Entanglement in non-Hermitian quantum theory

ARUN K PATI

Institute of Physics, Sachivalaya Marg, Bhubaneswar 751 005, India
E-mail: akpati@iopb.res.in

Abstract. Entanglement is one of the key features of quantum world that has no classical counterpart. This arises due to the linear superposition principle and the tensor product structure of the Hilbert space when we deal with multiparticle systems. In this paper, we will introduce the notion of entanglement for quantum systems that are governed by non-Hermitian yet $\mathcal{PT}$-symmetric Hamiltonians. We will show that maximally entangled states in usual quantum theory behave like non-maximally entangled states in $\mathcal{PT}$-symmetric quantum theory. Furthermore, we will show how to create entanglement between two $\mathcal{PT}$ qubits using non-Hermitian Hamiltonians and discuss the entangling capability of such interaction Hamiltonians that are non-Hermitian in nature.

Keywords. Entanglement; non-Hermitian Hamiltonians; $\mathcal{PT}$ symmetry.

PACS Nos 03.65.-w; 03.65.Ca; 03.67.-a

1. Introduction

Entanglement is one of the weirdest features of quantum mechanics. In quantum world, entanglement arises naturally when we have more than one particle at our disposal. There is no classical analog of quantum entanglement and that makes it more fascinating than anything else in physics. Though, there is a burst of activity in understanding the nature of entanglement, the concept by itself is not new. It was introduced by Schrödinger way back in 1935 and he has realized that *"entanglement is the characteristic trait of quantum mechanics, the one that enforces its entire departure from classical lines of thought"* [1]. In the emerging field of quantum information theory, entanglement plays a major role. This is also a very useful resource in the sense that using entanglement one can do many things in the quantum world which are usually impossible in ordinary classical world. Some of these tasks are quantum computing [2], quantum teleportation [3], quantum cryptography [4], remote state preparation [5], quantum communication [6], and so on. The fundamental carrier of information in the quantum world is a quantum bit or qubit. A qubit is any two-state quantum mechanical system that can exist simultaneously in both 0 and 1. It differs from a classical bit in many ways. Some important differences are that we can neither copy a qubit [7,8] nor can we delete a qubit from two identical copies [9].

Arun K Pati

In standard quantum mechanics the observables are represented by Hermitian operators and the evolution of a closed system is governed by unitary evolution. In recent years there is a considerable interest in quantum systems governed by non-Hermitian Hamiltonians [10–15]. In this context it was discovered that there exist a class of non-Hermitian Hamiltonians that possess real eigenvalues provided they respect $\mathcal{PT}$ symmetry and the symmetry is unbroken. In $\mathcal{PT}$-symmetric quantum mechanics the usual condition of Hermiticity of operators is replaced by the condition of $C\mathcal{PT}$ invariance, where C stands for charge conjugation, $\mathcal{P}$ for parity and $\mathcal{T}$ for time reversal [10]. In standard quantum theory, $C\mathcal{PT}$ symmetry and Hermiticity conditions are the same. The $C\mathcal{PT}$ invariance condition is a natural extension of Hermiticity condition that allows reality of observables and unitary dynamics. Using the operator C, Bender *et al* [11] have introduced an inner product structure associated with $C\mathcal{PT}$ which can have positive definite norms for quantum states.

In this paper we would like to introduce the notion of entanglement for quantum systems described by non-Hermitian Hamiltonians. Usually, with non-Hermitian Hamiltonians one may think that there will be dissipation in the system and one may not be able to create entanglement. Nevertheless, we will show how to create entanglement with interaction Hamiltonians that are non-Hermitian in nature. Towards the end, we will address what is the entangling capability of non-Hermitian interaction Hamiltonians. Before doing so, first we will give basic definitions of entanglement in standard quantum theory. Then we will introduce the notion of $\mathcal{PT}$-symmetric quantum bit ($\mathcal{PT}$qubit) and the notion of quantum entanglement in this theory. Because of the $C\mathcal{PT}$ inner product, orthogonal quantum states in ordinary quantum theory become non-orthogonal quantum states in non-Hermitian quantum theory. This has several consequences which will be explored in detail. Also, we will show that if we take an Einstein–Podolsky–Rosen (EPR) entangled state (which is known to be a maximally entangled state) in ordinary theory, that becomes a non-maximally entangled state in non-Hermitian quantum theory. We hope that the entanglement in $\mathcal{PT}$-symmetric quantum theory may provide new ways of processing information in the quantum world. We conclude our paper with some implications and open questions.

2. Entanglement in usual quantum theory

Let us consider a composite system that consists of two or more subsystems. The Hilbert space of a composite system is the tensor product of the individual Hilbert spaces. In the case of bipartite quantum system we have the joint Hilbert space $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2$. If the state of a composite system cannot be written as $|\Psi\rangle_{12} = |\psi\rangle_1 \otimes |\phi\rangle_2$, then it is an entangled state. Suppose $\{|\psi_n\rangle\} \in \mathcal{H}_1^N$ and $\{|\phi_m\rangle\} \in \mathcal{H}_2^M$ are the basis in the respective Hilbert spaces, then $\{|\psi_n\rangle_1 \otimes |\phi_m\rangle_2\} \in \mathcal{H}_1^N \otimes \mathcal{H}_2^M$ is a basis in the joint Hilbert space. A general pure bipartite state can be expressed as

$$|\Psi\rangle_{12} = \sum_{nm=1}^{NM} C_{nm}|\psi_n\rangle_1 \otimes |\phi_m\rangle_2. \tag{1}$$

The above state cannot be written in product form for general amplitudes, hence it is an entangled state. Thus, a generic pure bipartite state is actually an entangled state. There is a beautiful theorem called the Schmidt decomposition theorem which tells that any pure bipartite entangled state can be written as

$$|\Psi\rangle_{12} = \sum_{i=1}^{\min(N,M)} \sqrt{p_i}|a_i\rangle_1 \otimes |b_i\rangle_2, \tag{2}$$

where $p_i \geq o$ are the Schmidt coefficients, $|a_i\rangle, |b_i\rangle$ are the Schmidt vectors, and $\sum_i p_1 = 1$. It can be seen that if we have more than one non-zero Schmidt coefficients in the bipartite state then it is an entangled state. The Schmidt coefficients are invariant under local unitary transformations.

Now, if we want to define the state of the individual systems, then they are given by partial traces, i.e.,

$$\rho_1 = \mathrm{tr}_2(|\Psi\rangle_{1212}\langle\Psi|) = \sum_i p_i|a_i\rangle\langle a_i|$$

and

$$\rho_2 = \mathrm{tr}_1(|\Psi\rangle_{1212}\langle\Psi|) = \sum_i p_i|b_i\rangle\langle b_i|. \tag{3}$$

Note that ρ_1 and ρ_2 are no longer pure, i.e., $\rho_i^2 \neq \rho_i$ ($i = 1, 2$). This is another indication that the original state of the composite system is an entangled state. If it is not, then after performing partial trace the reduced density matrices will be still pure. The existence of the Schmidt decomposition for bipartite states guarantees that the reduced density matrices have equal spectrum, though the eigenvectors can be different. It may be stated that if we have an entangled state of three or more particles then there does not exist a Schmidt decomposition. The necessary and sufficient conditions for the existence of Schmidt decomposition was found in ref. [16]. If A is a linear Hermitian operator acting on $\mathcal{H}_1$ and if B is a linear Hermitian operator acting on $\mathcal{H}_2$, then the expectation values of these local observables are given by

$$_{12}\langle\Psi|A \otimes I|\Psi\rangle_{12} = \mathrm{tr}_1(\rho_1 A)$$

and

$$_{12}\langle\Psi|I \otimes B|\Psi\rangle_{12} = \mathrm{tr}_2(\rho_2 B). \tag{4}$$

This suggests that the expectation values of the local observables are completely determined by local (reduced) density matrices.

For any pure bipartite state one can quantify how much entanglement is there in a given state. The entropy of any one of the reduced density matrix is a very good measure of entanglement for any bipartite state $|\Psi\rangle$ [17]. It is given by

$$E(\Psi) = -\mathrm{tr}_1(\rho_1 \log \rho_1) = -\mathrm{tr}_2(\rho_2 \log \rho_2) = -\sum_i p_i \log p_i. \tag{5}$$

This measure of entanglement satisfies the following properties:

(i) $E(\Psi) = 0$ iff $|\Psi\rangle$ is separable.

(ii) $E(\Psi)$ is invariant under local unitary transformations, i.e., $E(\Psi) = E(U_1 \otimes V_2 \Psi)$.

(iii) $E(\Psi)$ cannot increase under local operation and classical communications (LOCC).

(iv) The entanglement content of n copies of $|\Psi\rangle$ is additive, i.e., $E(\Psi^{\otimes n}) = nE(\Psi)$.

The above ideas can be illustrated with two-qubits and two-qudits (qudit is a d-dimensional Hilbert space system) entangled states. One famous entangled state which has been extensively used in quantum information theory is the Einstein–Podolsky–Rosen (EPR) [18] state $|\Psi^-\rangle$ which is given by

$$|\Psi^-\rangle = \frac{1}{\sqrt{2}}(|0\rangle|1\rangle - |1\rangle|0\rangle). \tag{6}$$

This has one unit of entanglement or one (entangled bit) ebit (because $\rho_1 = \rho_2 = I/2$). This is also a maximally entangled state for two-qubits. In fact, any state which is locally equivalent to $|\Psi^-\rangle$ will have one unit of entanglement. Similarly, in a higher-dimensional Hilbert space $(d \times d)$ a maximally entangled state for two-qudits can be written as

$$|\Phi\rangle = \frac{1}{\sqrt{d}} \sum_{i=0}^{d-1} |i\rangle \otimes |i\rangle \tag{7}$$

which has $E(\Phi) = \log d$ ebits. Here also any other state such as $(U_1 \otimes V_2)|\Phi\rangle$ will have $\log d$ ebits of entanglement, where U_1 and V_2 are local unitary operators acting on $\mathcal{H}_1$ and $\mathcal{H}_2$, respectively.

In information theory (both classical and quantum) there is a famous slogan due to Landauer: *"Information is physical"*. In the same spirit, I would like to say that *Entanglement is Physical*. This is justified for the following reasons: Entanglement can be created, stored and consumed using physical systems and physical operations. Entanglement is independent of any particular representation. For example, one ebit can be stored in two photons, two electrons or two atoms. As said before, entanglement is a resource. One can do informational work like quantum computing, quantum teleportation, remote state preparation, quantum cryptography and many more.

Since I am not going to review all the details of entanglement here, let me mention some recent trends in entanglement theory. For the last several years, characterization and quantification of entanglement of multiparticle system is a vigorous area of research [19]. Understanding how well one can generate entanglement is another direction scientists are exploring. Also, there is an upsurge of interest in understanding the dynamics of entanglement. In this context many authors have investigated entanglement rate and entangling capabilities of non-local Hamiltonians [20], entangling power of quantum evolutions [21], various entangling operations [22], and simulation of one Hamiltonian by another using only local operations [23] and so on.

3. Non-Hermitian quantum theory

In this section we will give the basic formalism that is necessary to develop the notion of entanglement in non-Hermitian quantum theory. Recently, there has been a great deal of interest in studying $\mathcal{PT}$-symmetric quantum theory [10–15]. The operator $\mathcal{P}$ transforms $x \to -x$ and $p \to -p$. The operator $\mathcal{T}$ is anti-unitary and its effect is to transform $x \to x, p \to -p$ and $i \to -i$. In the earlier formulation of $\mathcal{PT}$-symmetric quantum theory, it turned out that $\mathcal{PT}$-symmetric quantum theory introduced states which have negative norms. This had no clear interpretation. This was cured by introducing another operator C called conjugation operator [10,11]. This operator commutes with the Hamiltonian and the operator $\mathcal{PT}$. Also $C^2 = I$, which implies that it has eigenvalues ± 1.

Bender *et al* [10,11] have shown that non-Hermitian Hamiltonians can have real eigenvalues if they possess $\mathcal{PT}$ symmetry, i.e., $[H, \mathcal{PT}] = 0$ and the symmetry is unbroken (if all of the eigenfunctions of H are simultaneous eigenfunctions of the operator $\mathcal{PT}$). Hamiltonians having unbroken $\mathcal{PT}$ symmetry can define a unitary quantum theory. Unitarity can be shown by the fact that such Hamiltonians possess a new symmetry called conjugation C with $[C, H] = 0$ and $[C, \mathcal{PT}] = 0$.

Quantum theory that deals with non-Hermitian Hamiltonians and respects $C\mathcal{PT}$ symmetry may be called non-Hermitian quantum theory. One can formalize this by stating the following postulates:

(i) A quantum system is a three-tuple $(\mathcal{H}, H, \langle \cdot | \cdot \rangle_{C\mathcal{PT}})$, where $\mathcal{H}$ is a physical Hilbert space with the $C\mathcal{PT}$ inner product $\langle \cdot | \cdot \rangle_{C\mathcal{PT}}$ having a positive norm, and H is the non-Hermitian Hamiltonian.

(ii) The state of a system is a vector $|\psi\rangle$ in $\mathcal{H}$. For any two vectors the $C\mathcal{PT}$ inner product is defined as $\langle \psi | \phi \rangle_{C\mathcal{PT}} = \int \, \mathrm{d}x [C\mathcal{PT}\psi(x)]\phi(x)$.

(iii) The time evolution of state vector is unitary with respect to $C\mathcal{PT}$ inner product.

(iv) An observable can be a linear operator O, provided it is Hermitian with respect to the $C\mathcal{PT}$ inner product, i.e., $\langle \cdot | O \cdot \rangle_{C\mathcal{PT}} = \langle O \cdot | \cdot \rangle_{C\mathcal{PT}}$.

(v) If we measure an observable O, then the eigenvalues are the possible outcomes.

(vi) If measurement gives an eigenvalue O_n, the states make a transition to the eigenstate $|\psi_n\rangle$ and the probability of obtaining the eigenvalues O_n (say) in a state $|\psi\rangle$ is given by

$$p_n = \frac{|\langle \psi | \psi_n \rangle_{C\mathcal{PT}}|^2}{||\psi||_{C\mathcal{PT}} ||\psi_n||_{C\mathcal{PT}}}, \tag{8}$$

where $||\psi||_{C\mathcal{PT}} = \sqrt{\langle \psi | \psi \rangle_{C\mathcal{PT}}}$.

(vii) If we have two quantum systems $(\mathcal{H}_1, H_1, \langle \cdot | \cdot \rangle_{C\mathcal{PT}})$ and $(\mathcal{H}_2, H_2, \langle \cdot | \cdot \rangle_{C\mathcal{PT}})$, then the state of the combined system will live in a tensor product Hilbert space $\mathcal{H}_1 \otimes \mathcal{H}_2$.

Some remarks are in the order. In our effort to introduce entanglement we are using $C\mathcal{PT}$ inner product and the above postulates. However, one can also use the pseudo-Hermiticity approach [13,14] and do similar thing. Incidentally, the physical

observable was defined as the one that is invariant under $C\mathcal{PT}$ operation [11]. It was shown to be inconsistent with the dynamics of the theory [15]. Then, it was modified and suggested that an observable should satisfy $O^{\mathrm{T}} = (C\mathcal{PT})O(C\mathcal{PT})$, where O^{T} is the transposition of O. This guarantees that the expectation value of O in any state is real. However, this definition restricts that Hamiltonian be not only $\mathcal{PT}$-symmetric but also symmetric [24].

4. $\mathcal{PT}$-symmetric quantum bit

In standard quantum mechanics, we say that any two-state system is a quantum bit or a qubit. For example, an arbitrary state of a spin-$\frac{1}{2}$ particle like $|\Psi\rangle = \alpha|\uparrow\rangle + \beta|\downarrow\rangle$ can represent a qubit. Here, $|\uparrow\rangle$ and $|\downarrow\rangle$ are the eigenstates of the Pauli matrix σ_z. Similarly, if we have a two-level atom, then an arbitrary superposition of the ground state and the first excited state will be a qubit. In fact, any arbitrary superposition of two orthogonal states can represent a qubit. In the same vein, in $\mathcal{PT}$-symmetric quantum mechanics if we store information in any two-state system, then we call it a $\mathcal{PT}$-symmetric quantum bit or in short $\mathcal{PT}$qubit. In general, a $\mathcal{PT}$qubit is different from a qubit.

In non-Hermitian quantum theory a general two-state system will be described by a 2×2 Hamiltonian which respects $C\mathcal{PT}$ symmetry. Following ref. [10], this Hamiltonian is given by

$$H = \begin{pmatrix} re^{i\theta} & s \\ t & re^{-i\theta} \end{pmatrix}, \tag{9}$$

with r, s, t and θ all are real numbers. This Hamiltonian is non-Hermitian yet it has real eigenvalues whenever we have $st > r^2 \sin^2 \theta$. Also, this Hamiltonian is invariant under $C\mathcal{PT}$. Two distinct eigenstates of this Hamiltonian are given by

$$|\psi_+\rangle = \frac{1}{\sqrt{2}\cos\alpha} \begin{pmatrix} e^{i\alpha/2} \\ e^{-i\alpha/2} \end{pmatrix} \quad \text{and} \quad |\psi_-\rangle = \frac{1}{\sqrt{2}\cos\alpha} \begin{pmatrix} e^{-i\alpha/2} \\ -e^{i\alpha/2} \end{pmatrix}, \tag{10}$$

where α is defined through $\sin\alpha = (r/\sqrt{st})\sin\theta$. With respect to the $C\mathcal{PT}$ inner product (which gives a positive definite inner product) we have $\langle\psi_\pm|\psi_\pm\rangle_{C\mathcal{PT}} = 1$ and $\langle\psi_\pm|\psi_\mp\rangle_{C\mathcal{PT}} = 0$. The $C\mathcal{PT}$ inner product for any two states of a $\mathcal{PT}$qubit is given by

$$\langle\psi|\phi\rangle = [(C\mathcal{PT})|\psi\rangle] \cdot \phi, \tag{11}$$

where $\langle\psi|$ is the $C\mathcal{PT}$ conjugate of $|\psi\rangle$. In the two-dimensional Hilbert space, the operator C is given by

$$C = \frac{1}{\sqrt{2}\cos\alpha} \begin{pmatrix} i\sin\alpha & 1 \\ 1 & -i\sin\alpha \end{pmatrix}. \tag{12}$$

The operator $\mathcal{P}$ is unitary and is given by

$$\mathcal{P} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{13}$$

Since the eigenstates $|\psi_\pm\rangle$ of the non-Hermitian Hamiltonian H span the two-dimensional Hilbert space, one can encode one bit of information in these orthogonal states. An arbitrary state can be represented as the superposition of these orthogonal states

$$|\Psi\rangle = \alpha|\psi_+\rangle + \beta|\psi_-\rangle = \alpha|0_{C\mathcal{PT}}\rangle + \beta|1_{C\mathcal{PT}}\rangle. \tag{14}$$

Thus, any arbitrary superposition of two orthogonal states of $\mathcal{PT}$-invariant Hamiltonian will be called $\mathcal{PT}$-quantum bit or $\mathcal{PT}$ qubit. In fact, any linear superposition of two orthogonal states of an observable O in $\mathcal{PT}$-symmetric quantum theory can represent a $\mathcal{PT}$ qubit.

5. Entanglement in non-Hermitian theory

Entanglement is one of the most important feature of quantum world [18]. As noted earlier, when we have more than one qubit then the state of the composite system may be found in an entangled state that has no classical analog. Now, in $\mathcal{PT}$-symmetric quantum theory we will have similar feature whenever we have more than one $\mathcal{PT}$ qubit. In this section, we introduce these basic notions.

Suppose we have two quantum systems with non-Hermitian Hamiltonians H_1 and H_2, where

$$H_1 = \begin{pmatrix} re^{i\theta} & s \\ s & re^{-i\theta} \end{pmatrix} \quad \text{and} \quad H_2 = \begin{pmatrix} r'e^{i\theta'} & s' \\ s' & r'e^{-i\theta'} \end{pmatrix}. \tag{15}$$

Let $\{|\psi_\pm\rangle\} \in \mathcal{H}_1$ and $\{|\psi'_\pm\rangle\} \in \mathcal{H}_2$ are the eigenfunctions of the Hamiltonians H_1 and H_2, respectively. Now, the state of the combined system will live in $\mathcal{H}_1 \otimes \mathcal{H}_2$ which is spanned by $\{|\psi_+\rangle \otimes |\psi'_+\rangle, |\psi_+\rangle \otimes |\psi'_-\rangle, |\psi_-\rangle \otimes |\psi'_+\rangle, |\psi_-\rangle \otimes |\psi'_-\rangle\}$. If the combined state cannot be written as $|\Psi\rangle = |\psi\rangle \otimes |\phi\rangle = |\psi\rangle|\phi\rangle$, then it is entangled. A general state of two $\mathcal{PT}$ qubit can be expanded using the joint basis in $\mathcal{H}_1 \otimes \mathcal{H}_2$ as

$$|\Psi\rangle = a|\psi_+\rangle \otimes |\psi'_+\rangle + b|\psi_+\rangle \otimes |\psi'_-\rangle + c|\psi_-\rangle \otimes |\psi'_+\rangle + d|\psi_-\rangle \otimes |\psi'_-\rangle. \tag{16}$$

For general values of the complex amplitudes a, b, c and d this is an entangled state. However, if $\frac{a}{b} = \frac{c}{d} = k$, then $|\Psi\rangle$ is not entangled. Now, we can quantify the entanglement content in $|\Psi\rangle$. It is given by the entropy of the reduced state of any one of the subsystem, i.e.,

$$E(\Psi) = -\lambda_+ \log \lambda_+ - \lambda_- \log \lambda_-, \tag{17}$$

where $\lambda_\pm = \frac{1}{2}(1 \pm \sqrt{X})$ and $X = 1 - 4[(|a|^2 + |b|^2)(|c|^2 + |d|^2) - |(ac^* + bd^*)|^2]$. For $\frac{a}{b} = \frac{c}{d} = k$, $E(\Psi) = 0$, as expected.

Now, the $C\mathcal{PT}$ inner product on the Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_1$ can be used to define the inner product on $\mathcal{H}_1 \otimes \mathcal{H}_1$. For any two arbitrary vectors $|\Psi\rangle, |\Phi\rangle \in \mathcal{H}_1 \otimes \mathcal{H}_2$, we define the inner product between them as

$$\langle\Psi|\Phi\rangle_{CPT} = [(CPT)\otimes(CPT)|\Psi\rangle]\cdot|\Phi\rangle. \tag{18}$$

Using this inner product we can calculate relevant physical quantities for the composite system under consideration.

One can generalize the notion of entanglement for more than two PT qubits. If we have n-PT qubits with individual Hamiltonians as $H_i(i = 1, 2, \ldots, n)$ with respective eigenbasis $\{|\psi_{\pm i}\rangle\}$, then the joint Hilbert spaces will be $\mathcal{H}_1\otimes\mathcal{H}_2\cdots\otimes\mathcal{H}_n$. If a joint state cannot be written as $|\psi\rangle_1\otimes|\phi\rangle_2\cdots\otimes|\chi\rangle_n$, then it will be an entangled state. A general n-PT qubit state can be written as

$$|\Psi\rangle = \sum_{k=0}^{2^n-1} \alpha_k|X_k\rangle, \tag{19}$$

where $|X_k\rangle$ is an n-bit string of all possible combinations of $|\psi_\pm\rangle$. Such a state will be generically an entangled state. However, in this paper we are not going to dwell upon multi-PT qubit systems.

In general, if we have two subsystems with non-Hermitian Hamiltonians in higher dimension ($\mathcal{H}^d\otimes\mathcal{H}^d$), then we can also introduce the notion of entanglement. A general state of two PT-symmetric quantum systems can be written as (note that for non-Hermitian quantum systems also we can write a Schmidt decomposition theorem)

$$|\Psi\rangle = \sum_i \sqrt{\lambda_i}|\psi_i\rangle\otimes|\phi_i\rangle. \tag{20}$$

Now the reduced states of the PT-symmetric particles 1 and 2 will be different if we calculate the partial traces in usual quantum theory and in non-Hermitian quantum theory. Because the inner products in ordinary and PT-symmetric quantum theory are different, the partial traces will also be different. For example, the reduced density matrix for particle 1 calculated in non-Hermitian quantum theory will be

$$\begin{aligned}
\rho_1 &= \sum_{ij} \sqrt{\lambda_i\lambda_j}|\psi_i\rangle\langle\psi_j|\mathrm{tr}_2(|\phi_i\rangle\langle\phi_j|)\\
&= \sum_{ij} \sqrt{\lambda_i\lambda_j}|\psi_i\rangle\langle\psi_j|[(CPT)|\phi_j\rangle]\cdot|\phi_i\rangle\\
&= \sum_i \lambda_i|\psi_i\rangle\langle\psi_i|.
\end{aligned} \tag{21}$$

But if we calculate the reduced density matrix of particle 1 in the usual quantum theory, then we will have

$$\rho_1 = \sum_{ij} \sqrt{\lambda_i\lambda_j}|\psi_i\rangle\langle\psi_j|\mathrm{tr}_2(|\phi_i\rangle\langle\phi_j|) = \sum_{ij} \sqrt{\lambda_i\lambda_j}|\psi_i\rangle\langle\psi_j|\langle\phi_j|\phi_i\rangle. \tag{22}$$

This is no more in diagonal form because $\langle\phi_j|\phi_i\rangle \neq \delta_{ij}$ in the usual sense. Similarly, one can check that the reduced density matrix of particle 2 will be different in two theories. The density for particle 2 in non-Hermitian theory will be

Entanglement in non-Hermitian quantum theory

$$\rho_2 = \sum_{ij} \sqrt{\lambda_i \lambda_j} |\phi_i\rangle\langle\phi_j| \text{tr}_1(|\psi_i\rangle\langle\psi_j|)$$

$$= \sum_{ij} \sqrt{\lambda_i \lambda_j} |\phi_i\rangle\langle\phi_j| \, [(C\mathcal{P}\mathcal{T})|\psi_j\rangle] \cdot |\psi_i\rangle$$

$$= \sum_i \lambda_i |\phi_i\rangle\langle\phi_i|. \tag{23}$$

But in the usual quantum theory, we will have

$$\rho_2 = \sum_{ij} \sqrt{\lambda_i \lambda_j} |\phi_i\rangle\langle\phi_j| \text{tr}_1(|\psi_i\rangle\langle\psi_j|) = \sum_{ij} \sqrt{\lambda_i \lambda_j} |\phi_i\rangle\langle\phi_j|\langle\psi_j|\psi_i\rangle. \tag{24}$$

As a consequence, the entanglement content of a bipartite state depends on the inner product being used to calculate the partial traces. In other words, $E(\Psi) = S(\rho_i)$ $(i = 1, 2)$ in usual quantum theory is not equal to $E(\Psi) = S(\rho_i)$ $(i = 1, 2)$ in the non-Hermitian quantum theory.

To illustrate the above idea, we can define a singlet state for two $\mathcal{P}\mathcal{T}$ qubits as

$$|\Psi^-_{C\mathcal{P}\mathcal{T}}\rangle = \frac{1}{\sqrt{2}}(|\psi_+\rangle|\psi_-\rangle - |\psi_-\rangle|\psi_+\rangle). \tag{25}$$

In $\mathcal{P}\mathcal{T}$-symmetric quantum theory, the entanglement content of $|\Psi^-_{C\mathcal{P}\mathcal{T}}\rangle$ is given by $E(\Psi^-_{C\mathcal{P}\mathcal{T}}) = 1$. Note that this is not the usual spin singlet $|\Psi^-\rangle$. This is because the entanglement content of $|\Psi^-\rangle$ in non-Hermitian quantum theory will be different.

This is one interesting aspect here. The singlet state in ordinary quantum theory has entanglement equal to one whereas in $\mathcal{P}\mathcal{T}$-symmetric quantum theory it will be less than one. Similarly, a singlet state in $\mathcal{P}\mathcal{T}$-symmetric quantum theory will have entanglement equal to one whereas in ordinary theory it will be less than one. This is because of the different nature of inner products in ordinary and non-Hermitian quantum theories. To see this clearly, let us consider the entangled state of spin-singlet in ordinary quantum theory. If we want to know the entanglement content in $\mathcal{P}\mathcal{T}$-symmetric quantum theory then we have to calculate the von Neumann entropy of the reduced density matrix in $\mathcal{P}\mathcal{T}$-symmetric theory. The reduced density matrix for particle 1 in non-Hermitian quantum theory is given by

$$\rho_1 = \text{tr}_2(|\Psi^-\rangle\langle\Psi^-|) = \frac{1}{2}[|0\rangle\langle0|\langle1|1\rangle_{C\mathcal{P}\mathcal{T}} - |0\rangle\langle1|\langle0|1\rangle_{C\mathcal{P}\mathcal{T}}$$

$$-|1\rangle\langle0|\langle1|0\rangle_{C\mathcal{P}\mathcal{T}} + |1\rangle\langle1|\langle0|0\rangle_{C\mathcal{P}\mathcal{T}}], \tag{26}$$

where the $C\mathcal{P}\mathcal{T}$ inner products are given by $\langle0|0\rangle_{C\mathcal{P}\mathcal{T}} = \langle1|1\rangle_{C\mathcal{P}\mathcal{T}} = 1/\cos\alpha$, $\langle0|1\rangle_{C\mathcal{P}\mathcal{T}} = i\tan\alpha$ and $\langle1|0\rangle_{C\mathcal{P}\mathcal{T}} = -i\tan\alpha$. Using this, the reduced density matrix for particle 1 is given by

$$\rho_1 = \text{tr}_2(|\Psi^-\rangle\langle\Psi^-|) = \frac{1}{2\cos^2\alpha} \begin{pmatrix} 1 + \sin^2\alpha & -2i\sin\alpha \\ 2i\sin\alpha & 1 + \sin^2\alpha \end{pmatrix}. \tag{27}$$

Note that ρ_1 is not normalized. In general, the reduced density matrices that come out of different inner product definitions are not normalized automatically. This is

because a normalized state in the sense of usual inner product is not normalized in the sense of $C\mathcal{PT}$ inner product. But, we can define a normalized density matrix $\tilde{\rho}_1 = \rho_1/\mathrm{Tr}\,\rho_1$, so that

$$\tilde{\rho}_1 = \frac{1}{2}\begin{pmatrix} 1 & -2i\sin\alpha \\ 2i\sin\alpha & 1 \end{pmatrix}. \tag{28}$$

The eigenvalues of the density matrix $\tilde{\rho}_1$ are given by $\lambda_\pm = \frac{1}{2}(1 \pm 2\sin\alpha)$. Now, the entanglement content of the usual singlet in $\mathcal{PT}$-symmetric quantum theory is given by

$$\begin{aligned} E(\Psi^-) &= -\lambda_1\log\lambda_1 - \lambda_2\log\lambda_2 \\ &= -\frac{1}{2}(1 + 2\sin\alpha)\log\frac{1}{2}(1 + 2\sin\alpha) \\ &\quad -\frac{1}{2}(1 - 2\sin\alpha)\log\frac{1}{2}(1 - 2\sin\alpha) \neq 1. \end{aligned} \tag{29}$$

This shows that if an entangled state in ordinary theory has one unit of entanglement, in non-Hermitian quantum theory it will have less than one unit of entanglement. This is the effect of non-Hermiticity on the quantum entanglement. In the Hermitian limit ($\alpha = 0$), $E(\Psi^-) = 1$. Similarly, one can check that the maximally entangled state $|\Psi^-_{C\mathcal{PT}}\rangle = \frac{1}{2}(|\psi_+\rangle|\psi_-\rangle - |\psi_-\rangle|\psi_+\rangle)$ of non-Hermitian quantum theory will have less than one unit of entanglement in ordinary quantum theory. One implication of such an effect is that if Alice and Bob share an EPR entangled state generated by $\mathcal{PT}$-symmetric quantum world, then they cannot use that for quantum teleportation in ordinary world, because, perfect quantum teleportation requires one ebit of entanglement.

6. Generation of entanglement with non-Hermitian Hamiltonian

We know that entanglement can be created between two systems via some interaction. In standard quantum theory, interactions are described by Hermitian Hamiltonians. One might think that with non-Hermitian Hamiltonians, one may tend to destroy entanglement. However, here we show how to create entanglement with such non-Hermitian Hamiltonians.

A general Hamiltonian for two particles in $\mathcal{PT}$-symmetric quantum theory is given by $H = H_1 \otimes I_2 + I_1 \otimes H_2 + H_{12}$, where H_1, H_2 and H_{12} could be non-Hermitian but respect $\mathcal{PT}$ symmetry. Total Hamiltonian must satisfy $[H, \mathcal{PT} \otimes \mathcal{PT}] = 0$. If $|\Psi(0)\rangle = |\psi(0)\rangle \otimes |\phi(0)\rangle$ evolves to $|\Psi(t)\rangle$ under the action of this non-local Hamiltonian, then the state at a later time could be entangled, i.e.,

$$|\Psi(t)\rangle = \mathrm{e}^{-iHt}|\psi(0)\rangle \otimes |\phi(0)\rangle \neq |\psi(t)\rangle \otimes |\phi(t)\rangle. \tag{30}$$

An important question is what is the best way to exploit the interaction to produce entanglement? First we will give a simple non-local Hamiltonian that is capable of creating entanglement. Consider an interacting Hamiltonian given by

$$H = \begin{pmatrix} re^{i\theta} & s \\ s & re^{-i\theta} \end{pmatrix} \otimes \begin{pmatrix} r'e^{i\theta'} & s' \\ s' & r'e^{-i\theta'} \end{pmatrix} \tag{31}$$

which satisfies $[H, \mathcal{PT} \otimes \mathcal{PT}] = 0$. Using the Pauli matrices we can write H as

$$H = \left[r\cos\theta I + \frac{\omega}{2}\sigma \cdot \mathbf{n} \right] \otimes \left[r'\cos\theta' I + \frac{\omega'}{2}\sigma \cdot \mathbf{n}' \right], \tag{32}$$

where $\mathbf{n} = (2/\omega)(s, 0, ir\sin\theta)$, $\omega = 2s\cos\alpha$, similarly for $\mathbf{n}'$ and ω'. This interaction Hamiltonian consists of local and non-local terms. To see this we write it explicitly as

$$H = rr'\cos\theta\cos\theta'(I \otimes I) + r\cos\theta\frac{\omega'}{2}(I \otimes \sigma \cdot \mathbf{n}')$$

$$+ r'\cos\theta'\frac{\omega}{2}(\sigma \cdot \mathbf{n} \otimes I) + \frac{\omega\omega'}{4}(\sigma \cdot \mathbf{n} \otimes \sigma \cdot \mathbf{n}'). \tag{33}$$

In the above expression, first, second and third terms are local terms. We know that the local terms cannot create entanglement, and so they can be transformed away. The only term which is capable of creating entanglement is $(\omega\omega'/4)(\sigma \cdot \mathbf{n} \otimes \sigma \cdot \mathbf{n}')$. Therefore, the entangling evolution operator is given by

$$U(t) = \exp\left[-i\frac{\omega\omega't}{4}(\sigma \cdot \mathbf{n} \otimes \sigma \cdot \mathbf{n}') \right]$$

$$= \cos\frac{\omega\omega't}{4}I - i\sin\frac{\omega\omega't}{4}(\sigma \cdot \mathbf{n} \otimes \sigma \cdot \mathbf{n}'). \tag{34}$$

If the initial state of two $\mathcal{PT}$ qubit $|\Psi(0)\rangle = |0\rangle \otimes |0\rangle$, then at a later time t the state is given by

$$|\Psi(t)\rangle = e^{-i\frac{\omega\omega't}{4}(\sigma \cdot \mathbf{n} \otimes \sigma \cdot \mathbf{n}')}|0\rangle \otimes |0\rangle$$

$$= \alpha(t)|0\rangle|0\rangle + \beta(t)|0\rangle|1\rangle + \gamma(t)|1\rangle|0\rangle + \delta(t)|1\rangle|1\rangle, \tag{35}$$

where

$$\alpha(t) = \cos\left(\frac{\omega\omega't}{4}\right) + i\sin\left(\frac{\omega\omega't}{4}\right)\frac{4}{\omega\omega'}rr'\sin\theta\sin\theta',$$

$$\beta(t) = \frac{4}{\omega\omega'}\sin\left(\frac{\omega\omega't}{4}\right)s'r\sin\theta,$$

$$\gamma(t) = \frac{4}{\omega\omega'}\sin\left(\frac{\omega\omega't}{4}\right)sr'\sin\theta',$$

and

$$\delta(t) = -i\frac{4ss'}{\omega\omega'}\sin\left(\frac{\omega\omega't}{4}\right).$$

It is clear that for the above values of the amplitudes $|\Psi(t)\rangle$ is indeed an entangled state. Note that $|\Psi(t)\rangle$ is not normalized as the initial state that we have chosen is also not normalized (under $C\mathcal{PT}$ inner product).

7. Entangling capability of non-Hermitian Hamiltonians

Given an interaction Hamiltonian, what is the most efficient way of entangling particles? For Hermitian interaction Hamiltonians it is known that [20] (i) it is better to start with initial entangled state, (ii) the best initial entanglement is independent of the physical process, (iii) one can improve the capability if we allow fast local operations and (iv) in some cases, the capability improves by using ancillas. Now the question is whether similar facts hold for non-Hermitian Hamiltonians that respect $\mathcal{PT}$ symmetry? In this section, we will define the entanglement rate for non-Hermitian Hamiltonians. But we do not yet know if all these hold for non-Hermitian case. It is plausible that the above facts may still hold.

Let an initial state $|\Psi(0)\rangle$ evolves to $|\Psi(t)\rangle$ via an interaction Hamiltonian H which is non-Hermitian. Now, $|\Psi(t)\rangle$ can be entangled and the ability to create entanglement depends on the nature of interaction and on the initial state. To quantify the entanglement production, define the entanglement rate $\Gamma(t) = \mathrm{d}E(t)/\mathrm{d}t$, where $E(t) = E(\Psi)$ is the entanglement measure for the state $|\Psi(t)\rangle$. For example, the entanglement measure can be the von Neumann entropy of the reduced density matrix.

Let the state of two $\mathcal{PT}$ qubits at time t is

$$|\Psi(t)\rangle = \sqrt{\lambda_1}(t)|a_1(t)\rangle|b_1(t)\rangle + \sqrt{\lambda_2}(t)|a_2(t)\rangle|b_2(t)\rangle \tag{36}$$

with $\langle a_1(t)|a_2(t)\rangle_{CPT} = \langle b_1(t)|b_2(t)\rangle_{CPT} = 0$ and $\lambda_1 + \lambda_2 = 1$. The amount of entanglement at time t is given by the entropy of the reduced density matrix (with $\lambda = \lambda_1$)

$$E(\Psi(t)) = -\lambda(t)\log\lambda(t) - (1 - \lambda(t))\log(1 - \lambda(t)). \tag{37}$$

The entanglement rate is given by

$$\Gamma(t) = \frac{\mathrm{d}E(\Psi)}{\mathrm{d}\lambda}\frac{\mathrm{d}\lambda}{\mathrm{d}t}. \tag{38}$$

Using the Schrödinger equation we have

$$\frac{\mathrm{d}\lambda}{\mathrm{d}t} = 2\sqrt{\lambda(1 - \lambda)}\,\mathrm{Im}\langle a_1(t)|\langle b_1(t)|H|a_2(t)\rangle|b_2(t)\rangle_{CPT}. \tag{39}$$

Therefore, the entanglement rate is

$$\Gamma(t) = f(\lambda)|h(H, a_1, b_1)|, \tag{40}$$

where $f(\lambda) = 2\sqrt{\lambda(1 - \lambda)}(\mathrm{d}E/\mathrm{d}\lambda)$ and $h(H, a_1, b_1) = \langle a_1(t)|\langle b_1(t)|H|a_2(t)\rangle |b_2(t)\rangle_{CPT}$. Let $h_{\max}$ is the maximum value of $|h(H, a_1, b_1)|$. Then $h_{\max} = \max_{||a_1||,||b_1||=1}|\langle a_1(t)|\langle b_1(t)|H|a_2(t)\rangle|b_2(t)\rangle_{CPT}|$. As in the Hermitian case, if we solve for $\mathrm{d}\lambda/\mathrm{d}t$ we have $\lambda(t) = \sin^2(h_{\max}t + \phi_0)$, with $\lambda_0 = \sin^2(\phi_0)$. The evolution of entanglement is characterized by $h_{\max}$ which depends on the interaction Hamiltonian. Thus, for a given H, $h_{\max}$ measures the capability of creating entanglement. The entanglement rate satisfies

$$\Gamma(t) \leq \log[(1-\lambda)/\lambda]h_{\max}, \tag{41}$$

showing that the bound is proportional to the entangling capability for a given Schmidt number.

In future, we will investigate the entanglement rate for two entangled $\mathcal{PT}$-symmetric quantum systems in higher dimension and see if all known results for Hermitian case also hold for non-Hermitian case.

8. Conclusions

In this paper we have introduced the notion of entanglement for quantum systems described by non-Hermitian Hamiltonians. We have introduced the notion of $\mathcal{PT}$ qubit in the non-Hermitian quantum theory. States of qubit which are orthogonal in ordinary quantum theory become non-orthogonal in $\mathcal{PT}$-symmetric quantum theory and vice versa. More interestingly, the entanglement property of quantum states also change if we go from one theory to another. We have shown that a maximally entangled state that has von Neumann entropy equal to unity in the ordinary theory will have less entropy in $\mathcal{PT}$-symmetric quantum theory and vice versa. One implication is that if there is a source that emits maximally entangled state in the sense of ordinary theory and two observers are now in non-Hermitian quantum world then they cannot use the entangled state for quantum teleportation. This is so, because quantum teleportation requires one unit of entanglement and in the non-Hermitian quantum world the entanglement content is not equal to unity. We have illustrated how to create entanglement between two $\mathcal{PT}$ qubits using non-Hermitian Hamiltonians. Furthermore, we have discussed the entangling capability of interaction Hamiltonians that are non-Hermitian in nature. In future, we would like to apply these ideas in the context of entangled brachistocrone problem in $\mathcal{PT}$-symmetric quantum theory. We hope that the fascinating field of entanglement will take a new turn in the non-Hermitian quantum world. In particular, it will be interesting to see if $\mathcal{PT}$-symmetric entanglement can offer something new for quantum information processing and in sharpening our understanding of quantum channels.

Before ending, the following remarks are in order. Early formulation of $\mathcal{PT}$-symmetric quantum theory aimed to offer a genuine extension of usual quantum theory. Later, mathematical unitary equivalence has been shown between pseudo-Hermitian quantum theory and the usual quantum theory for single quantum systems [13]. However, entangled quantum systems may offer new insights into the nature of this equivalence. Because of the fact that the equivalence property of entangled states are different under joint unitary and under local unitary transformations, it is conjectured that under local unitary transformations (or more generally under LOCC paradigm) equivalence between pseudo-Hermitian and the usual quantum theory may not exist. One hopes to discover something new in such situations.

Arun K Pati

Note added

After the completion of this work, A Mostafazadeh informed the author in Mumbai during the International Conference on Non-Hermitian Hamiltonians in Quantum Physics (Jan. 13–16, 2009) about ref. [25], where compound systems have been described using pseudo-Hermitian quantum theory.

References

[1] E Schrödinger, *Cambridge Philosophical Society* **31**, 555 (1935); **32**, 446 (1936)
[2] C H Bennett and D DiVicenzo, *Nature* (*London*) **404**, 247 (2000)
[3] C H Bennett, G Brassard, C Crepeau, R Jozsa, A Peres and W K Wooters, *Phys. Rev. Lett.* **70**, 1895 (1993)
[4] *The physics of quantum information*: *Quantum cryptography, quantum teleportation, quantum computation* edited by D Bouwmeester, A K Ekert, A Zeilinger (Springer, 2001)
[5] A K Pati, *Phys. Rev.* **A63**, 014320 (2001)
[6] *Quantum communications and cryptography* edited by A V Sergienko (CRC Press, Taylor and Francis Group, 2006)
[7] W K Wootters and W H Zurek, *Nature* (*London*) **299**, 802 (1982)
[8] D Dieks, *Phys. Lett.* **A92**, 271 (1982)
[9] A K Pati and S L Braunstein, *Nature* (*London*) **404**, 164 (2000)
[10] C M Bender and S Boettcher, *Phys. Rev. Lett.* **80**, 5243 (1998)
[11] C M Bender, D C Brody and H F Jones, *Phys. Rev. Lett.* **89**, 270401 (2002)
[12] C M Bender, *Contemp. Phys.* **46**, 277 (2005)
[13] A Mostafazadeh, *J. Phys. A: Math. Gen.* **37**, 7081 (2003)
[14] A Mostafazadeh and A Batal, *J. Phys.* **A37**, 11645 (2004)
[15] A Mostafazadeh, *Czech. J. Phys.* **54**, 1125 (2004)
[16] A K Pati, *Phys. Lett.* **A278**, 118 (2000)
[17] S Popescu and D Rohrlich, *Phys. Rev.* **A56**, R3319 (1997)
[18] A Einstein, B Podolsky and N Rosen, *Phys. Rev.* **47**, 777 (1935)
[19] M B Plenio and S Virmani, *Quant. Inf. Comp.* **7**, 1 (2007)
[20] W Dür, G Vidal, J I Cirac, N Linden and S Popescu, *Phys. Rev. Lett.* **87**, 137901 (2001)
[21] P Zanardi, C Zalka and L Faoro, *Phys. Rev.* **A62**, 030301 (2001)
[22] J I Cirac, W Dür, B Kraus and M Lewenstein, *Phys. Rev. Lett.* **86**, 544 (2001)
[23] C H Bennett, J I Cirac, M S Leifer, D W Leung, N Linden, S Popescu and G Vidal, *Phys. Rev.* **A66**, 012305 (2002)
[24] A Mostafazadeh, arXiv:quant-ph/0407070 (2004)
[25] G Solarici and L Solombrino, *J. Phys. A: Math. Gen.* **42**, 055303 (2009)