VANISHING OF CHERN CLASSES OF THE DE RHAM BUNDLES FOR SOME FAMILIES OF MODULI SPACES

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Abstract. Given a family of nonsingular complex projective surfaces, there is a corresponding family of Hilbert schemes of zero dimensional subschemes. We prove that the Chern classes, with values in the rational Chow groups, of the de Rham bundles for such a family of Hilbert schemes vanish. A similar result is proved for any relative moduli space of rank one sheaves over any family of complex projective surfaces.

1. Introduction

Let $\pi : \mathcal{X} \longrightarrow T$ be a smooth algebraic family of complex projective manifolds of dimension d such that the parameter space T is a nonsingular variety. Consider the local systems $R^k\pi_*\mathbb{C}$, $0 \leq k \leq 2d$, and the associated vector bundles $\mathcal{H}^k := (R^k\pi_*\mathbb{C}) \bigotimes_{\mathbb{C}} \mathcal{O}_T$ over T. These vector bundles are equipped with the Gauss–Manin connection. The Gauss–Manin connection, which we will denote by ∇, is flat. This flat vector bundle $(\mathcal{H}^k\, , \nabla)$ is an algebraic bundle and it is called the *de Rham bundle* of weight k.

By the Chern–Weil theory, the de Rham Chern classes

$$c_i^{dR}(\mathcal{H}^k) \,\in\, H^{2i}_{dR}(T)$$

vanish. Let

$$c_i^{Ch}(\mathcal{H}^k) \,\in\, CH^i(T) \otimes_{\mathbb{Z}} \mathbb{Q} \,=:\, CH^i(T)_{\mathbb{Q}}$$

be the Chern classes in the rational Chow groups. A question posed in [Es] asks whether $c_i^{Ch}(\mathcal{H}^k)$ vanishes for each $i \geq 1$ (see [Es, pp. 22, 3.1(1)]).

The known cases where the above question has an affirmative answer are as follows. In [Mu], Mumford proved this for any family of stable curves. In [vdG], van der Geer proved that $c_i^{Ch}(\mathcal{H}^1)$ is trivial when $\mathcal{X} \longrightarrow T$ is a family of principally polarized abelian varieties. For any family of principally polarized abelian varieties of dimension g, the rational Chern classes (in the Chow group) on a good compactification of the parameter space were proved to be trivial by Iyer under the assumption that $g \,\leq\, 5$, [Iy], and by Esnault and Viehweg for all $g \,>\, 0$ [EV].

Our aim here is to check the vanishing of $c_i^{Ch}(\mathcal{H}^k)$, where $i, k \,\geq\, 1$, for two types of families that are described below.

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Let

$$\mathcal{S} \longrightarrow T \tag{1}$$

be a family of smooth surfaces. For any integer $n \geq 1$, we have the relative Hilbert scheme

$$\mathcal{X} := \mathcal{S}^{[n]} \longrightarrow T$$

of zero dimensional subschemes of length n. We prove that $c_i^{Ch}(\mathcal{H}^k)$ vanishes for all i and $k > 0$ (Theorem 2.3).

Let

$$\mathcal{X} := \mathcal{M}_S \longrightarrow T$$

be a relative moduli space of rank one stable sheaves over the family of surfaces $\mathcal{S}$ in (1). We prove that $c_i^{Ch}(\mathcal{H}^k)$ vanishes for all i and k (see Proposition 3.1).

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2. Hilbert scheme of points on surfaces

Let S be a nonsingular projective surface defined over the field of complex numbers. Let $S^{[n]}$ denote the Hilbert scheme of zero dimensional subschemes of S of length n. We know that $S^{[n]}$ is a nonsingular projective variety [Fo, pp. 517, Theorem 2.4]. Furthermore, the map to the symmetric product

$$\rho : S^{[n]} \longrightarrow S^{(n)} := S^n/\sigma_n , \tag{2}$$

where σ_n is the symmetric group of n letters, is a resolution of singularities [Fo, Proposition 2.3, Corollary 2.6].

Let $P(n)$ denote the set of all partitions of $\{1, \cdots, n\}$; so any $\alpha \in P(n)$ is of the form $(n_1, \cdots, n_l)$ with $1 \leq n_i \leq n$ and $\sum_{i=1}^{l} n_i = n$. Given a partition

$$\alpha = (n_1, \cdots, n_l) \in P(n) , \tag{3}$$

the corresponding locally closed stratum $S^{(n)}_\alpha$ of $S^{(n)}$ is the locus defined by elements $n_1[x_1] + \cdots + n_l[x_l]$, with $x_1, \cdots, x_l$ distinct points of S. We put $|\alpha| := l$.

Consider a smooth algebraic family of projective surfaces

$$\pi_{\mathcal{S}} : \mathcal{S} \longrightarrow T , \tag{4}$$

where the parameter space T is nonsingular.

For any $r \in \mathbb{N}$, we have the fiber product

$$\pi^r_{\mathcal{S}} : \mathcal{S}^r := \overbrace{\mathcal{S} \times_T \cdots \times_T \mathcal{S}}^{r-\text{times}} \longrightarrow T , \tag{5}$$

and also have the relative symmetric product

$$\pi^r_s \,:\, \mathcal{S}^{(r)} \longrightarrow T \tag{6}$$

which is the quotient of $\mathcal{S}^r$ for the natural action of the symmetric group σ_r of r letters. For any $\alpha = (n_1, \cdots, n_l) \in P(n)$, let

$$\pi^\alpha_{\mathcal{S}} \,:\, \mathcal{S}^{(\alpha)} := \mathcal{S}^{(n_1)} \times_T \mathcal{S}^{(n_2)} \times_T \cdots \times_T \mathcal{S}^{(n_l)} \longrightarrow T \tag{7}$$

be the fiber product constructed from (6).

There is a relative Hilbert scheme

$$\pi_H \,:\, \mathcal{S}^{[n]} \longrightarrow T \tag{8}$$

whose fiber $\pi_H^{-1}(t)$ over any rational point $t \in T$ is the Hilbert scheme parametrizing zero dimensional subschemes of length n on the complex projective surface $\pi_{\mathcal{S}}^{-1}(t)$. Let

$$\begin{aligned} \mathcal{H}^k_H &:= (R^k\pi_{H*}\mathbb{C}) \otimes_{\mathbb{C}} \mathcal{O}_T\,, \\ \mathcal{H}^k_{\mathcal{S}^r} &:= (R^k\pi^r_{\mathcal{S}*}\mathbb{C}) \otimes_{\mathbb{C}} \mathcal{O}_T\,, \\ \mathcal{H}^k_{\mathcal{S}^r,\,s} &:= (R^k\pi^r_{s\,*}\mathbb{C}) \otimes_{\mathbb{C}} \mathcal{O}_T\,, \\ \mathcal{H}^k_\alpha &:= (R^k\pi^\alpha_{\mathcal{S}*}\mathbb{C}) \otimes_{\mathbb{C}} \mathcal{O}_T \end{aligned}$$

be the de Rham bundles of weight k over T, where $\alpha \in P(n)$, and the projections π_H, $\pi^r_{\mathcal{S}}$, π^r_s and $\pi^\alpha_{\mathcal{S}}$ are defined in (8), (5), (6) and (7) respectively.

For any $\alpha \in P(n)$ and $t \in T$, there is a canonical morphism

$$\kappa_\alpha \,:\, S^{(\alpha)}_t \longrightarrow \overline{(S^{(n)}_\alpha)_t} \tag{9}$$

to the closure $\overline{(S^{(n)}_\alpha)_t}$ of the stratum $(S^{(n)}_\alpha)_t \subset S^{(n)}_t$, and hence there is a map

$$S^{(\alpha)}_t \longrightarrow \overline{(S^{(n)}_\alpha)_t} \hookrightarrow S^{(n)}_t$$

(see [GS, §3, pp. 236] for the details). This defines a morphism over T of the relative universal schemes

$$\Delta_\alpha \,:\, \mathcal{S}^{(\alpha)} \longrightarrow \overline{\mathcal{S}^{(n)}_\alpha}\,. \tag{10}$$

Here $\overline{\mathcal{S}^{(n)}_\alpha}$ is the normalization of the subscheme obtained after taking closure of the fibers $(S^{(n)}_\alpha)_t$.

There is a natural isomorphism

$$H^*(S^{[n]}_t,\, \mathbb{Q}) = \bigoplus_{\alpha\in P(n)} H^*(S^{(\alpha)}_t,\, \mathbb{Q}) \tag{11}$$

[Go, pp. 613, Theorem 1.1], [GS, pp. 236, Theorem 2]. For any integer $k \geq 0$, set $k_\alpha \in \mathbb{N}$ such that $H^k(S^{[n]}_t, \mathbb{Q})$ corresponds to $H^{k_\alpha}(S^{(\alpha)}_t, \mathbb{Q})$ under the above isomorphism.

Lemma 2.1. *There is a canonical direct sum decomposition of the vector bundle*

$$\mathcal{H}^k_H = \bigoplus_{\alpha\in P(n)} \mathcal{H}^{k_\alpha}_\alpha$$

over T.

Proof. This follows from [Go, pp. 613, Theorem 1.1]. We note that a similar result is also proved in [dCM]. The isomorphism is constructed using $\sum_{\alpha\in P(n)}(\Delta_\alpha)_*$, where Δ_α is the map in (10); the details of the construction of the isomorphism are given in [Go, Proposition 3.1]. $\square$

Lemma 2.2. *Take any $\alpha \in P(n)$. The Chern classes $c_i(\mathcal{H}^k_\alpha) \in CH^*(T)_{\mathbb{Q}}$ vanish for all $i, k \geq 1$.*

Proof. Using the Künneth decomposition we obtain

$$\mathcal{H}^k_{\mathcal{S}^r} = \bigoplus_{\sum_{j=1}^r i_j = k} \mathcal{H}^{i_1}_{\mathcal{S}} \otimes \mathcal{H}^{i_2}_{\mathcal{S}} \otimes \cdots \otimes \mathcal{H}^{i_r}_{\mathcal{S}} ,$$

where $\mathcal{S}^r$ is defined in (5).

For any $t \in T$, the cohomology of the fiber $\mathcal{S}^{(r)}_t$ is isomorphic to the space of invariants

$$H^*(\mathcal{S}^r_t, \mathbb{Q})^{\sigma_r} \subset H^*(\mathcal{S}^r_t, \mathbb{Q})$$

for the action of the symmetric group σ_r of r letters [Gr, Theorem 5.3.1]; see [Ma, Part I, §3, pp. 564] for a description of the action of σ_r. Hence

$$\mathcal{H}^k_{\mathcal{S}^r, s} = (\mathcal{H}^k_{\mathcal{S}^r})^\sigma .$$

Combining these we conclude that the σ_r–invariant subbundle $(\mathcal{H}^k_{\mathcal{S}^r})^{\sigma_r}$ consists of the direct summands which are of the type

$$\mathrm{Sym}^{j_1}\mathcal{H}^{p_1}_{\mathcal{S}} \otimes \mathrm{Sym}^{j_2}\mathcal{H}^{p_2}_{\mathcal{S}} \otimes \cdots \otimes \mathrm{Sym}^{j_s}\mathcal{H}^{p_s}_{\mathcal{S}} \otimes \Lambda^{l_1}\mathcal{H}^{q_1}_{\mathcal{S}} \otimes \Lambda^{l_2}\mathcal{H}^{q_2}_{\mathcal{S}} \otimes \cdots \otimes \Lambda^{l_t}\mathcal{H}^{q_t}_{\mathcal{S}} ,$$

where p_i are even integers and q_i are odd integers (see [dB, pp. 116, Proposition 3.8]). Here

$$\mathcal{H}^i_{\mathcal{S}} := (R^i\pi_{\mathcal{S}*}\mathbb{C}) \bigotimes_{\mathbb{C}} \mathcal{O}_T ,$$

where $\pi_{\mathcal{S}}$ is the projection in (4), and Sym (respectively, Λ) denotes the symmetric power (respectively, exterior power).

The Chern classes of $\mathrm{Sym}^j\mathcal{H}^p_{\mathcal{S}}$ and $\Lambda^l\mathcal{H}^q_{\mathcal{S}}$ are determined in terms of the Chern classes of the vector bundles $\mathcal{H}^p_{\mathcal{S}}$ and $\mathcal{H}^q_{\mathcal{S}}$ respectively [Fu, pp. 55]. We also know that $c_i(\mathcal{H}^m_{\mathcal{S}}) \in CH^i(T)_{\mathbb{Q}}$ vanishes for each m and $i > 0$ [BE, pp. 950, Example 7.3]. Consequently, the Chern classes of $\mathcal{H}^k_{\mathcal{S}^r, s} = (\mathcal{H}^k_{\mathcal{S}^r})^\sigma$ in the rational Chow groups of T vanish.

Since $\mathcal{S}^{(\alpha)} = \mathcal{S}^{(n_1)} \times_T \mathcal{S}^{(n_2)} \times_T \cdots \times_T \mathcal{S}^{(n_l)}$, using the Künneth decomposition, and the additivity property of the Chern character for a direct sum, we deduce that the Chern classes of $\mathcal{H}^k_\alpha$ vanish in the rational Chow groups. This completes the proof of the lemma. $\square$

Theorem 2.3. *The Chern classes $c_i(\mathcal{H}^k_H) \in CH^*(T)_{\mathbb{Q}}$ vanish for all $i, k \geq 1$.*

Proof. We use the decomposition in Lemma 2.1 together with the additivity property of the Chern character map to obtain

$$ch(\mathcal{H}_H^k) = \sum_{\alpha\in P(n)} ch(\mathcal{H}_\alpha^{k_\alpha}).$$

Lemma 2.2 says that $ch(\mathcal{H}_\alpha^{k_\alpha}) \in CH^0(T)_{\mathbb{Q}}$ for all $\alpha \in P(n)$. This implies that $ch(\mathcal{H}_H^k) \in CH^0(T)_{\mathbb{Q}}$, and the proof of the theorem is complete. □

3. Moduli spaces of rank one sheaves

Let S be a smooth projective surface defined over $\mathbb{C}$. Take any nonnegative integer n. The moduli space of stable sheaves E over S of rank one and $c_2(E) = n$ is $\mathrm{Pic}^0(S)\times S^{[n]}$; if $n = 0$, then consider $S^{[n]}$ to be a single point. This identification is constructed by sending any $(L, Z) \in \mathrm{Pic}^0(S) \times S^{[n]}$ to the rank one sheaf $L \bigotimes_{\mathcal{O}_S} I_Z$, where $I_Z \subset \mathcal{O}_S$ is the ideal of Z.

As in (4), let

$$\pi : \mathcal{S} \longrightarrow T$$

be a smooth algebraic family of smooth projective surfaces. Fix a nonnegative integer n. Let

$$\pi_{\mathcal{M}} : \mathcal{M} \longrightarrow T$$

be the relative moduli space of stable sheaves of rank one and second Chern class n over $\mathcal{S}$. So for any point $t \in T$, the fiber $\pi_{\mathcal{M}}^{-1}(t)$ parametrize all stable sheaves E over $\pi^{-1}(t)$ with $\mathrm{rank}(E) = 1$ and $c_2(E) = n$.

Consider the relative Hilbert scheme

$$\pi_H : \mathcal{S}^{[n]} \longrightarrow T$$

and the relative Picard variety $\pi_J : \mathrm{Pic}^0_T(\mathcal{S}) \longrightarrow T$. Let

$$\pi_{J,n} : \mathrm{Pic}^0_T(\mathcal{S}) \times_T \mathcal{S}^{[n]} \longrightarrow T$$

be the fiber product over T. We have

$$\mathcal{M} = \mathrm{Pic}^0_T(\mathcal{S}) \times_T \mathcal{S}^{[n]}. \tag{12}$$

We have the associated de Rham bundles

$$\begin{aligned}
\mathcal{H}^k_{\mathcal{S}^{[n]}} &:= (R^k\pi_{H*}\mathbb{C}) \otimes \mathcal{O}_T\,,\\
\mathcal{H}^k_J &:= (R^k\pi_{J*}\mathbb{C}) \otimes \mathcal{O}_T\,,\\
\mathcal{H}^k_{J,n} &:= (R^k\pi_{J,n*}\mathbb{C}) \otimes \mathcal{O}_T\,,\\
\mathcal{H}^k_{\mathcal{M}} &:= (R^k\pi_{\mathcal{M}*}\mathbb{C}) \otimes \mathcal{O}_T
\end{aligned}$$

over T.

Proposition 3.1. *The Chern classes $c_i(\mathcal{H}^k_{\mathcal{M}}) \in CH^*(T)_{\mathbb{Q}}$ vanish, where $i, k \geq 1$.*

Proof. Using (12), we have an isomorphism of the de Rham bundles

$$\mathcal{H}^k_{\mathcal{M}} \simeq \mathcal{H}^k_{J,n}\,.$$

Using the Künneth decomposition we have

$$\mathcal{H}^k_{J,n} \,=\, \sum_{p+q=k} \mathcal{H}^p_J \otimes \mathcal{H}^q_{\mathcal{S}^{[n]}}\,. \tag{13}$$

Using (13), the Chern classes of $\mathcal{H}^k_{J,n}$ are expressed in terms of the Chern classes of $\mathcal{H}^p_J$ and $\mathcal{H}^q_{\mathcal{S}^{[n]}}$. We recall that Theorem 2.3 says that the Chern classes of $\mathcal{H}^q_{\mathcal{S}^{[n]}}$ vanish, and [vdG] and [EV] say that the Chern classes of $\mathcal{H}^p_J$ vanish. Consequently, $c_i(\mathcal{H}^k_{\mathcal{M}}) \in CH^i(T)_{\mathbb{Q}}$ vanishes for each $i, k > 0$. This completes the proof of the proposition. □

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