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# Boundary regularity of correspondences in $\mathbb{C}^n$

RASUL SHAFIKOV[1] and KAUSHAL VERMA[2]

[1]Department of Mathematics, Middlesex College, University of Western Ontario, London, Ontario N6A 5B7, USA
[2]Department of Mathematics, Indian Institute of Science, Bangalore 560 012, India

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**Abstract.** Let $M$, $M'$ be smooth, real analytic hypersurfaces of finite type in $\mathbb{C}^n$ and $\hat{f}$ a holomorphic correspondence (not necessarily proper) that is defined on one side of $M$, extends continuously up to $M$ and maps $M$ to $M'$. It is shown that $\hat{f}$ must extend across $M$ as a locally proper holomorphic correspondence. This is a version for correspondences of the Diederich–Pinchuk extension result for CR maps.

**Keywords.** Correspondences; Segre varieties.

## 1. Introduction and statement of results

### 1.1 *Boundary regularity*

Let $U, U'$ be domains in $\mathbb{C}^n$ and let $M \subset U$, $M' \subset U'$ be relatively closed, connected, smooth, real analytic hypersurfaces of finite type (in the sense of D'Angelo). A recent result of Diederich and Pinchuk [DP3] shows that a continuous CR mapping $f\colon M \to M'$ is holomorphic in a neighbourhood of $M$. The purpose of this note is to show that their methods can be adapted to prove the following version of their result for correspondences. We assume additionally that $M$ (resp. $M'$) divides the domain $U$ (resp. $U'$) into two connected components $U^+$ and $U^-$ (resp. $U'^{\pm}$).

**Theorem 1.1.** *Let $\hat{f}\colon U^- \to U'$ be a holomorphic correspondence that extends continuously up to $M$ and maps $M$ to $M'$, i.e., $\hat{f}(M) \subset M'$. Then $\hat{f}$ extends as a locally proper holomorphic correspondence across $M$.*

We recall that if $\mathcal{D} \subset \mathbb{C}^p$ and $\mathcal{D}' \subset \mathbb{C}^m$ are bounded domains, a holomorphic correspondence $\hat{f}\colon \mathcal{D} \to \mathcal{D}'$ is a complex analytic set $A \subset \mathcal{D} \times \mathcal{D}'$ of pure dimension $p$ such that $\overline{A} \cap (\mathcal{D} \times \partial\mathcal{D}') = \emptyset$, where $\partial D'$ is the boundary of $D'$. In this situation, the natural projection $\pi\colon A \to \mathcal{D}$ is proper, surjective and a finite-to-one branched covering. If in addition the other projection $\pi'\colon A \to \mathcal{D}'$ is proper, the correspondence is called proper. The analytic set $A$ can be regarded as the graph of the multiple valued mapping $\hat{f} := \pi' \circ \pi^{-1}\colon \mathcal{D} \to \mathcal{D}'$. We also use the notation $A = \text{Graph}(\hat{f})$.

The branching locus $\sigma$ of the projection $\pi$ is a codimension one analytic set in $\mathcal{D}$. Near each point in $\mathcal{D} \setminus \sigma$, there are finitely many well-defined holomorphic inverses of $\pi$. The symmetric functions of these inverses are globally well-defined holomorphic functions on $\mathcal{D}$. To say that $\hat{f}$ is continuous up to $\partial\mathcal{D}$ simply means that the symmetric functions extend continuously up to $\partial\mathcal{D}$. Thus in Theorem 1.1 the various branches of $\hat{f}$ are continuous up to $M$ and each branch maps points on $M$ to those on $M'$.

We say that $\hat{f}$ in Theorem 1.1 extends as a *holomorphic correspondence* across $M$ if there exist open neighbourhoods $\tilde{U}$ of $M$ and $\tilde{U}'$ of $M'$, and an analytic set $\tilde{A} \subset \tilde{U} \times \tilde{U}'$ of pure dimension $n$ such that (i) Graph$(\hat{f})$ intersected with $(\tilde{U} \cap U^-) \times (\tilde{U}' \cap U')$ is contained in $\tilde{A}$ and (ii) the projection $\tilde{\pi}\colon \tilde{A} \to \tilde{U}$ is proper. Without condition (ii), $\hat{f}$ is said to extend as an *analytic set*. Finally, the extension of $\hat{f}$ is a proper holomorphic correspondence if in addition to (i) and (ii), $\tilde{\pi}'\colon \tilde{A} \to \tilde{U}'$ is also proper.

COROLLARY 1.1

*Let $D$ and $D'$ be bounded pseudoconvex domains in $\mathbb{C}^n$ with smooth real-analytic boundary. Let $\hat{f}\colon D \to D'$ be a holomorphic correspondence. Then $\hat{f}$ extends as a locally proper holomorphic correspondence to a neighbourhood of the closure of $D$.*

The corollary follows immediately from Theorem 1.1 and [BS] where the continuity of $\hat{f}$ is proved. This generalizes a well-known result of [BR] and [DF] where the extension past the boundary of $D$ is proved for holomorphic mappings.

1.2 *Preservation of strata*

Let $M_s^+$ (resp. $M_s^-$) be the set of strongly pseudoconvex (resp. pseudoconcave) points on $M$. The set of points where the Levi form $\mathcal{L}_\rho$ has eigenvalues of both signs on $T^{\mathbb{C}}(M)$ and no zero eigenvalue will be denoted by $M^{\pm}$ and finally $M^0$ will denote those points where $\mathcal{L}_\rho$ has at least one zero eigenvalue on $T^{\mathbb{C}}(M)$. $M^0$ is a closed real analytic subset of $M$ of real dimension at most $2n-2$. Then

$$M = M_s^+ \cup M_s^- \cup M^{\pm} \cup M^0.$$

Further, let $M^+$ (resp. $M^-$) be the pseudoconvex (resp. pseudoconcave) part of $M$, which equals the relative interior of $\overline{M_s^+}$ (resp. $\overline{M_s^-}$). For non-negative integers $i, j$ such that $i + j = n - 1$, let $M_{i,j}$ denote those points at which $\mathcal{L}_\rho$ has exactly $i$ positive and $j$ negative eigenvalues on $T^{\mathbb{C}}(M)$. Each (non-empty) $M_{i,j}$ is relatively open in $M$ and semi-analytic whose relative boundary is contained in $M^0$. With this notation, $M_{0,n-1} = M_s^-$ and $M_{n-1,0} = M_s^+$. Moreover, $M^{\pm}$ is the union of all (non-empty) $M_{i,j}$ where both $i, j$ are at least 1 and $i + j = n - 1$. Note that points in $M_s^-, M^{\pm}$ are in the envelope of holomorphy of $U^-$. Following [B], there is a semi-analytic stratification for $M^0$ given by

$$M^0 = \Gamma_1 \cup \Gamma_2 \cup \Gamma_3 \cup \Gamma_4, \tag{1.1}$$

where $\Gamma_4$ is a closed, real analytic set of dimension at most $2n-4$ and $\Gamma_2 \cup \Gamma_3 \cup \Gamma_4$ is also a closed, real analytic set of dimension at most $2n-3$. Further, $\Gamma_1, \Gamma_2, \Gamma_3$ are either empty or smooth, real analytic manifolds; $\Gamma_2, \Gamma_3$ have dimension $2n-3$, and $\Gamma_1$ has dimension $2n-2$. Finally, $\Gamma_2$ and $\Gamma_3$ are CR manifolds of complex dimension $n-2$ and $n-3$ respectively. The set of points, denoted by $\Gamma_h^1$ in $\Gamma_1$ where the complex tangent space to $\Gamma_1$ has dimension $n-1$ is semi-analytic and has real dimension at most $2n-3$, as otherwise there would exist a germ of a complex manifold in $M$ contradicting the finite type hypothesis. Then $\Gamma_1 \setminus \Gamma_h^1$ is a real analytic manifold of dimension $2n-2$ and has CR dimension $n-2$. Using the same letters to denote the various strata of $M^0$, there exists a refinement of (1.1), so that $\Gamma_1$, $\Gamma_2$, $\Gamma_3$ are all smooth, real analytic manifolds of dimensions $2n-2$, $2n-3$, $2n-3$ respectively, while the corresponding CR dimensions are $n-2$, $n-2$, and $n-3$. Finally, $\Gamma_4$ is a closed, real analytic set of dimension at most $2n-4$.

**Theorem 1.2.** *With the hypothesis of Theorem* 1.1, *the extended correspondence $\hat{f}: M \to M'$ satisfies the additional properties*: $\hat{f}(M^+) \subset M'^+$, $\hat{f}(M^+ \cap M^0) \subset M'^+ \cap M'^0$ *and* $\hat{f}(M^-) \subset M'^-$, $\hat{f}(M^- \cap M^0) \subset M'^- \cap M'^0$. *Moreover*, $\hat{f}(M^+ \cap \Gamma_j) \subset M'^+ \cap \Gamma'_j$ *and* $\hat{f}(M^- \cap \Gamma_j) \subset M'^- \cap \Gamma'_j$ *for* $j = 1, 3, 4$. *Finally, $\hat{f}$ maps the relative interior of $\overline{M}^{\pm}$ to the relative interior of $\overline{M}'^{\pm}$.*

Preservation of $\Gamma_2$ is not always possible even for holomorphic mappings as the following example shows: the domain $\Omega = \{(z_1, z_2) \colon |z_1|^2 + |z_2|^4 < 1\}$ is mapped to the unit ball in $\mathbb{C}^2$ by the proper holomorphic mapping $f(z_1, z_2) = (z_1, z_2^2)$. Points of the form $\{(e^{i\theta}, 0)\} \subset \partial\Omega$ are weakly pseudoconvex and in fact form $\Gamma_2 \subset \partial\Omega$, and $f$ maps them to strongly pseudoconvex points.

## 2. Segre varieties

We will write $z = ('z, z_n) \in \mathbb{C}^{n-1} \times \mathbb{C}$ for a point $z \in \mathbb{C}^n$. The word 'analytic' will always mean complex analytic unless stated otherwise. The techniques of Segre varieties will be used and here is a synopsis of the main properties that will be needed. The proofs of these can be found in [DF] and [DW]. As described above, let $M$ be a smooth, real analytic hypersurface of finite type in $\mathbb{C}^n$ that contains the origin. If $U$ is small enough, the complexification $\rho(z, \overline{w})$ of $\rho$ is well-defined by means of a convergent power series in $U \times U$. Note that $\rho(z, \overline{w})$ is holomorphic in $z$ and anti-holomorphic in $w$. For any $w \in U$, the associated Segre variety is defined as

$$Q_w = \{z \in U \colon \rho(z, \overline{w}) = 0\}.$$

By the implicit function theorem, it is possible to choose neighbourhoods $U_1 \subset\subset U_2$ of the origin such that for any $w \in U_1$, $Q_w$ is a closed, complex hypersurface in $U_2$ and

$$Q_w = \{z = ('z, z_n) \in U_2 \colon z_n = h('z, \overline{w})\},$$

where $h('z, \overline{w})$ is holomorphic in $'z$ and anti-holomorphic in $w$. Such neighbourhoods will be called a standard pair of neighbourhoods and they can be chosen to be polydiscs centered at the origin. It can be shown that $Q_w$ is independent of the choice of $\rho$. For $\zeta \in Q_w$, the germ $Q_w$ at $\zeta$ will be denoted by ${}_{\zeta}Q_w$. Let $\mathcal{S} := \{Q_w \colon w \in U_1\}$ be the set of all Segre varieties, and let $\lambda \colon w \mapsto Q_w$ be the so-called Segre map. Then $\mathcal{S}$ admits the structure of a finite dimensional analytic set. It can be shown that the analytic set

$$I_w := \lambda^{-1}(\lambda(w)) = \{z \colon Q_z = Q_w\}$$

is contained in $M$ if $w \in M$. Consequently, the finite type assumption on $M$ forces $I_w$ to be a discrete set of points. Thus $\lambda$ is proper in a small neighbourhood of each point of $M$. For $w \in U_1^+$, the symmetric point ${}^s w$ is defined to be the unique point of intersection of the complex normal to $M$ through $w$ and $Q_w$. The component of $Q_w \cap U_2^-$ that contains the symmetric point is denoted by $Q_w^c$.

Finally, for all objects and notions considered above, we simply add a prime to define their corresponding analogs in the target space.

## 3. Localization and extension across an open dense subset of *M*

In the proof of Theorem 1.1 in order to show extension of $\hat{f}$ as a holomorphic correspondence, it is enough to consider the problem in an arbitrarily small neighbourhood of any point $p \in M$. The reason is the following. Firstly, since the projection $\pi\colon \mathrm{Graph}(\hat{f}) \to U^-$ is proper, the closure of $\mathrm{Graph}(\hat{f})$ has empty intersection with $U^- \times \partial U'$. Therefore, by [C] § 20.1, to prove the continuation of $\hat{f}$ across $M$ as an analytic set, it is enough to do that in a neighbourhood of any point in $M$. Secondly, once the extension of $\hat{f}$ as a holomorphic correspondence in a neighbourhood of any point $p \in M$ is established, then globally there exists a holomorphic correspondence defined in a neighbourhood $\tilde{U}$ of $M$ which extends $\hat{f}$. To see that simply observe that if $F \subset \tilde{U} \times \tilde{U}'$ is an analytic set extending $\hat{f}$, then by choosing smaller $\tilde{U}$ we may ensure that the projection to the first component is proper, as otherwise there would exist a point $z$ on $M$ such that $\hat{F}(z)$ has positive dimension (here $\hat{F}$ is the map associated with the set $F$). This however contradicts local extension of $\hat{f}$ near $z$ as a holomorphic correspondence.

Since the projection $\pi$: $\mathrm{Graph}(\hat{f}) \to U^-$ is proper, $\mathrm{Graph}(\hat{f})$ is contained in the analytic set $A \subset U^- \times U'$, defined by the zero locus of holomorphic functions $P_1(z, z_1'), P_2(z, z_2'), \ldots, P_n(z, z_n')$ given by

$$P_j(z, z_j') = {z_j'}^{l} + a_{j1}(z){z_j'}^{l-1} + \cdots + a_{jl}(z), \tag{3.1}$$

where $l$ is the generic number of images in $\hat{f}(z)$, and $1 \le j \le n$ (for details, see [C]). The coefficients $a_{\mu\nu}(z)$ are holomorphic in $U^-$ and extend continuously up to $M$. This is the definition of continuity of the correspondence $\hat{f}$ up to $M$ which is equivalent to that given in §1.1.

The discriminant locus is $\{R_j(z) = 0\}$, $1 \le j \le n$, where $R_j(z)$ is a universal polynomial function of $a_{j\mu}(z)$ $(1 \le \mu \le l)$ and hence by the uniqueness theorem, it follows that $\overline{\{R_j(z) = 0\}} \cap M$ is nowhere dense in $M$, for all $j$. The set of points $S$ on $M$ which do not belong to $\overline{\{R_j(z) = 0\}} \cap M$ for any $j$ is therefore open and dense in $M$. Near each point $p$ on $S$, $\hat{f}$ splits into well-defined holomorphic maps $f_1(z), f_2(z), \ldots, f_l(z)$ each of which is continuous up to $M$.

If $p \in S \cap (M^- \cup M^{\pm})$, the functions $a_{\mu\nu}(z)$ extend holomorphically to a neighbourhood of $p$ and hence $\hat{f}$ extends as a holomorphic correspondence across $p$. It is therefore sufficient to show that $\hat{f}$ extends across an open dense subset of $S \cap M^+$. But this follows from Lemma 3.2 and Corollary 3.3 in [DP3]. We denote by $\Sigma \subset M$ the non-empty open dense subset of $M$ across which $\hat{f}$ extends as a holomorphic correspondence.

## 4. Extension as an analytic set

Fix $0 \in M$ and let $p_1', p_2', \ldots, p_k' \in \hat{f}(0) \cap M'$. The continuity of $\hat{f}$ allows us to choose neighbourhoods $0 \in U_1$ and $p_i' \in U_i'$ and local correspondences $\hat{f}_i\colon U_1^- \to U_i'$ that are irreducible and extend continuously up to $M$. Moreover, $\hat{f}_i(0) = p_i'$ for all $1 \le i \le k$. It will suffice to focus on one of the $\hat{f}_i'$'s, say $\hat{f}_1'$ and to show that it extends holomorphically across the origin. Abusing notation, we will write $\hat{f}_1' = \hat{f}$, $U_1' = U'$ and $p_1' = 0'$ . Thus $\hat{f}\colon U_1^- \to U'$ is an irreducible holomorphic correspondence and $\hat{f}(0) = 0'$. Define

$$V^+ = \{(w, w') \in U_1^+ \times U'\colon \hat{f}(Q_w^c) \subset Q'_{w'}\}.$$

Then $V^+$ is non-empty. Indeed, $\hat{f}$ extends across an open dense set near the origin and [V] shows that the invariance property of Segre varieties then holds. Moreover, a similar argument as in [S2] shows that $V^+ \subset U_1^+ \times U'$ is an analytic set of dimension $n$ and exactly the same arguments as in Lemmas 4.2 – 4.4 of [DP3] show that: first, the projection $\pi\colon V^+ \to U^+ := \pi(V^+) \subset U_1^+$ is proper (and hence that $U^+ \subset U_1^+$ is open) and second, the projection $\pi'\colon V^+ \to U'$ is locally proper. Thus, to $V^+$ is associated a correspondence $F^+\colon U^+ \to U'$ whose branches are $\hat{F}^+ = \pi' \circ \pi^{-1}$.

Let $a \in M$ be a point close to the origin, across which $\hat{f}$ extends as a holomorphic correspondence. If $\hat{f}$ is well-defined in the ball $B(a,r), r > 0$ and $w \in B(a,r)^-$, it follows from Theorem 4.1 in [V] that all points in $\hat{f}(w)$ have the same Segre variety. By analytic continuation, the same holds for all $w \in U_1^-$. Using this observation, it is possible to define another correspondence $F^-\colon U_1^- \to U'$ whose branches are $\hat{F}^-(w) = (\lambda')^{-1} \circ \lambda' \circ \hat{f}(w)$. Let $U := U_1^- \cup U^+ \cup (\Sigma \cap U_1)$. The invariance property of Segre varieties shows that the correspondences $\hat{F}^+$, $\hat{F}^-$ can be glued together near points on $\Sigma \cap U_1$. Hence, there is a well-defined correspondence $\hat{F}\colon U \to U'$ whose values over $U^+$ and $U_1^-$ are $\hat{F}^+$ and $\hat{F}^-$ respectively. Note that

$$F := \mathrm{Graph}(\hat{F}) = \{(w, w') \in U \times U'\colon w' \in \hat{F}(w)\}$$

is an analytic set in $U \times U'$ of pure dimension $n$, with proper projection $\pi\colon F \to U$. Once again, the invariance property shows that all points in $\hat{F}(w)$, $w \in U$, have the same Segre variety.

*Lemma* 4.1. *The correspondence $\hat{F}$ satisfies the following properties*:

(i) *For* $w_0 \in \partial U \cap U_1^+$, $\mathrm{cl}_{\hat{F}}(w_0) \subset \partial U'$.
(ii) $\mathrm{cl}_{\hat{F}}(0) \subset Q'_{0'}$.
(iii) *If* $\mathrm{cl}_{\hat{F}}(0) = \{0'\}$, *then* $0 \in \Sigma$.
(iv) $F \subset (U_1 \setminus (M \setminus \Sigma)) \times U'$ *is a closed analytic set.*

*Proof.*

(i) Choose $(w_j, w'_j) \in F$ converging to $(w_0, w'_0) \in (\partial U \cap U_1^+) \times \overline{U}'$. Then $\hat{f}(Q^c_{w_j}) \subset Q'_{w'_j}$ for all $j$. If $w'_0 \in U'$, then passing to the limit, we get $\hat{f}(Q^c_{w_0}) \subset Q'_{w'_0}$ which shows that $(w_0, w'_0) \in F$ and hence $w_0 \in U$, which is a contradiction. This also proves (iv).

(ii) Choose $w_j \in U$ converging to 0. There are two cases to consider. First, if $w_j \in U_1^- \cup (\Sigma \cap U_1)$ for all $j$, it follows that $\hat{f}(w_j) \to 0'$. Moreover, for any $w'_j \in \hat{F}(w_j)$, $Q'_{w'_j} = Q'_{\hat{f}(w_j)}$. If $U'$ is small enough, the equality $Q'_{w'} = Q'_{0'}$ implies that $w' = 0'$ and thus we conclude that $w'_j \to 0' \in Q'_{0'}$. Second, if $w_j \in U^+$ for all $j$, then $\hat{f}(Q^c_{w_j}) \subset Q'_{w'_j}$ for any $w'_j \in \hat{F}(w_j)$. Let $w'_j \to w'_0 \in U'$. If $\zeta \in Q^c_{w_j}$, then $\hat{f}(\zeta) \in Q'_{w'_j} \to Q'_{w'_0}$. But $w_j \to 0$ implies that $\mathrm{dist}(Q^c_{w_j}, 0) \to 0$ and hence $\hat{f}(\zeta) \to 0'$. Thus $0' \in Q'_{w'_0}$ which shows that $w'_0 \in Q'_{0'}$.

(iii) If $\mathrm{cl}_{\hat{F}}(0) = \{0'\}$, then (i) shows that $0 \notin \partial U \cap U_1^+$. Let $B(0,r)$ be a small ball around the origin such that $B(0,r) \cap \partial U = \emptyset$. The correspondence $\hat{F}$ over $B(0,r)^+$ is the union of some components of the zero locus of a system of monic pseudo-polynomials whose coefficients are bounded holomorphic functions on $B(0,r)^+$. By Trepreau's theorem, all these coefficients extend holomorphically to $B(0,r)$, and the extended zero locus contains the graph of $\hat{f}$ near the origin since $\Sigma$ is dense. It follows that $0 \in \Sigma$. □

Following [S1], for any $w_0 \in U$, it is possible to find a neighbourhood $\Omega$ of $w_0$, relatively compact in $U$ and a neighbourhood $V \subset U_1$ of $Q_{w_0} \cap U_1$ such that for $z \in V$, $Q_z \cap \Omega$ is non-empty and connected. Associated with the pair $(\Omega, V)$ is

$$\tilde{F} := \tilde{F}(w_0, \Omega, V) = \{(z, z') \in V \times U' : \hat{F}(Q_z \cap \Omega) \subset Q'_{z'}\} \tag{4.1}$$

which (see [DP4]) is an analytic set of dimension at most $n$. If $w_0 \in \Sigma$, then Corollary 5.3 of [DP3], shows that $F \cap (V \times U')$ is the union of irreducible components of $\tilde{F}$ of dimension $n$. As in [DP3] we call $(w_0, z_0) \in U \times Q_{w_0}$ a pair of reflection if there exist neighbourhoods $\Omega(w_0) \ni w_0$ and $\Omega(z_0) \ni z_0$ such that for all $w \in \Omega(w_0)$, $\hat{F}(Q_w \cap \Omega(z^0)) \subset Q'_{\hat{F}(w)}$. It follows from the invariance property of Segre varieties that the definition of the pair of reflection is symmetric. As an example we note that if the set $\tilde{F}$ defined in (4.1) contains $F \cap (V \times U')$, then $(w_0, z)$ is a point of reflection for any point $z$ in a connected component of $Q_{w_0} \cap U$ containing $w_0$.

Let $w_0 \in U$, $z_0 \in Q_{w_0} \cap \Sigma$ be a pair of reflection. Fix $B(z_0, r)$, a small ball around $z_0$ where $\hat{f}$ is well-defined and let $S(w_0, z_0) \subset \tilde{F} \cap ((Q_{w_0} \cap U_1) \times U')$ be the union of those irreducible components that contain Graph$(\hat{f})$ over $Q_{w_0} \cap B(z_0, r)$. Note that $S(w_0, z_0)$ is an analytic set of dimension $n-1$ and is contained in $(Q_{w_0} \cap U_1) \times U'$ and moreover, the invariance property shows that

$$S(w_0, z_0) \subset ((Q_{w_0} \cap U_1) \times (Q'_{\hat{F}(w_0)} \cap U')).$$

Furthermore, from the above considerations it follows that for any $z \in \pi(S(w_0, z_0))$ the point $(w_0, z)$ is a pair of reflection. Finally, let the cluster set of a sequence of closed sets $\{C_j\} \subset \mathcal{D}$, where $\mathcal{D}$ is some domain, be the set of all possible accumulation points in $\mathcal{D}$ of all possible sequences $\{c_j\}$ where $c_j \in C_j$.

PROPOSITION 4.1

*Let* $\{z_\nu\} \in \Sigma$ *converge to* 0. *Suppose that the cluster set of the sequence* $\{S(z_\nu, z_\nu)\}$ *contains a point* $(\zeta_0, \zeta'_0) \in U \times U'$. *Then* $\hat{f}$ *extends as an analytic set across the origin.*

*Proof.* First, the pair $(z_\nu, z_\nu)$ is an example of a pair of reflection and hence $S(z_\nu, z_\nu)$ is well-defined. Also, note that $(z_\nu, \hat{f}(z_\nu)) \to (0, 0')$. Choose $(\zeta_\nu, \zeta'_\nu) \in S(z_\nu, z_\nu)$ that converges to $(\zeta_0, \zeta'_0) \in U \times U'$. It follows that $(\zeta_\nu, z_\nu)$ is a pair of reflection. Let $\Omega$, $V$ be neighbourhoods of $\zeta_0$ and $Q_{\zeta_0}$ as in the definition of $\tilde{F}(\zeta_0, \Omega, V)$. Since $\zeta_0 \in U$, it follows that $\tilde{F}(\zeta_0, \Omega, V)$ is a non-empty, analytic set in $V \times U'$. Shrinking $U_1$ if needed, $Q_{\zeta_\nu} \cap U_1 \subset V$ and $\zeta_\nu \in \Omega$ for all large $\nu$. This shows that $\tilde{F}(\zeta_\nu, \Omega, V) = \tilde{F}(\zeta_0, \Omega, V)$ for all large $\nu$. Lemma 5.2 of [DP3] shows that $\tilde{F}(\zeta_\nu, \Omega, V)$ contains the graph of all branches of $\hat{f}$ near $z_\nu$ and hence $\tilde{F}(\zeta_0, \Omega, V)$ contains the graph of $\hat{f}$ near $(0, 0')$. Therefore, $\tilde{F}(\zeta_0, \Omega, V)$ extends the graph of $\hat{f}$ across the origin. □

*Remarks.* First, as in [DP3] this proposition will be valid if the pair $(z_\nu, z_\nu)$ were replaced by a pair of reflection $(w_\nu, z_\nu) \in U \times \Sigma$ that converges to $(0, 0')$ and $\hat{F}(w_\nu)$ clusters at some point in $U'$. Second, this proposition shows the relevance of studying the cluster set of a sequence of analytic sets (see [SV] also). In general, the hypothesis that the cluster set of $\{S(z_\nu, z_\nu)\}$ (or $S(w_\nu, z_\nu)$ in case $(w_\nu, z_\nu)$ is a pair of reflection) contains a point in $U \times U'$ cannot be guaranteed since the projection $\pi\colon S(z_\nu, z_\nu) \to U$ is not known to be proper. However, the following version of Lemma 5.9 in [DP3] holds.

*Lemma* 4.2. *There are sequences $(w_\nu, z_\nu) \in U \times \Sigma$, $w'_\nu \in \hat{F}(w_\nu)$ and analytic sets $\sigma_\nu \subset U$ of pure dimension $p \geq 1$ ($p$ independent of $\nu$) such that*:

(i) $(w_\nu, z_\nu) \to (0, 0)$ *and $(w_\nu, z_\nu)$ is a pair of reflection for all $\nu$.*
(ii) $w'_\nu \to w'_0 \in U'$ *and* $z_\nu \in \sigma_\nu \subset \pi(S(w_\nu, z_\nu))$.

*Proof.* Choose a sequence $z_\nu \in \Sigma$ that converges to the origin. If the projections $\pi\colon S(z_\nu, z_\nu) \to U$ were proper for all $\nu$, then for some fixed $r > 0$ and $\nu$ large enough, let $\sigma_\nu := Q_{z_\nu} \cap B(z_\nu, r)$, $w_\nu = z_\nu$ and $w'_\nu \in \hat{f}(z_\nu)$. It can be seen that the lemma holds with these choices. On the other hand, if $\pi$ is not known to be proper on $S(z_\nu, z_\nu)$, no fixed value of $r$, as described above, exists. Hence, for arbitrarily small values of $r' > 0$, there exist $(w_\nu, w'_\nu) \in S(z_\nu, z_\nu) \cap (U^+ \times U')$ such that $w_\nu \to 0$ and $w'_\nu \to w'_0$ with $|w'_0| = r'$. Since $M'$ is of finite type, we may assume that $Q'_{w'_0} \neq Q'_{0'}$. Moreover, note that $w'_0 \in Q'_{0'} \cap U'$ (which shows that $0' \in Q'_{w'_0}$) and $(w_\nu, z_\nu)$ is a pair of reflection for all $\nu$. By making a small holomorphic perturbation of coordinates in the target space, if needed, it follows that $0' \in Q'_{w'_0} \cap \{z' \in U'\colon z'_2 = \cdots = z'_n = 0\}$ is an isolated point. Therefore, there exists an $\epsilon > 0$ such that after shrinking $U'$, if needed, $q'_0 := Q'_{w'_0} \cap \{z' \in U'\colon z'_2 = \cdots = z'_{n-1} = 0, |z'_n| < \epsilon\}$ (which is an analytic set of dimension 1 in $U' \cap \{|z'_n| < \epsilon\}$ containing the origin) has no limit points on $\partial U' \cap \{|z'_n| < \epsilon\}$. Let $l$ be the multiplicity of $\hat{f}\colon U_1^- \to U'$. Let $\hat{f}(z_\nu) = \{\zeta^j_\nu\}$, $1 \leq j \leq l$ counted with multiplicity. For large $\nu$, the $l$ sets

$$q'_{\nu,j} = Q'_{w'_\nu} \cap \{z' \in U'\colon z'_k = (\zeta^j_\nu)_k, \quad 2 \leq k \leq n-1, \quad |z'_n| < \epsilon\}$$

are analytic, of dimension 1, in $U' \cap \{|z'_n| < \epsilon\}$ without limit points on $\partial U' \cap \{|z'_n| < \epsilon\}$ and clearly contain $(z_\nu, \zeta^j_\nu)$. Since $\pi'(S(w_\nu, z_\nu)) \subset Q'_{w'_\nu}$,

$$s_{\nu,j} := S(w_\nu, z_\nu) \cap \{(z, z')\colon z'_k = (\zeta^j_\nu)_k, \quad 2 \leq k \leq n-1\}$$

are analytic sets of dimension at least 1 in $U_1 \times (U' \cap \{|z'_n| < \epsilon\})$ for all $1 \leq j \leq l$. By construction, the analytic sets $q'_{\nu,j}$ do not have limit points on $\partial U' \cap \{|z'_n| < \epsilon\}$ and hence $s_{\nu,j}$ do not have limit points on $U_1 \times (\partial U' \cap \{|z'_n| < \epsilon\})$. By Lemma 4.1, $\mathrm{cl}_{\hat{F}}(0) \subset Q'_{0'} = \{z'_n = 0\}$ and by shrinking $U_1$ if needed, this shows that $s_{\nu,j}$ have no limit points on $U_1 \times (U' \cap \{|z'_n| = \epsilon\})$. Thus for large $\nu$ and all $j$, the projections $\pi\colon s_{\nu,j} \to U_1$ are proper and their images $\sigma_{\nu,j} := \pi(s_{\nu,j})$ are analytic sets in $U_1$ of dimension at least 1 and $z_\nu \in \sigma_{\nu,j}$ for all $\nu$, $j$. It remains to pass to subsequences if necessary to choose $\sigma_{\nu,j}$ with constant dimension. □

One conclusion that follows now is: if $\hat{f}$ does not extend as an analytic set across the origin, then $\mathrm{cl}(\sigma_\nu) \subset M \setminus \Sigma$. Indeed, if there exists $\zeta_0 \in \mathrm{cl}(\sigma_\nu) \cap (U_1 \setminus (M \setminus \Sigma))$, let $(\zeta_\nu, \zeta'_\nu) \in S(w_\nu, z_\nu)$ converge to $(\zeta_0, \zeta'_0) \in U_1 \times U'$. Proposition 4.1 now shows that $\zeta_0 \in \partial U \cap U_1$. But since $\zeta_0 \notin M \setminus \Sigma$, it follows from Lemma 4.1 that $\zeta'_0 \in \partial U'$ which is a contradiction.

The goal will now be to show that $\hat{f}$ extends as an analytic set across the origin. For this, choose $\{z_\nu\} \in \Sigma$ converging to the origin and consider the analytic sets $S(z_\nu, z_\nu)$. By Proposition 4.1, it suffices to show that $\pi(\mathrm{cl}(S(z_\nu, z_\nu)) \cap U \neq \emptyset$. Let

$$S' := \pi'(\mathrm{cl}(S(z_\nu, z_\nu)) \cap (\{0\} \times U')) \subset Q'_{0'}$$

and let $m$ be the dimension of $\hat{S}'$ – the smallest closed analytic set containing $S'$ (the so-called Segre completion of [DP3]). If $m = 0$, then $0'$ is an isolated point in $S'$ and after shrinking $U_1, U'$ suitably, it follows that $\mathrm{cl}(S(z_\nu, z_\nu))$ has no limit points on $U_1 \times \partial U'$. Thus $\pi\colon S(z_\nu, z_\nu) \to U_1$ are proper projections and therefore $\pi(S(z_\nu, z_\nu)) = Q_{z_\nu} \cap U_1$ for all large $\nu$. Hence $\pi(\mathrm{cl}(S(z_\nu, z_\nu))) = Q_0 \cap U_1$. If $\hat{f}$ did not extend as an analytic set across the origin, the aforementioned remark shows that with $\sigma := Q_{z_\nu} \cap U_1$, $Q_0 \cap U_1 = \mathrm{cl}(\sigma_\nu) \subset M \setminus \Sigma \subset M$. This cannot happen as $M$ is of finite type. Hence $\hat{f}$ extends as an analytic set across the origin in case $m = 0$. We may therefore suppose that $m > 0$. We recall the following lemma proved by Diederich and Pinchuk:

*Lemma* 4.3 ([DP3], *Lemma* 9.8). *Let $S'$ be a subset of $Q'_{0'}$, $0' \in S'$ and $m = \dim \hat{S}'$. Then after possibly shrinking $U_1$, there are points $w'^1, \ldots, w'^k \in S'$ $(k \le n-1)$ such that one of the following holds*:

(1) $k = m$ *and* $\dim(\hat{S}' \cap Q'_{w'^1} \cap \cdots \cap Q'_{w'^k}) = 0$;
(2) $k \ge 2m - n + 1$ *and* $\dim(\hat{S}' \cap Q'_{w'^1} \cap \cdots \cap Q'_{w'^k}) = m - k$.

Thus there are two cases to consider.

*Case* 1. Choose $(w_{1\nu}, w'_{1\nu}), (w_{2\nu}, w'_{2\nu}), \ldots, (w_{m\nu}, w'_{m\nu}) \in S(z_\nu, z_\nu)$ so that $w_{\mu\nu} \to 0$ and $w'_{\mu\nu} \to w'_\mu$ for all $1 \le \mu \le m$. A generic choice of $w_{\mu\nu}$ (see p. 136 in [DP3]) ensures that $q_{m\nu} := Q_{w_{1\nu}} \cap Q_{w_{2\nu}} \cap \cdots \cap Q_{w_{m\nu}}$ has dimension $n - m$. Each $(w_{\mu\nu}, z_\nu)$ is a pair of reflection and hence the analytic set

$$S_\nu^m := \bigcap_{1 \le \mu \le m} S(w_{\mu\nu}, z_\nu) \subset (q^{m\nu} \times q'^{m\nu}) \cap (U_1 \times U')$$

is well-defined. If $m = n - 1$, then Lemma 9.7 of [DP3] shows that the germ of $q'^{(n-1)}$ at the origin has dimension 1. Moreover, $\hat{S}' = Q'_{0'}$ and Lemma 4.3 implies that $q'^{(n-1)} \cap Q'_{0'}$ contains $0'$ as an isolated point. Since $\mathrm{cl}_{\hat{F}}(0) \subset Q'_{0'}$, it follows that $0'$ is an isolated point of

$$\pi'(\mathrm{cl}(S_\nu^{n-1}) \cap (\{0\} \times U')) \subset q'^{(n-1)} \cap Q'_{0'} = \{0'\}.$$

Shrinking $U_1$, the projection $\pi\colon S_\nu^{n-1} \to U_1$ becomes proper and $\pi(S_\nu^{n-1}) = q^{n-1,\nu} \cap U_1$. By Theorem 7.4 of [DP3], there is a subsequence of $q^{n-1,\nu} \cap U_1$ that converges to an analytic set $A \subset U_1$ of pure dimension 1 and contains the origin. $A$ contains points $\zeta_0$ that do not belong to $M$ because of the finite type assumption and $\zeta_0 \in \pi(\mathrm{cl}(S_\nu^{n-1})) \subset \pi(\mathrm{cl}(S(w_{\mu\nu}, z_\nu)))$. By Proposition 4.1, $\hat{f}$ extends as an analytic set across the origin.

If $m < n - 1$, the dimension of $S_\nu^m \cap S(z_\nu, z_\nu)$ is at least $n - m - 1 > 0$. Now

$$\pi'(\mathrm{cl}(S_\nu^m \cap S(z_\nu, z_\nu)) \cap (\{0\} \times U')) \subset q'^m \cap \hat{S}' = \{0'\},$$

the last equality following from Lemma 4.3. The projection $\pi\colon S_\nu^m \cap S(z_\nu, z_\nu) \to U_1$ is therefore proper for small $U_1$ and that $\pi(S_\nu^m \cap S(z_\nu, z_\nu)) = q^{m\nu} \cap Q_{z_\nu} \cap U_1$. Again, by

Theorem 7.4 of [DP3], there is a subsequence of $q^{m\nu} \cap Q_{z_\nu} \cap U_1$ that converges to an analytic set $A \subset U_1$ of positive dimension and as before this shows that $\hat{f}$ extends as an analytic set across the origin.

*Case* 2. As before, choose $(w_{1\nu}, w'_{1\nu}), (w_{2\nu}, w'_{2\nu}), \dots, (w_{k\nu}, w'_{k\nu}) \in S(z_\nu, z_\nu)$ such that $w_{\mu\nu} \to 0$ and $w'_{\mu\nu} \to w'_\mu$ for all $1 \le \mu \le k$ and $q_{k\nu} = Q_{w_{1\nu}} \cap Q_{w_{2\nu}} \cap \dots \cap Q_{w_{k\nu}}$, $\tilde{q}^{k\nu} := Q_{z_\nu} \cap q^{k\nu}$ have dimension $n-k$ and $n-k-1$ respectively. Now note that $\dim(S^k_\nu \cap S(z_\nu, z_\nu)) \ge n-k-1 > 1$. Indeed, the inequalities $2m-n+1 \le k < m$ show that $m \le n-2$ and hence $k < n-2$. Since the dimension of $\hat{S}' \cap q'^k$ is $m-k$, choose coordinates so that

$$\hat{S}' \cap q'^k \cap \{z' \in U' : z'_1 = z'_2 = \dots = z'_{m-k} = 0\} = \{0'\}.$$

Let $\hat{f}(z_\nu) = \{\zeta^j_\nu\}$, $1 \le j \le l$, $l$ being the multiplicity of $\hat{f}$. The $l$ sets

$$T_{\nu,j} = \{(z,z') \in S^k_\nu \cap S(z_\nu, z_\nu) : z'_1 = (\zeta^j_\nu)_1, \\ z'_2 = (\zeta^j_\nu)_2, \dots, z'_{m-k} = (\zeta^j_\nu)_{m-k}\},$$

where $1 \le j \le l$ are analytic sets in $U_1 \times U'$ and have dimension at least $n-k-1-(m-k) = n-m-1 > 0$. By construction,

$$\pi'(\mathrm{cl}(T_{\nu,j}) \cap (\{0\} \times U')) \subset \hat{S}' \cap q'^k \\ \cap \{z' \in U' : z'_1 = z'_2 = \dots = z'_{m-k} = 0\} = \{0'\}$$

and hence by shrinking $U_1, U'$, the projections $\pi : T_{\nu,j} \to U_1$ are proper and the images $\sigma_{\nu,j} := \pi(T_{\nu,j}) \subset U_1$ are analytic and have dimension $n-m-1$. Moreover $\sigma_{\nu,j} \subset \tilde{q}^{k\nu}$, and since $\tilde{q}^{k\nu}$ depend anti-holomorphically on the $k$-tuple defining it, Theorem 7.4 of [DP3] shows that $\tilde{q}^{k\nu}$ converges to an analytic set $\tilde{A} \subset U_1$ of dimension $n-k-1$, after passing to a subsequence. Working with this subsequence, we see that $\mathrm{cl}(\sigma_{\nu,j}) \subset \tilde{A}$. On the other hand, since $2m-n+1 \le k$, it follows, as in [DP3], that

$$\dim \tilde{A} = n-k-1 \le 2(n-m-1) = 2 \dim \sigma_{\nu,j}.$$

Proposition 8.3 of [DP3] shows that $\mathrm{cl}(\sigma_{\nu,j}) \not\subset M$ and hence by Proposition 4.1, it follows that $\hat{f}$ extends as an analytic set across the origin.

To complete the proof, it suffices to show that extension as an analytic set implies extension as a locally proper holomorphic correspondence. This is achieved in the next lemma.

*Lemma* 4.4. *There exist neighbourhoods* $U$ *of* 0 *and* $U'$ *of* $0'$ *such that* $F \subset U \times U'$ *is a proper holomorphic correspondence which extends* $\hat{f}$.

*Proof.* Extension as a holomorphic correspondence essentially follows from [DP4]. All nuances in the proof of Proposition 2.4 in [DP4] work in this situation as well provided the following two modifications are made. Let $U, U'$ be neighbourhoods of $0, 0'$ respectively and suppose that $F \subset U \times U'$ extends $\hat{f}$ as an analytic set in $U \times U'$. Then it needs to be checked that $F \cap (U^+ \times U') \ne \emptyset$ and that there exists a sequence $\{z_\nu\} \in M$ converging to 0 such that $\hat{f}$ extends as a correspondence across each $z_\nu$.

Suppose that $F \cap (U^+ \times U') = \emptyset$. In this case, the proof of Proposition 3.1 (or even Proposition 4.1 in [SV]) shows that $(0, 0')$ is in the envelope of holomorphy of $\overline{U^-} \times U'$. The coefficients $a_{\mu\nu}(z)$ in (3.1) can be regarded as holomorphic functions on $U^- \times U'$ (i.e., independent of the $z'$ variables) and thus each $a_{\mu\nu}(z)$ extends holomorphically across $(0, 0')$. This extension must be independent of the $z'$ variables by the uniqueness theorem and hence $a_{\mu\nu}(z)$ extends holomorphically across the origin. This shows that $\hat{f}$ extends as a holomorphic correspondence across the origin. To show the existence of the sequence $\{z_\nu\}$ claimed above, let $\pi\colon F \to U$ be the natural projection and define

$$A = \{(z, z') \in F\colon \dim\, (\pi^{-1}(z))_{(z,z')} \geq 1\},$$

where $(\pi^{-1}(z))_{(z,z')}$ denotes the germ of the fiber over $z$ at $(z, z')$. Then $A$ is an analytic subset of $F$, and since $F$ contains the graph of $\hat{f}$ over $U^-$, it follows that the dimension of $A$ is at most $n - 1$. Since Lipschitz maps do not increase Hausdorff dimension, it follows that the Hausdorff dimension of $\pi(A)$ is at most $2n - 2$. Pick $p \in M \setminus \pi(A)$. The fiber $F \cap \pi^{-1}(p)$ is discrete and this shows that $\hat{f}$ extends as a holomorphic correspondence across $p$.

Finally, we show that $U'$ can be chosen so small that the projection $\pi'\colon F \to U'$ is also proper. Indeed, for $z' \in M'$, $\pi'^{-1}(z')$ is an analytic subset of $F$. Since $\pi$ is proper, it follows by Remmert's theorem that $\hat{F}^{-1}(z') = \pi \circ \pi'^{-1}(z')$ is an analytic set. The invariance property of Segre varieties yields $\hat{F}(Q_z \cap U) \subset Q'_{z'}$ for any $z \in \hat{F}^{-1}(z')$. Since $M$ is of finite type, the set $\cup_{z \in \hat{F}^{-1}(z')} Q_z$ has Hausdorff dimension $n$, and therefore cannot be mapped by $\hat{F}$ into $Q'_{z'}$ which has dimension $n-1$. This shows that projection $\pi'$ has discrete fibers on $M'$. It follows from the Cartan–Remmert theorem that there exists a neighbourhood $U'$ of $M'$ such that $\pi'$ has only discrete fibers, and therefore the projection $\pi'$ from $F$ to $U'$ will be proper.

This completes the proof of Theorem 1.1. □

## 5. Preservation of strata

Fix $p \in M$ and let $p'_1, p'_2, \ldots, p'_k \in \hat{f}(p) \subset M'$. Choose neighbourhoods $U, U'$ of $p, p'_1$ respectively and let $\hat{f}_1\colon U^- \to U'$ be a component of $\hat{f}$ such that $\hat{f}_1(p) = p'_1$. Then $\hat{f}_1$ extends as a holomorphic correspondence $F \subset U \times U'$ and to prove Theorem 1.2, it suffices to focus on $\hat{f}_1$, which will henceforth be denoted by $\hat{f}$. The following two general observations can be made in this situation. First, the branching locus $\hat{\sigma}$ of $\hat{F}$ is an analytic set in $U$ and the finite-type assumption on $M$ shows that the real dimension of $\hat{\sigma} \cap M$ is at most $2n - 3$. The branching locus of $\hat{f}$ denoted by $\sigma$, is contained in $\hat{\sigma} \cap U^-$. Second, the invariance property of Segre varieties in [DP1], [V] shows that $\hat{F}$, the extended correspondence, preserves the two components $U^{\pm}$. That is, after possibly re-labelling $U'^{\pm}$, it follows that $\hat{F}(U^{\pm}) \subset U'^{\pm}$ and $\hat{F}(M) \subset M'$. The same holds for $\hat{G} := \hat{F}^{-1}\colon U' \to U$.

*Proof of Theorem* 1.2. Let $p \in M^+$ and suppose that $\{\zeta'_j\} \in M'$ is a sequence converging to $p'_1$ with the property that the Levi form $\mathcal{L}_\rho$ restricted to the complex tangent space to $M$ at $\zeta'_j$ has at least one negative eigenvalue. Fix $\zeta'_{j_0} \in U'$ for some large $j_0$. By shifting $\zeta'_{j_0}$ slightly, we may assume that $\zeta'_{j_0} \notin \hat{\sigma}' \cup \hat{F}(M^0 \cap U)$, where $\hat{\sigma}'$ is the branching locus of $\hat{G}$, and at the same time retain the property of having at least one negative eigenvalue. Let $g_1$ be a locally biholomorphic branch of $\hat{G}$ near $\zeta'_{j_0}$. Then $g_1(\zeta'_{j_0})$ is clearly a pseudoconvex point and this contradicts the invariance of the Levi form. This shows that $\hat{f}(M^+) \subset M'^+$. The same arguments show that $\hat{f}(M^-) \subset M'^-$.

Let $p \in M^+ \cap M^0$ and suppose that $p'_1 \in M_s'^+$. The extending correspondence $\hat{F}: U \to U'$ satisfies the invariance property, namely $\hat{F}(Q_w) \subset Q'_{w'}$ for all $(w, w') \in (U \times U') \cap \mathrm{Graph}(\hat{F})$. But near $p'_1$, the Segre map $\lambda$ is injective and this shows that $\hat{F}$, and hence $\hat{f}$, is a single valued, proper holomorphic mapping, say $f: U \to U'$ with $f(p) = p'_1$. Two observations can be made at this stage: first, $f$ cannot be locally biholomorphic near $p$ due to the invariance of the Levi form. Second, if $V_f \subset U$ is the branching locus of $f$ defined by the vanishing of the Jacobian determinant of $f$, then $V_f$ intersects both $U^{\pm}$. Indeed, suppose that $V_f \cap U^- = \emptyset$. Choose a branch of $f^{-1}$ near some fixed point $a' \in U'^-$ and analytically continue it along all paths in $U'^-$ to get a well-defined mapping, say $g: U'^- \to U^-$. The analytic set $F \subset U \times U'$ extends $g$ as a correspondence and hence [DP2] $g$ is a well-defined holomorphic mapping in $U'$ and this must be the single valued inverse of $f$. Thus $f$ is locally biholomorphic near $p$ and this is a contradiction. The same argument works to show that $V_f$ must intersect $U^+$ as well. Note that $V_f \cap M$ has real dimension at most $2n - 3$. If $p \in \Gamma_1$, choose $U$ so small that $M^0 \cap U \subset \Gamma_1$. Then there exists $q \in \Gamma_1 \backslash (V_f \cap M)$ near $p$, where $f$ is locally biholomorphic. Thus $q$ is mapped locally biholomorphically to $f(q)$ which is a strongly pseudoconvex point and this is a contradiction. If $p \in \Gamma_3$, then again we shrink $U$ so that $M^0 \cap U \subset \Gamma_3$ and $(M \cap U) \backslash \Gamma_3 \subset M_s^+$. Then $f$ is locally biholomorphic near all points in $(M \cap U) \backslash \Gamma_3$ and therefore $V_f \cap U^-$ must cluster only along $\Gamma_3$. Since the CR dimension of $\Gamma_3 = n - 3 < (n-1) - 1$ which is one less than the dimension of $V_f$, it follows (Theorem 18.5 in [C]) that $\overline{V_f \cap U^-}$ is a closed, analytic set in $U$. Thus $V_f \cap U^-$ has two analytic continuations, namely $V_f$ and $\overline{V_f \cap U^-}$ and therefore they must be the same. This shows that $V_f$ cannot intersect $U^+$ which is a contradiction. The same argument works if $p \in \Gamma_4$, the only difference being that $\overline{V}_f \subset \overline{U^-}$ is analytic because of Shiffman's theorem. Thus if $p \in M^+ \cap M^0$, then $p'_1 \in M'^+ \cap M'^0$.

To study this further, suppose that $p \in M^+ \cap \Gamma_1$ and $p'_1 \in M'^+ \cap \Gamma'_2$. Choose $U, U'$ small enough so that $M^0 \cap U \subset \Gamma_1$ and $M'^0 \cap U' \subset \Gamma'_2$. Pick $q \in \Gamma_1 \backslash (\hat{\sigma} \cap M)$. Then $\hat{f}$ splits near $q$ into finitely many well-defined holomorphic mappings each of which extends across $q$. Moving $q$ slightly, if needed, on $\Gamma_1 \backslash (\hat{\sigma} \cap M)$, each of these holomorphic mappings are even locally biholomorphic near $q$. Working with one of these mappings, say $f_1$, it follows that $f_1(q) \notin M_s'^+$ due to the invariance of the Levi form. This means that $f_1(q) \in \Gamma'_2$. In the same way, all points in $\Gamma_1$ that are sufficiently near $q$ are mapped locally biholomorphically by $f_1$ to $\Gamma'_2$. This cannot happen as $\Gamma'_2$ has strictly smaller dimension than $\Gamma_1$. The same argument shows that $p'_1 \notin \Gamma'_3 \cup \Gamma'_4$. Hence $p'_1 \in M'^+ \cap \Gamma'_1$.

Suppose that $p \in M^+ \cap \Gamma_2$ and $p'_1 \in M'^+ \cap \Gamma'_1$. Considering $\hat{f}^{-1}: U' \to U$, the arguments used in the preceding lines show that this cannot happen. The case when $p'_1 \in \Gamma'_4$ can be dealt with similarly. Now suppose that $p'_1 \in \Gamma'_3$. As always, $U, U'$ will be small enough so that $M^0 \cap U \subset \Gamma_2$ and $M'^0 \cap U' \subset \Gamma'_3$. The arguments used above show that the cluster set of points in $M_s^+ \cap U$ is contained in $M_s'^+ \cap U'$ and hence $\hat{f}$ splits into finitely well-defined mappings each of which is locally biholomorphic near points in $M_s^+ \cap U$. This shows that the branching locus $\sigma \subset U^-$ of $\hat{f}$ clusters only along $\Gamma_2$. Then $\hat{F}(\sigma)$ is an analytic set of dimension $n - 1$ in $U'^-$. There are two cases to consider: first, if $\hat{F}(\sigma)$ clusters only along $\Gamma'_3$, then arguing as above, $\overline{\hat{F}(\sigma)} \subset \overline{U'^-}$ is a closed, analytic set in $U'$. The strong disk theorem shows that $p'_1$ is in the envelope of holomorphy of $\overline{U'^-}$ and this is a contradiction. Second, if there are points in $\overline{\hat{F}(\sigma)} \cap M_s'^+$, this means that $(\overline{\hat{F}(\hat{\sigma})} \cap M') \cap \Gamma'_3$

has real dimension at most $2n-4$. Pick $q' \in \Gamma'_3 \backslash (\overline{\hat{F}(\hat{\sigma})} \cap M')$ and note that the continuity of $\hat{f}$ implies that $\hat{f}^{-1}(q') \in M_s^+$. As seen above, this cannot happen. Thus $p'_1 \in \Gamma'_2$ or $M_s'^+$. Similar arguments show that if $p \in M^+ \cap \Gamma_3$ or $M^+ \cap \Gamma_4$, then $p'_1 \in M'^+ \cap \Gamma'_3$ or $M'^+ \cap \Gamma'_4$ respectively.

By reversing the roles of $U^\pm$, the same arguments used in the preceding paragraphs can be applied to show that $\hat{f}(M^- \cap M^0) \subset M'^- \cap M'^0$ with the preservation of $M^- \cap \Gamma_j$ for $j = 1, 3, 4$.

Finally, fix integers $i, j$ both at least 1 such that $i + j = n - 1$ and suppose that $p \in M_{i,j}$. Then there exists a point $p_0$, in $U$ (chosen so small that $M \cap U \subset M_{i,j}$) and arbitrarily close to $p$, where all branches of $\hat{f}$ are well-defined and locally biholomorphic. The invariance of the Levi form shows that the images of $p_0$ under any of the branches of $\hat{f}$ should all be in $M_{i,j}$. Note that each of these images is close to $p'_1$. This cannot happen if $p'_1$ is in $M'^+$, $M'^-$ or in $M'_{i',j'}$ for $i \neq i'$ and $j \neq j'$. The only possibility is that $p'_1$ is in the relative interior of $\overline{M}'_{i,j}$. The same argument works if $p$ is in the relative interior of $\overline{M}_{i,j}$. □

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