

# The Wilsonian Revolution in Statistical Mechanics and Quantum Field Theory

*Gautam Mandal*

We describe how Wilson's concept of the renormalization group revolutionized our understanding of the physics of phase transitions, and of quantum field theory in general. We underline the key ideas of Wilson, based on earlier ideas of Landau, Ginzburg, Kadanoff and many others, which gave us an insight into the observed universality of diverse physical phenomena, and the remarkable efficacy of a few simple models to describe them.

## 1. Systems with Many Degrees of Freedom

In physics as well as in many other disciplines, we are often confronted with solving problems that involve many coupled degrees of freedom.

For example, let us consider the degrees of freedom involved in a block of solid. This has typically an Avogadro number ( $10^{23}$ ) of atoms. Each of the atoms has a number of electrons, and a nucleus consisting of the same number of protons and a similar number of neutrons. Each of these nucleons, in turn, are composites of quarks. It turns out that in discussing, e.g., the elastic or thermal properties of the solid, the nuclei can be regarded effectively as point particles with a certain mass and electric charge, which interact with electrons and other nuclei via electromagnetic interactions. The nuclear forces which bind the protons and neutrons together in a single nucleus, with a range of a femtometre, can essentially be ignored between the nuclei of different atoms which are at a distance of few nanometers.

This is an example of a situation where one has different Hamiltonians to describe phenomena at different scales. At atomic



Gautam Mandal works in the Department of Theoretical Physics, TIFR, Mumbai. He finished his PhD from TIFR in 1989 and did postdoctoral studies at the Institute for Advanced Studies, Princeton. His research and teaching involve various topics in quantum field theory, string theory, gravitational physics and statistical mechanics.

### Keywords

Ferromagnets, Ising model, quantum field theory, critical phenomena, renormalization, group scale invariance, universality.



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This is not always the case, though.

distances, the Hamiltonian involves charged particles (electrons and nuclei) interacting only through electromagnetic forces; at nuclear distances, both nuclear and electromagnetic forces are present. Carrying on to slightly larger distances, at the level of a few molecules, one has an effective van der Waals interaction between molecules which are regarded as electric dipoles. Zooming out to galactic scales, a Hamiltonian involving only gravitational forces often suffices.

### *Continuously Varying Scales*

The above discussion teaches us that simplifications can happen at larger distances in terms of an effectively new description of the degrees of freedom described by a Hamiltonian relevant for that scale. The separation of scales happens because of the presence of short range forces or different sizes of compact objects.

This is not always the case, though. Let us get back to the thermal properties of a solid. As remarked above, we can ignore the subnuclear degrees of freedom. Finding the energy levels then requires solving for the eigenvalues of a Hamiltonian involving an Avogadro number of atoms or molecules! The problem, at low energies, is not as hopeless as it sounds. In *Box 1*, it is shown that the degrees of freedom, expressed in terms of phonon modes, are approximately decoupled for low momenta. The mode-coupling, however, grows at higher momenta, giving us back a complicated interacting theory for wave-vectors of an order of the inverse lattice spacing.

***Elementary particles:*** The example of the phonon is qualitatively similar to elementary particles. Electromagnetic radiation consists of massless photons which interact with electrons and other charged particles. Weak nuclear interactions are mediated by the massive W and Z bosons between electrons, muons and neutrinos. The momentum modes of these particles range from zero up to some high value (we will discuss shortly what this means).



**Box 1. Phonons**

A block of solid can be modelled as a 3D lattice of mass points (atoms or molecules) connected by springs (in case of metals, there are of course the free electrons which we are ignoring at the moment). Call the position of the lattice sites  $\mathbf{x} = \mathbf{m}a_0$ , where  $a_0$  is the lattice spacing and  $\mathbf{m} = (m_1, m_2, m_3)$  with  $m_i$ 's integers. Let  $\phi(\mathbf{m})$  be the displacement from equilibrium of the atom at the lattice point  $\mathbf{m}$ . The potential energy of the 'spring' connecting two neighbouring points  $\mathbf{m}, \mathbf{m}'$  is of the form  $U(\phi(\mathbf{m}) - \phi(\mathbf{m}'))$ . For small fluctuations, it is enough to expand  $U$  to a quadratic order (Hooke's law regime); in this case the many-body Hamiltonian becomes decoupled in terms of the following 'normal coordinates' which are (discrete) Fourier modes,

$$\mathbf{u}(\mathbf{k}) = \sum_{\mathbf{m}} \phi(\mathbf{m}) e^{ia_0\mathbf{k}\cdot\mathbf{m}},$$

and behave like independent harmonic oscillators with angular frequency  $\omega(\mathbf{k})$ . The oscillator modes, also called 'phonons', describe propagation of sound of various momenta. The momenta  $\mathbf{k}$ , range from very low values (inverse of the system size), taken as zero in the thermodynamic limit, up to  $O(1/a_0)$ . The energy of the phonon,  $E(\mathbf{k}) = \hbar\omega(\mathbf{k})$ , vanishes, as  $|\mathbf{k}| \rightarrow 0$ ; hence the system is 'gapless'. For larger positional fluctuations of the atoms, the potential  $U$  needs to be expanded beyond the quadratic order (the 'springs' do not obey Hooke's law), leading to cubic and higher degree terms in the  $\phi(\mathbf{m})$ 's. A cubic term, rewritten in terms of the normal modes, is typically of the form,

$$\sum_{\mathbf{k}_1, \mathbf{k}_2} \mathbf{k}_1 \cdot \mathbf{u}(\mathbf{k}_1) \mathbf{k}_2 \cdot \mathbf{u}(\mathbf{k}_2) (\mathbf{k}_1 + \mathbf{k}_2) \cdot \mathbf{u}(-\mathbf{k}_1 - \mathbf{k}_2). \tag{1}$$

This represents a mode coupling between all phonon modes\*. The system remains gapless, as phonons can be identified as massless\*\* modes associated, by Goldstone's theorem, with spontaneous breaking of translational symmetry by the solid.

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\* Unlike the mode-couplings for ferromagnets, e.g., in (6), these couplings are unimportant at low momenta because of the momentum factors (they are irrelevant in the Wilsonian sense explained later), precluding any critical behaviour of interacting phonons.

\*\* Throughout this paper we would identify 'mass' of a system as the energy gap at zero momentum.

The interactions couple all momentum modes, like in the example of phonons. For massive intermediate particles such as W and Z bosons, at energies far lower than those masses, one can ignore their effects or incorporate them in a simple effective interaction as in the Fermi theory of beta decay. For long range forces such as electromagnetic or gravitational, such a method is not applicable.

Thus, both in the case of phonons and of massless elemen-



The physics of large distances is intricately mixed up here with the short distance physics, because of the mode-coupling.

tary particles, it seems that the physics of large distances (low energies) is intricately mixed up with the short distance physics, because of the mode-coupling described above. The other important class of systems with such properties, discussed next, involve the phenomenon of second order phase transitions where all length scales coexist up to arbitrarily high distance scales. This would seem to make such problems impossibly difficult to tackle. It is here that Wilson's work, building on earlier works of Landau, Ginzburg, Kadanoff, Gell-Mann, Low and others, provided the right technique of addressing these problems and led to revolutionary insights into quantum field theory and statistical mechanics.

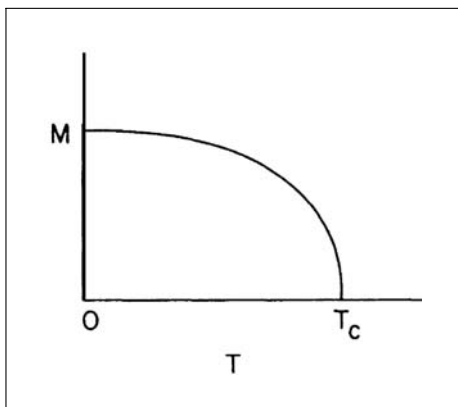
### *Second Order Phase Transitions*

Before continuing to describe these developments, it is important to introduce and explain one of the most important phenomenon that was a cornerstone in Wilson's work – the second order phase transition. We will describe two examples: the Curie transition in ferromagnets and the liquid–gas critical point [1].

**Ferromagnets:** In a ferromagnet, the atoms behave like small magnetic moments because of unpaired electrons, with mutual interactions that favour alignment. At high temperatures, the moments are randomly oriented because of thermal fluctuations and the system shows no magnetic order. However, as the temperature is brought down towards the Curie temperature, the aligning tendencies start to dominate more, and alignments occur over considerably large distances; the average such distance is called the correlation length  $\xi$  (measured by spin-spin correlation functions at a distance  $r$  which decay as  $e^{-r/\xi}$ ). At Curie temperature, aligned domains exist in all sizes and the correlation length diverges, as  $\xi \propto (T - T_c)^{-\nu}$ . As  $T$  goes below  $T_c$ , regions with one preferential alignment grow in size over those with other alignments, leading to a net magnetization  $\mathbf{M}$ . The temperature-dependence of  $|\mathbf{M}|$  is shown in *Figure 1*. Just below  $T_c$ , the magnetization is proportional to  $(T_c - T)^\beta$ . The exponents  $\nu, \beta$  are

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**Figure 1.** Magnetization [M] as a function of temperature [2].

called critical exponents. Experimentally these are given approximately by  $2/3$  and  $1/3$  for a wide variety of second order phase transitions (see *Table 1*).

**Liquid–Gas Critical Point:** In the case of the liquid–gas phase transition, such as boiling of water, the density of water changes

**Table 1.** Table of critical exponents (in this article we have discussed only  $\nu$  and  $\beta$ . The column ‘QFT’ refers to the Wilsonian calculation described in this article. ‘Lattice’ refer to high-temperature expansions for lattice statistical mechanics models. ‘Experiment’ refers to experiments on critical points of the systems alluded to.  $N$  refers to the symmetry of the order parameter (the quantity which distinguishes phases). Thus,  $N = 1$  for scalar order parameter (e.g., density of liquid);  $N = 2$  for two-dimensional vector (with planar rotational symmetry, e.g., for superfluids);  $N = 3$  for three-dimensional vector order parameter (e.g., magnetization) [4].

Exponent	Landau	QFT	Lattice	Experiment	
<i>N = 1 Systems:</i>					
$\gamma$	1.0	1.241 (2)	1.239 (3)	1.240 (7)	binary liquid
				1.22 (3)	liquid-gas
				1.24 (2)	$\beta$ -brass
$\nu$	0.5	0.630 (2)	0.631 (3)	0.625 (5)	binary liquid
				0.65 (2)	$\beta$ -brass
$\alpha$	0.0	0.110 (5)	0.103 (6)	0.113 (5)	binary liquid
				0.12 (2)	liquid-gas
$\beta$	0.5	0.325 (2)	0.329 (9)	0.325 (5)	binary liquid
				0.34 (1)	liquid-gas
$\eta$	0.0	0.032 (3)	0.027(5)	0.016 (7)	binary liquid
				0.04 (2)	$\beta$ -brass
<i>N = 2 Systems:</i>					
$\gamma$	1.0	1.316 (3)	1.32 (1)		
$\nu$	0.5	0.670 (3)	0.674 (6)	0.672 (1)	superfluid $^4\text{He}$
$\alpha$	0.0	-0.007 (6)	0.01 (3)	-0.013 (3)	superfluid $^4\text{He}$
<i>N = 3 Systems:</i>					
$\gamma$	1.0	1.386 (4)	1.40 (3)	1.40 (3)	EuO, EuS
				1.33 (3)	Ni
				1.40 (3)	RbMnF <sub>3</sub>
$\nu$	0.5	0.705 (3)	0.711 (8)	0.70 (2)	EuO, EuS
				0.724 (8)	RbMnF <sub>3</sub>
$\alpha$	0.0	-0.115 (9)	-0.09 (6)	-0.011 (2)	Ni
$\beta$	0.5	0.365 (3)	0.37 (5)	0.37 (2)	EuO, EuS
				0.348 (5)	Ni
				0.316 (8)	RbMnF <sub>3</sub>
$\eta$	0.0	0.033 (4)	0.041 (14)		

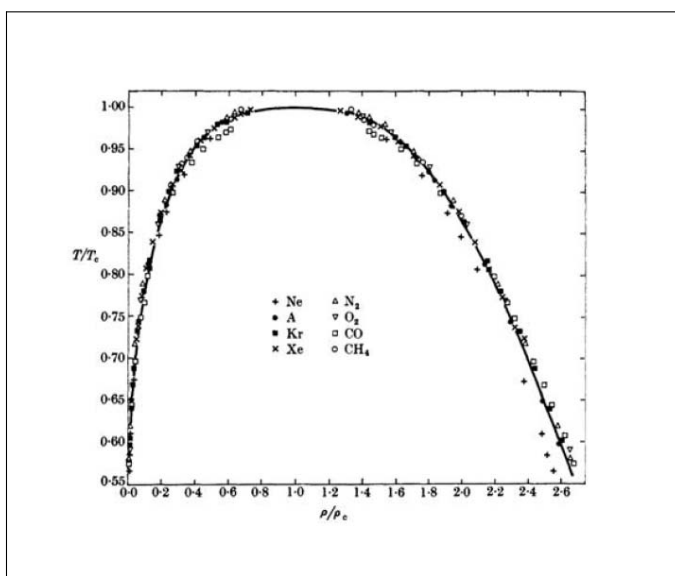


discontinuously to that of steam at the normal boiling point ( $100^{\circ}\text{C}$ ). However, at the critical point (214 atm pressure and  $374^{\circ}\text{C}$ ), the density of both phases coincide and the liquid meniscus disappears. Regions comprising regions of higher and lower density exist at all scales, which is spectacularly shown in the phenomenon of critical opalescence. Like in case of the ferromagnet, the correlation length again diverges. The correlation length exponent  $\nu$ , for a wide variety of critical liquid–gas transitions, is found to be  $2/3$ . Even more spectacularly, the density–temperature curve, plotted in units of the critical values, appear to coincide for a large number of liquids for a range of temperatures around the critical temperature (see *Figure 2*).

We are thus led to the remarkable phenomenon of universality near second order phase transitions amongst a wide variety of physical systems. A major theoretical challenge is to understand this universality and explain the critical exponents. This is what we undertake in the next section.

**The Kondo Model:** Another model which played a major role in Wilson’s work was the Kondo model, which deals with the magnetic impurities in a non-magnetic metal. These materials

**Figure 2.** Densities of coexisting liquid and gas phases of a variety of substances, plotted against temperature, with both densities and temperatures scaled to their value at the critical point, resulting in a striking superposition of all data on to a single curve regardless of the substance. (From: E A Guggenheim, *J. Chem. Phys.*, Vol.13, No.253, 1945, as reprinted in the *Resonance* article [1]).



are famous for the so-called Kondo effect, which is a logarithmic dependence of resistivity on the absolute temperature, leading to an apparent divergence at  $T = 0$ . The logarithmic dependence can eventually be traced to the fact that each magnetic impurity is coupled to electron–hole excitations of all wavelengths, from atomic scales to a very high scale. We will not elaborate on this further, except to say that Wilson provided a solution to this problem by bringing in insights from his thesis work on meson scattering off fixed nucleons. The method involved a simple use of Wilson’s RG transformations (to be discussed later).

## 2. Landau’s Mean Field Description and Thermodynamics

The general theme in the previous section was that systems exhibiting well-separated scales were amenable to different effective descriptions at different scales. Such a result does not immediately seem applicable to gapless systems with degrees of freedom at continuously varying energy scales. However, the apparent existence of a thermodynamic description for most systems, in terms of a few averaged variables (average energy or temperature, average density, etc.) leads one to hope that at very large distance scales it may be possible to completely average out smaller scale fluctuations of various physical quantities. Landau’s mean field description, as we describe below, is based on this idea. We will find that it generically works for finite correlation lengths, but fails near phase transitions, where fluctuations exist at all scales and the process of averaging proceeds *ad infinitum*; to treat these requires new insights and techniques embodied in Wilson’s renormalization group, as we will detail below.

Let us return to the ferromagnets to illustrate Landau’s mean field approach. It is helpful to keep in mind the Ising model (*Box 2*).

As for the Ising model, let us assume the magnetic moments to be aligned along a particular axis. We will consider, following Landau, a continuum magnetization variable  $M(\mathbf{x})$  which can be regarded as a local average of the atomic magnetic moments where the averaging is done over atomic distances. To a first approxima-



**Box 2. The Ising Model and Mean Field Approximation**

Consider a ferromagnet, modelled by a  $d$ -dimensional square lattice of  $N$  atoms (where  $d = 1, 2$  or  $3$ ). The lattice sites are labelled by integers  $\mathbf{m}$ , as in *Box 1*. For simplicity, we will assume that the magnetic moments (we will henceforth call them ‘spins’) are aligned along some particular axis (we will discuss the general case later). We will denote the spins as  $s(\mathbf{m})$  with possible values  $\pm 1$  (in appropriate units). The ferromagnetic interaction is modelled by the following Hamiltonian,

$$H = -J \sum_{\langle \mathbf{m}\mathbf{m}' \rangle} s(\mathbf{m})s(\mathbf{m}') - h \sum_{\mathbf{m}} s(\mathbf{m}), \quad J > 0,$$

where the first sum is over nearest neighbours, we have included an applied magnetic field  $h$ . This is called the Ising model.

The ‘mean field’, or the self-consistent field approximation to the Ising model assumes that each spin sees all its  $2d$  neighbouring spins as approximately equal to the thermodynamic average  $m = 1/N \sum_{\mathbf{m}} \langle s(\mathbf{m}) \rangle$ . This leads to the following decoupled Hamiltonian,

$$H_{mf} = -h_{total} \sum_{\mathbf{m}} s(\mathbf{m}), \quad h_{total} = 2dmJ + h.$$

For consistency,  $m$  must equal the thermodynamic average computed from this Hamiltonian. It is easy to see that this gives,

$$m = \tanh((2dmJ + h)/(kT)).$$

For  $h = 0$ , we have a solution  $m = 0$  if  $T > T_c = (2dJ)/k$  and  $\neq 0$  if  $T < T_c$ . For  $T \lesssim T_c$ , we get  $m \propto (T_c - T)^{1/2}$  giving a mean field exponent  $\beta_{mf} = 1/2$ .

It is easy to see that the self-consistent equation follows from a mean-field ‘free energy’,

$$F = N \left( \frac{m^2}{2kT} - \frac{1}{kT_c} \log \cosh \frac{mkT_c + h}{kT} \right) = N (rm^2 + um^4 + a_1hm + O(hm^3, m^5, h^2)),$$

where  $r \propto (T - T_c)$ ,  $u > 0$ ,  $a_1 < 0$ . It is not difficult to generalize this to the original vector magnetic moment. The free energy becomes,

$$F = N (r\mathbf{m}^2 + u(\mathbf{m}^2)^2 + a_1\mathbf{h}\cdot\mathbf{m} + \dots).$$

tion, we will also assume that one has also averaged over spatial fluctuations, leaving a constant magnetization  $M$ . The thermodynamic average of  $M$  is assumed to be obtained by classically minimizing a free energy function<sup>1</sup>  $\mathcal{F}(M)$  (compare with the mean field treatment of Ising model in *Box 2*). Assuming a symmetry between the two allowed directions of the magnetic field (in the

<sup>1</sup>This is defined as the conventional free energy  $F$  divided by  $kT$ .





absence of external magnetic fields), the free energy density will be of a generic form (up to fourth order)

$$\mathcal{F} = \text{Volume} \times (rM^2 + uM^4).$$

It is an easy exercise to find that the minimization of  $\mathcal{F}$  gives,

$$\overline{M} = 0, \quad r > 0; \quad \overline{M} = \pm \sqrt{-2r/u}, \quad r < 0. \quad (2)$$

Clearly  $r > 0$  reflects the paramagnetic range of temperatures  $T > T_c$ , where  $r < 0$  reflects temperatures  $T < T_c$ . A crucial assumption of Landau is that the parameters of free energy which are obtained by a partition sum of the kind  $\sum e^{-E/kT}$  cannot have non-analyticities in  $T$  except perhaps the obvious one at  $T = 0$ . With this assumption, and the fact that  $r = 0$  implies  $T = T_c$ , we must have,

$$r = \alpha(T - T_c), \quad \alpha > 0.$$

This gives

$$\overline{M} = (T_c - T)^{\beta_{\text{mf}}}, \beta_{\text{mf}} = 1/2,$$

which agrees with the mean field treatment of the Ising model in *Box 2*, as expected. The mean field result disagrees with experimental and theoretical results which indicate that  $\beta$  is about 1/3 (see the table of critical exponents in *Table 1*).

**Long Wavelength Modes: Landau–Ginzburg theory:** It is not necessary to assume in Landau’s theory that all fluctuations have been averaged out. Let us assume instead that only short wavelength fluctuations have been averaged out, and consider a magnetization function  $M(\mathbf{x})$  which has slow spatial variations (long wavelengths). In this case the above free energy generalizes to the Landau–Ginzburg form,

$$\mathcal{F} = \int d^3\mathbf{x}[(\nabla M)^2 + rM(\mathbf{x})^2 + uM(\mathbf{x})^4 - B(\mathbf{x})M(\mathbf{x})]. \quad (3)$$

where we have added a small local magnetic field  $B(\mathbf{x})$ . The gradient term represents the lowest rotationally invariant term involving derivatives, we will ignore higher derivative terms since they are small for long wavelength modes.



We will continue to assume that the magnetization  $M(\mathbf{x})$  is again obtained by minimizing the  $F$  above. Without the magnetic field, it is easy to see that the minimum is again given by (2). In the paramagnetic phase, a small magnetic field will shift this minimum from zero by a small amount  $\delta M$ , which will be given by the linearized variational equation,

$$-\nabla^2 \delta M + r \delta M = 2B(\mathbf{x}).$$

This is mathematically identical to Poisson's equation with a 'photon mass term'  $m^2 = r$ . The response function  $\frac{\partial \delta M(x)}{\partial B(y)}$  is then of the Yukawa form  $\propto e^{-m|x-y|}$ , leading to a correlation length,

$$\xi = 1/\sqrt{r} \propto (T - T_c)^{-\nu_{mf}}, \quad \nu_{mf} = 1/2.$$

The mean field exponent, again, is different from the approximate value  $2/3$  obtained from experiment and lattice calculations.

The generalization of these calculations to the original problem of vector-valued magnetization  $\mathbf{M}(\mathbf{x})$  is easy to work out, which involves the following Landau-Ginzburg free energy (see *Box 2*)

$$\mathcal{F} = \int d^3 \mathbf{x} [(\nabla \mathbf{M})^2 + r \mathbf{M}(\mathbf{x})^2 + u (\mathbf{M}(\mathbf{x})^2)^2 - \mathbf{B}(\mathbf{x}) \cdot \mathbf{M}(\mathbf{x})].$$

In the ferromagnetic phase ( $r < 0$ ) the non-zero magnetization picks up a specific direction  $\hat{\mathbf{e}}$ :  $\mathbf{M} = \sqrt{-2r/u} \hat{\mathbf{e}}$ . This amounts to a spontaneous symmetry breaking, leading to the presence of associated massless Goldstone bosons which are called magnons.

### 3. Wilson's Renormalization Group

Why does Landau theory fail to reproduce the correct features of a second order phase transition?

The main assumption in Landau theory was that the thermodynamic average of magnetization was obtained by minimizing the free energy function of a slowly varying magnetization variable  $M(\mathbf{x})$ . The free energy was assumed to be obtained by an averaging over short wavelength modes, which was supposed to be done away with once for all. As long as the correlation length  $\xi$  is



finite, such a statement is reasonable. One needs to compute the average of various quantities over fluctuations out to a distance  $\xi$  and for larger distances one can keep using those averages. The problem arises when the correlation length diverges (at the second order phase transition temperature); the system becomes gapless and there is no canonical notion of what is a short wavelength and what is a long wavelength. In other words, the process of averaging appears to go on forever.

In order to address the problem correctly, we need to understand in a quantitative way the ‘process of averaging over shorter wavelengths’. In *Box 3* we gave an explicit example of such a process for the one-dimensional Ising model.

**The Renormalization Group:** The example of the 1D Ising model teaches us that starting from a microscopic Hamiltonian at very short length scales, it is possible to construct new Hamiltonians at a slightly larger length scale so that the physics (e.g., the partition function and various correlations) remains unaltered. The sequence of Hamiltonians  $H_0 \rightarrow H_1 \rightarrow H_2 \rightarrow \dots$  at successive lattice spacings  $a_0, a_1, a_2, \dots$  satisfies a ‘group’ property. Let us say that we change the lattice spacing  $a \rightarrow \kappa a$ , and the corresponding change in the Hamiltonian is  $H \rightarrow f_\kappa(H)$ . Then it can be verified that  $f_{\kappa_1 \kappa_2}(H) = f_{\kappa_1}(f_{\kappa_2}(H))$ . In case of the 1D Ising model, the group property is easily verified from the fact that  $\tanh(\mathbf{J}(2^{n_1+n_2}a)) = \tanh(\mathbf{J}(a))^{n_1+n_2}$ .

The scale transformation between the effective Hamiltonians at various scales is called a ‘renormalization group’, or simply RG. The word ‘renormalization’, for scale transformation, has a historic origin in a similar procedure in the Quantum Field Theory embodied in the Gell-Mann–Low equation (see later).



**Box 3. RG Flow of Coupling in the 1D Ising Model**

Let us consider the  $d = 1$  version of the Ising model of *Box 2*. This describes a chain of  $N$  magnetic moments (spins), at lattice sites  $x = ia_0, i = 1, 2, \dots, N$ . For zero external magnetic field, the Hamiltonian is,

$$H_0 = -J_0 \sum_{i=1}^N s_i s_{i+1}, \quad J_0 > 0, \quad s_{N+1} \equiv s_1.$$

We have added a subscript 0 as a reminder that the lattice spacing is  $a_0$ .

Suppose we divide all  $N$  spins (assume  $N$  even) into blocks of spin pairs as in *Figure A*.

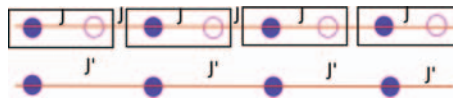


Figure A. Decimation of Ising chain: doing a partial sum over even spins (represented by hollow pink circles) leads to a new Hamiltonian for odd spins at a new lattice spacing  $a = 2a_0$ .

We will define a process of summing over each spin block to arrive at a new Hamiltonian describing an interaction between half the original number of spins, at a new lattice spacing  $a = 2a_0$ . The new Hamiltonian must ensure that the physical properties, e.g., the partition function, remains the same as that obtained from the old Hamiltonian. The process we will employ will be called decimation, where we will do a partial sum over all even spins. In the old partition function, the terms involving  $s_2$  are,

$$\sum_{s_2 = \pm 1} e^{J_0/(kT)[s_1 s_2 + s_2 s_3]}.$$

It is easy to evaluate this sum which turns out to be,

$$e^{J/(kT)[s_1 s_3] + K}, \quad \tanh(J/kT) = \tanh^2(J_0/kT), \quad K = \ln 2 + J/(kT),$$

corresponding to a new interaction strength (and a constant term). Doing this for every even spin, we arrive at a new Hamiltonian involving only the odd spins, at a new lattice spacing  $a = 2a_0$

$$H = -J \sum_{i=1,3,\dots} s_i s_{i+1} + \text{constant}.$$

In this simple example, the process of partial summing does not lead to new couplings, however, in more complicated models, additional terms involving, e.g. next to nearest neighbour, multilinear, etc., couplings are generated. In this model, the process of decimation can be continued *ad infinitum*, and by repeatedly using the above recursion formula (marked in blue above) between  $\mathbf{J}$  and  $\mathbf{J}_0$  (with  $\mathbf{J}$  the dimensionless coupling  $J/kT$ ), we get a flow of couplings  $\mathbf{J}(a_0) \rightarrow \mathbf{J}(2a_0) \rightarrow \dots \rightarrow \mathbf{J}(2^m a_0) \rightarrow \dots$  which is driven towards  $\mathbf{J} = 0$ :

Box 3. *Continued...*



Box 3. *Continued...*



Figure B. Flow of coupling in 1D Ising model (the coupling  $\mathbf{J}$  here is the dimensionless quantity  $J/(kT)$ ). Note the two fixed points of the recursion relation written in blue above. The flow is called a renormalization group (RG) flow (see text). The large distance property is controlled by the attractor fixed point, called the infra-red fixed point, at  $J/kT = 0$  which corresponds to a disordered phase.

**Fixed points:** Another important lesson from the 1D Ising model is that the RG flow has fixed points, which are the limiting points as  $L/a \rightarrow \infty$ . The fixed points determine the long distance behaviour of the theory. If we are precisely at the fixed point (which corresponds to the second order phase transition point), the correlations are given by power laws since there are no scales in the theory ( $\xi = \infty$ ).

RG flow has fixed points, which determine the long distance behaviour of the theory.

Let us see how to apply these ideas to the problem of the ferromagnet (in the next section we will apply this idea to quantum field theory). Suppose we have performed an averaging over atomic fluctuations to a scale  $a$ , enough to recover rotational invariance. As in the case of Landau theory, let us call the local magnetization  $M(\mathbf{x})$ . However, unlike Landau theory, the partial sum over these variables from the distance scale  $a$  to the scale  $L$  of experiments, is yet to be done. Since we do not know the microscopic details at the atomic scale (nor do we need to care, it will turn out), let us write the most general effective Hamiltonian at the scale  $a$  consistent with rotational invariance of space and the reflection invariance  $M \rightarrow -M$  (this is the remnant of rotational invariance of magnetic moments when restricted to a single axis)

$$H = \int d^d \mathbf{x} \left[ (\nabla M)^2 + g_{4,2} (\nabla^2 M)^2 + g_{4,4} ((\nabla M)^2)^2 + g_{0,2} M^2 + g_{0,4} M^4 + g_{0,6} M^6 + \dots \right], \quad (4)$$

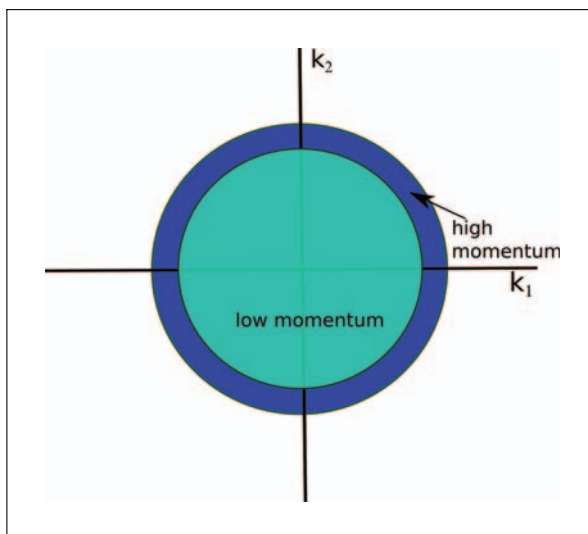
where the omitted terms involve higher powers of derivatives and



$M$ . We have allowed for arbitrary  $d$ ; for classical statistical mechanics, the physical values are  $d = 1, 2, 3$ , whereas for quantum statistical mechanics or QFT,  $d = 1, 2, 3, 4$  (for computational purposes, one can treat  $d$  as a parameter, as Wilson did). In principle, one can have literally any number of terms in  $H$  at this scale, depending on the precise microscopic interactions – shape and size of lattice, nature of the exchange interaction in the particular metal, etc. The notation  $g_{n,m}$  represents the coupling constant, characterizing an interaction term containing  $n$  powers of derivative and  $m$  powers of  $M$ .

How does one construct the effective Hamiltonian at a much larger scale  $L \gg a$ ? The method is in principle similar to the one in *Box 3* adopted for the 1D Ising model; however, the partial summation is done here in momentum space. One writes the Hamiltonian (4) in momentum space (similar to *Box 1* (1)), and integrates over a high momentum shell depicted in *Figure 3*.

This leaves us with new values of coupling terms with a lower ultraviolet cut-off  $|\mathbf{k}| < 1/(a + \delta a)$ , amounting to a scale transformation  $a \rightarrow a + \delta a$ . It is important to appreciate that this exercise has to be carried out in the space of all couplings allowed by the symmetries, as in (4), since the momentum-shell integra-



**Figure 3.** RG transformation involves integrating over high momentum modes  $1/(a + \delta a) \leq |\mathbf{k}| \leq 1/a$ . For clarity, we have taken  $d = 2$ .

tion typically generates all couplings irrespective of whether such couplings were there or not at the scale  $a$ . This is where Wilson’s treatment departs from all previous treatments of renormalization group (more later), and is in principle, complete.

We do not have the scope here to detail all the steps (see *Box 4* for some more details). The first importance consequence of this RG exercise is that for small couplings,  $g_{m,n}$  scale as

$$g_{n,m}(L) = (L/a)^{d-m-(d-2)n/2} g_{n,m}(a). \quad (5)$$

For  $d = 4$ , the exponent is  $4 - m - n$ . Hence only the  $M^2$ -term in (4) grows (is ‘relevant’) at large distances while the  $M^4$  coupling stays constant (at higher orders, this coupling decreases logarithmically with distance). All other couplings in (4) die out at large distances; hence they are called ‘irrelevant couplings’. If one continues  $d$  to slightly below 4,  $d = 4 - \epsilon$  (see more details in *Box 4* and under “RG techniques”) the couplings  $g_{0,4} \equiv u, g_{0,2} \equiv r$  are the only relevant couplings at the free-field fixed point (see *Box 4, Figure A*). At large distances, we reach a fixed point (the Wilson–Fisher fixed point). Near this fixed point the physics is essentially governed by the Hamiltonian,

$$H = \int d^d \mathbf{x} [(\nabla M)^2 + rM^2 + uM^4], \quad (6)$$

where the other couplings of (4) are ignored at leading order in  $\epsilon$  in the sense of *Box 4*. Note that the above Hamiltonian, for  $d = 3$ , resembles the Landau free energy (3) for scalar (Ising type) magnetization and zero magnetic field (although, very importantly, the couplings are dependent on the length scale). The Wilson–Fisher fixed point, therefore, represents the second order phase transition of Ising magnets.

The robustness of the above analysis implies that all second order phase transitions characterized by a scalar ‘order parameter’ (i.e., with a scalar quantity distinguishing phases, e.g., density of a liquid or gas, or an Ising-type magnetization, etc.), must have the same critical exponents! It does not matter at all what the detailed short distance Hamiltonian is. This phenomenon is called

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**Box 4. Critical Exponents**

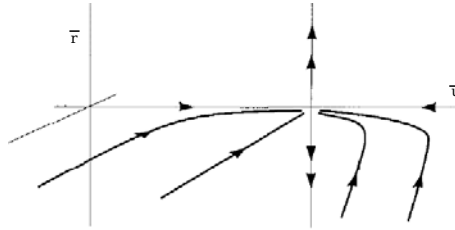


Figure A. The RG Flow of the Couplings in (5) and Wilson–Fisher point.

We will briefly show how Wilson’s RG methods can be used to compute critical exponents. Since both  $r$  and  $u$  in (6), grow at large distances, the formula (5) for variation of couplings is not sufficient. Defining the dimensionless coupling  $\bar{u}(L) = L^{4-d}u(L)$ , one can find the following  $\beta$ -function (differential change of  $\bar{u}$  under scale transformation),

$$Ld\bar{u}/dL = (4 - d)\bar{u} - \frac{3}{16\pi^2}(\bar{u})^2 + O(\bar{u})^3$$

where the first term follows from (5) and the second term follows from a ‘one-loop integration’ over the high momentum shell (Figure 3). For  $d = 4 - \epsilon$ , the  $\beta$ -function vanishes at,

$$\bar{u}_* = \epsilon \frac{16\pi^2}{3} + O(\epsilon^2).$$

Similarly, the  $\beta$ -function for the dimensionless  $\bar{r} = L^2 r$  is  $Ld\bar{r}/dL = 2\bar{r} - \frac{\bar{u}}{16\pi^3}(\bar{r} - 1)$ . The  $\beta$ -functions define a flow of the couplings in the  $\bar{u}$ - $\bar{r}$  plane, as in Figure A. The flows have two fixed points: one at zero coupling, called the ‘free-field’ or Gaussian fixed point, which determines the short distance behaviour, and a second one, call the Wilson–Fisher fixed point which determines the long distance behaviour of the theory. The RG flow, of course, happens in the space of all couplings in (4); in the present discussion they are ignored here since they do not affect the  $O(\epsilon)$  computations. Solving these two differential equations near  $\bar{u}_*$ , we get,

$$r(L) = r(a)(L/a)^{(d-4)/3}.$$

**Running Coupling** As in Landau theory, the correlation length  $\xi$  is identified with  $r^{-1/2}$ , but where  $r$  itself is now to be taken at the length scale  $L = \xi$ , giving us,

$$\xi = r(\xi)^{-1/2}, \Rightarrow \xi \propto r(a)^{-1/2} \xi^{(d-4)/3}.$$

Landau’s argument of analyticity, applied to the scale  $a$ , implies  $r(a) = \alpha(T - T_c)$ ,  $\alpha > 0$ . Hence we get,

$$\xi^{(d+2)/3} = [\alpha(T - T_c)]^{-\nu}, \nu = 3/(d + 2) = 0.6.$$

In the last step we have put  $d = 3$ . Since  $4 - d = \epsilon = 1$  is not a small parameter one needs to improve on this calculation by computing higher order terms in  $\epsilon$ . This procedure, called the  $\epsilon$ -expansion, gives the results quoted in Table 1.





Universality. The universality class characterized by a scalar order parameter is called the Ising universality class. Similarly, two-dimensional and three-dimensional vector order parameters define other universality classes, again independent of the short-distance description. This is illustrated in *Table 1*.

The theoretical explanation of the universality of critical phenomena is one of the most significant aspects of Wilson's work. We will elaborate on this further shortly.

**RG techniques:** The practicality of integrating out higher momentum modes, although conceptually completely clear, can pose a formidable mathematical problem. Wilson, in collaboration with Michael Fisher, found one way of solving this problem by introducing variable dimensionality of space  $d = 4 - \epsilon$ , in which the RG transformation can be worked out order by order in  $\epsilon$  (see *Box 4*)<sup>2</sup>. A second, very useful, method is the large  $N$  technique invented by Abe and Hikami, where the magnetization variable  $M(\mathbf{x})$  was taken to be an  $N$ -component vector (in real ferromagnets  $N = 3$ ) and did the RG calculations in a  $1/N$  expansion. In absence of these techniques, however, one has to appeal to numerical formulation of RG on computers. It should be emphasized that Wilson's approach is fundamentally different from a weak-coupling expansion, although these two approaches overlap in case of the  $\epsilon$ -expansion.

#### 4. Quantum Field Theory

It is perhaps in Quantum Field Theory, that the impact of Wilson's work is at its most profound. Indeed it is fair to say that it is the Wilsonian paradigm [3] which taught us how to understand quantum field theory with as much clarity as we understand a lattice of interacting atoms. It provided a definition of quantum field theory which was both tangible and precise, enough to be understood by a computer (as epitomized by Wilson's lattice gauge theory). In the process, it demystified the prevailing notion of 'renormalization' of quantum field theory, which was essentially a set of rules

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<sup>2</sup>Fractional dimensions are, of course, a formal device, introduced by Wilson and Fisher in their famous paper [2] titled *Critical Exponents in 3.99 Dimensions!* The idea is to use an expansion in the small parameter  $\epsilon$  to eventually put  $\epsilon = 1$  to find out about  $d = 3$ .

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for subtracting infinities and making sense of them in a limited class of renormalizable models.

The key idea here is as follows. Consider a quantum field theory (such as the theory of electrons and photons), in  $d$  Euclidean space time dimensions. Computations of observable quantities in such a theory reduce to computing a Feynman ‘path integral’, which amounts to summing over  $d$ -dimensional field configurations. One way to define such a sum would be to regard space time as a discrete lattice (with, say lattice spacing  $a_0$ ), where we would eventually take  $a_0 \rightarrow 0$  (see below). The sum over field configurations reduces, in principle, to a statistical mechanics problem, where the Euclidean action is identified with a Hamiltonian of the kind (4) (for scalar field theories). The Boltzman sum  $\sum e^{-H/kT}$ , which represents thermal fluctuations, is now replaced by  $\sum e^{-S/\hbar}$  which represents quantum fluctuations.

Now, it is important to note that the limit  $a_0 \rightarrow 0$  is formally the same as limit  $L/a_0 \rightarrow \infty$  discussed in the context of the Wilsonian RG flow in the last section. The limiting points of the RG flows, the fixed points, then clearly define Euclidean quantum field theories, to be precise, a particular class of them defined by power-law correlations and scale invariance. Other quantum field theories can be described by deforming such theories by relevant couplings with appropriate dependence on the scale, as dictated by the  $\beta$ -functions. Such a procedure is explicitly carried out in *Box 4*, in which a physical spin-spin correlation is computed slightly away from the fixed point. Of course, to define the Euclidean QFT, we must specify one constant for each relevant coupling (e.g. the constant  $r(a)$ , or equivalently  $\alpha$ , in *Box 4*). This gives us the following astonishing definition of all conceivable quantum field theories and critical phenomena!

Suppose that there was a way of enumerating all possible scale-invariant field theories (such an enumeration program does exist and is called the conformal bootstrap; originally started by Polyakov and Migdal, and is a very active program of research using clever analytical and numerical methods). Then any quantum field theory can be labelled by (i) a scale-invariant field the-

The set of fixed points would yield the critical behaviour of all conceivable material systems!



ory and (ii) a number of parameters corresponding to values of each relevant coupling. In terms of statistical mechanics, the set of fixed points, would yield the critical behaviour of all conceivable material systems!

*The Relevance of ‘Renormalizable’ QFTs*

With the above discussion, let us try to understand the relevance of textbook examples of 4D quantum field theories. For e.g., the famous  $\lambda\phi^4$  given by a Euclidean action,

$$S = \int d^4x [\partial_\mu\phi\partial^\mu\phi + m^2\phi(x)^2 + \lambda\phi(x)^4] \quad (7)$$

What is so special about these terms? Why do we not include higher derivatives or higher powers?

Similarly in quantum electrodynamics, we construct the action by using the principle of minimal coupling (carried out by the replacement  $\mathbf{p} \rightarrow \mathbf{p} - \mathbf{A}$ ). Why do we not include other terms, e.g. a coupling of the spin current to the field strength  $F_{\mu\nu}\bar{\Psi}[\gamma^\mu, \gamma^\nu]\Psi$  which are allowed by both relativistic invariance and gauge invariance?

The answer found in usual QFT textbooks is that such terms make the theory ‘non-renormalizable’, in the sense that the usual recipe of subtracting infinities using a finite number of counterterms works only for some terms (called renormalizable) and not the others. Such an answer is not satisfactory, since the so-called non-renormalizable couplings arise after quantum corrections even if we started with none. This arbitrary truncation to a finite dimensional space of couplings, in fact, was a major lacuna of renormalization theory, such as developed by Gell-Mann and Low in quantum field theory (further developed by Callan and Symanzik), and by Kadanoff and others in statistical mechanics, before Wilson.

The correct answer, as we know now, thanks to Wilson, is that generally no systematic truncation to a finite number of couplings is possible (all couplings consistent with the allowed symmetries of the problem should be allowed, as in (4)). However, as we



discussed above, the continuum limit  $L/a \rightarrow \infty$  of a QFT is defined by specifying the value of one constant for each relevant coupling at some specified length scale  $L$  (the other couplings are truly irrelevant!). The two couplings in (7), which can be identified with those in (6) as  $m^2 = r$ ,  $\lambda = u$ , are the only two relevant ones (considering the theory at  $4 - \epsilon$  dimensions) at the free-field fixed point. Thus, the importance of the so-called renormalizable Lagrangians is that these contain ‘relevant’ (or marginal) couplings, and other terms die out at large  $L/a$ . Inclusion of the non-renormalizable or irrelevant terms is indeed desirable in fact, when we consider theories that are meant to be effective theories valid only up to some finite high momentum cut-off (or finite  $a$ ), and we need to estimate physical phenomena at distances  $L$  not too large compared to  $a$ . In the discussion of physics beyond the standard model, such irrelevant couplings are routinely studied for consequences.

***Infinites demystified:*** As evident from the above discussion, the traditional treatment of divergence in the  $a_0 \rightarrow 0$  limit of quantum field theory is replaced by whether a field theory can be viewed as deformation of a scale-invariant field theory. All asymptotically free field theories (e.g., quantum chromodynamics, the microscopic theory of strong interactions) are in this category. On the other hand, the  $\phi^4$  theory (7) in  $d = 4$ , except at the free limit  $\lambda = 0$ , cannot be viewed as a relevant deformation of a scale-invariant field theory (without adding other ingredients). Such a theory, therefore, can only be regarded as an effective theory, valid up to some finite short distance cut-off  $a_0$ , and there is no need to be concerned about divergences in the  $a_0 \rightarrow 0$  since beyond some value of  $a_0$ , the theory needs to be superseded by some other, more fundamental, quantum field theory, or something else (e.g., a string theory, a lattice theory, etc.).

### ***Universality***

As is clear from the above, fixed points with a finite number of relevant operators represent universality classes of critical phe-

The traditional treatment of divergence in the  $a_0 \rightarrow 0$  limit of quantum field theory is replaced by whether a field theory can be viewed as deformation of a scale-invariant field theory.



nomena because microscopic details are all encoded in the irrelevant operators which are washed out at large distances. Thus, (6) or equivalently (7) at  $d = 3$  describes the Ising universality class, which represents a wide class of critical phenomena as already described in the first section. The classification of fixed points, in fact, is the same as the classification of universality classes of possible critical behaviour (described by the so-called conformal theories). In this regard, note that  $d = 2$  represents a surprisingly rich variety of universality classes, essentially because in  $d = 2$  any coupling  $g_{0,n}\phi^n$  has the property that all  $g_{0,n}$  are relevant (they grow as  $(L/a)^2$ , as follows from (5)). A similar reasoning in higher dimensions leads one to expect a rather sparse set of RG fixed points, which seems to be borne out by various classification programs, including conformal bootstrap. In fact, there is no known theoretical model of a fixed point in  $d > 6$ . The listing of RG fixed points is an outstanding problem which is clearly important as it teaches us about all possible critical behaviours and all possible quantum field theories.

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### A Bit of History

It is important to emphasize, as indeed Wilson himself had done in his Nobel lecture, several ingredients of what has been described above, had been around in some form or the other well before Wilson. Kadanoff's block transformation, described in the *Resonance* article [1], is regarded as the precursor to Wilson's renormalization group. However, Kadanoff had missed three important ingredients: (a) the importance of fixed points, (b) the fact that scale transformations were only consistent in a multidimensional space of coupling constants, and (c) a workable scheme of carrying out the RG transformation such as the  $\epsilon$ -expansion<sup>3</sup>. The fact that Wilson was trained as a particle physicist (with Murray Gell-Mann as his advisor) helped him find his way through this difficult problem. Indeed, his obsession with strong interactions, especially the problem of scattering of mesons off a fixed nucleon, led him to a remarkable solution of the Kondo problem by a similar treatment of the scattering of conduction electrons in

<sup>3</sup>This was conveyed by Leo Kadanoff in a personal conversation to Spenta Wadia at the University of Chicago after Wilson's Nobel Prize. I thank Spenta for sharing this information with me.



a metal off magnetic impurities. The other most influential work for Wilson, from quantum field theory, was the Gell-Mann–Low equation which described an older notion of scale transformation applied to a single coupling. As mentioned above, it was Wilson who first emphasized that scale transformations could not be truncated to a finite set of couplings. With this insight, and a very original understanding of works ranging from those of Landau, Ginzburg, Dyson, Gell-Mann and Low, Kadanoff and others, Wilson was able to build the elegant edifice of present-day quantum field theory and the theory of critical phenomena.

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#### Address for Correspondence

Gautam Mandal  
Senior Professor  
Department of Theoretical  
Physics  
Tata Institute of Fundamental  
Research  
Mumbai 400 005, India.  
Email:  
mandal@theory.tifr.res.in

### Suggested Reading

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