-1/N}$$

when $|z|$ is large and $\phi \circ \Phi^{-1}(z_1, z_2) \in V$. Recall that $\phi \circ \Phi^{-1}(a_j, b_j) \in V$ for j large. Let $(a_j, z_2) \in \Omega$ be such that $\phi \circ \Phi^{-1}(a_j, z_2) \in V$. The analytic discs $f_j : \mathcal{H} \to D$ where

$$f_j(\tau) = \Phi^{-1}\bigl(a_j, z_2 + |c_j|\tau\bigr)$$

and $c_j = 2\Re z_2 + P(a_j, \bar{a}_j)$ are then well defined and satisfy $f_j(0) \in \phi^{-1}(V)$ for all j. It follows from (2.4) that

$$f_j\big(\Delta(0, 3/4)\big) \subset \phi^{-1}(V)$$

for all large j. Therefore the discs $\phi \circ f_j : \Delta(0, 3/4) \to V$ satisfy the attraction property (cf. [7]) of order $1/2m$ and this yields

$$\left|\frac{\partial}{\partial z_2}\big(\phi \circ \Phi^{-1}\big)(a_j, z_2)\right| \leqslant C|c_j|^{-1}\big(|a_j|^{2m} + |z_2|^2\big)^{-1/(2mN)}.$$

This can be integrated on paths of the form $t \mapsto (a_j, b_j \pm \imath t)$ as in [8] and it shows that for all large j, $\lim \Phi^{-1}(a_j, b_j \pm \imath t) = \zeta_0$ as $t \to \infty$. Now given any other $(z_1, z_2) \in \Omega$, note that $\Phi^{-1}(z_1, z_2 \pm \imath t) = g_t(\Phi^{-1}(z_1, z_2))$. Fix J large so that $\lim \Phi^{-1}(a_J, b_J \pm \imath t) = \zeta_0$. It follows from Proposition 2.1 that $g_t(K) \to \zeta_0$ for all compact $K \subset D$. Taking $K = \Phi^{-1}(z_1, z_2)$ gives the desired claim. $\square$

Since ζ_0 is a parabolic point, it follows that if $\mathcal{X}$ has the form (3.6), then $\alpha = 0$. Indeed, if not then depending on whether $\alpha > 0$ or < 0, ζ_0 will become a repelling or attracting fixed point for $\mathcal{X}$, which is a contradiction. Thus either (3.7) holds, or (3.6) holds and $\alpha = 0$. In both cases, $H(z_1, \bar{z}_1) = \lambda |z_1|^{2m}$. The constant λ can be absorbed in $H(z_1, \bar{z}_1)$ by a global change of variables. The defining function ρ for $\phi(U \cap \partial D)$ near the origin now has the form

$$\rho(z_1, z_2) = 2\Re z_2 + |z_1|^{2m} + \cdots.$$

By the change of variables $z_1 = \tilde{z}_1$, $z_2 = \tilde{z}_2 - \tilde{z}_2^2$, this becomes

$$\rho(\tilde{z}_1, \tilde{z}_2) = 2\Re\tilde{z}_2 + |\tilde{z}_1|^{2m} + \Im\tilde{z}_2^2 + \cdots \tag{3.8}$$

and thus in these coordinates the domain $\phi(U \cap D)$ is convex. This coordinate change will be absorbed in ϕ in the sequel. Using the fact that $\phi \circ \psi_j(p)$ converges to the origin and Proposition 2.1, it is possible to apply the methods of [16] and [22] to get that

$$\lim_{j\to\infty} \big(d(\phi \circ \psi_j)(p)\big)^{-1}\big(\phi \circ \psi_j(z)\big)$$

exists uniformly on compact subsets of D, the limit being a convex, hyperbolic domain biholomorphic to D and is of the form $\tilde{\Omega} = \{(z_1, z_2) \in \mathbf{C}^2\colon\ 2\Re z_2 + \tilde{P}(z_1, \bar{z}_1) < 0\}$ and $\tilde{P}(z_1, \bar{z}_1)$ is a positive, homogeneous polynomial of degree $2m$. Henceforth, for the sake of brevity we write Ω, $P(z_1, \bar{z}_1)$ instead of $\tilde{\Omega}$, $\tilde{P}(z_1, \bar{z}_1)$ respectively and $\Phi : D \to \Omega$ is the biholomorphic equivalence. The mapping $(z_1, z_2) \mapsto (z_1, 1/z_2)$ is well defined on Ω since $P(z_1, \bar{z}_1) \geqslant 0$ (which implies that $\Re z_2 < 0$ on Ω) and it transforms Ω into a domain of the form $\tilde{\Omega} = \{(z_1, z_2) \in \mathbf{C}^2\colon 2\Re z_2 + |z_2|^2 P(z_1, \bar{z}_1) < 0\}$ and let $\tilde{\Phi} : D \to \tilde{\Omega}$ be the biholomorphic equivalence.

Choose $(0, b) \in \Omega$ with $\Re b < 0$ and since $\tilde{\Phi}^{-1}(0, 1/(b \pm \imath t)) = \Phi^{-1}(0, b \pm \imath t) \to \zeta_0$ as $t \to \infty$, it follows from Propositions 4.1 and 4.2 of [3] that $\phi \circ \tilde{\Phi}^{-1} : \tilde{\Omega} \to \phi(U \cap D) \subset \mathbf{C}^2$ extends continuously to a neighborhood of the origin in $\partial\tilde{\Omega}$ and $\phi \circ \tilde{\Phi}^{-1}(0) = 0$. Furthermore $|(\phi \circ \tilde{\Phi}^{-1})_2(0, -\delta)| = O(\delta)$ where $(\phi \circ \tilde{\Phi}^{-1})_i$ for $i = 1, 2$ are the components of the mapping $\phi \circ \tilde{\Phi}^{-1}$. Now apply Julia's theorem to the function $z_2 \mapsto (\phi \circ \tilde{\Phi}^{-1})_2(0, z_2)$ which shows that for any $K > 0$, there exists $\epsilon > 0$ such that the image of the circle $(0, 1/(-\epsilon \pm \imath t))$ under $(\phi \circ \tilde{\Phi}^{-1})_2$ lies in the disc $x_2 + K(x_2^2 + y_2^2) < 0$. Observe that $z(t) = (\phi \circ \tilde{\Phi}^{-1})(0, 1/(-\epsilon \pm \imath t))$ is the image under ϕ of the orbit of $\Phi^{-1}(0, -\epsilon)$ under the action of (g_t), which stays near ζ_0 for all $|t|$. It follows from (3.8) that $|\Im z_2(t)|^2 \leqslant |\Re z_2(t)|/K$ and $|z_1(t)|^{2m} \leqslant C|\Re z_2(t)|$. Consider the transformation

$$S^t(z_1, z_2) = \left(\frac{z_1}{|\Re z_2(t)|^{1/2m}}, \frac{z_2 - \imath\Im z_2(t)}{|\Re z_2(t)|}\right).$$

For a given compact $K \subset D$, the mapping $\phi \circ g_t$ is well defined on K for large $|t|$ by Proposition 2.1 and hence $S^t \circ \phi \circ g_t : K \to \mathbf{C}^2$ makes sense. The image $S^t \circ \phi \circ g_t(K)$ is contained in the domain $\{\rho^t(z) = (|\Re z_2(t)|)^{-1}\rho \circ (S^t)^{-1}(z) < 0\}$. Using (3.8) it follows that

$$\rho^t(z) = 2\Re z_2 + |z_1|^{2m} + \big|\Im z_2(t)\big|^2/\big|\Re z_2(t)\big| + \cdots$$

where the lower dots indicate terms that converge to zero uniformly on compact sets of $\mathbf{C}^2$. Note that $S^t \circ \phi \circ g_t(\phi^{-1}(z(0))) = S^t(z(t))$ and hence that $\Re((S^t \circ \phi \circ g_t(\phi^{-1}(z(0)))) = -1$ for all time t. It follows from the defining function $\rho^t(z)$ that $S^t \circ \phi \circ g_t$ is a normal family on K and by exhausting D by compact sets we get a holomorphic mapping $g : D \to \bar{\Omega}_m = \{(z_1, z_2) \in \mathbf{C}^2\colon\ 2\Re z_2 + |z_1|^{2m} + c < 0\}$ where $0 \leqslant c \leqslant 1/K$. If $g(\phi^{-1}(z(0))) \in \Omega$, the maximum principle shows that $g : D \to \Omega$. Also g is biholomorphic since the inverses $(S^t \circ \phi \circ g_t)^{-1}$ are defined on larger and larger compact subsets of Ω_m and they take values in D which is hyperbolic. The sequence of inverses will then converge to $g^{-1} : \Omega \to D$. It is evident that Ω_m is equivalent to $\{(z_1, z_2) \in \mathbf{C}^2\colon\ 2\Re z_2 + |z_2|^{2m} < 0\}$. In case $g(\phi^{-1}(z(0))) = -1 + \imath\beta \in \partial\Omega_m$, it follows that $\partial\Omega_m$ is strongly pseudoconvex at that point by taking K sufficiently large. This is then the situation of Proposition 4.3 in [3] (which is a version of the Wong–Rosay theorem) for domains in a complex manifold and it follows from [18] that D is equivalent to $\mathbf{B}^2$.

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