POTENTIAL DENSITY FOR SOME FAMILIES OF HOMOGENEOUS SPACES

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Abstract. For a smooth, projective family of homogeneous varieties defined over a number field, we show that if potential density holds for the rational points of the base, then it also holds for the total space. A conjecture of Campana and Peternell, known in dimension at most 4 and for certain higher dimensional cases, would then imply potential density for the rational points of smooth projective varieties over number fields whose tangent bundle is nef.

Introduction

Let k be a number field. A geometrically integral variety X over the field k satisfies potential density if there exists a finite field extension K/k such that the set $X(K)$ of rational points of X is Zariski dense in $X_K = X \otimes_k K$. One hopes that this property only depends on the geometry of the variety X over an algebraically closed field containing k, for instance over the complex numbers. It has been known for some time that Abelian varieties satisfy potential density (see [Has, Prop. 4.2]). It is an open problem whether potential density holds for rationally connected varieties, in particular for Fano varieties.

For an overview of problems and results regarding potential density over number field, as of 2003, including work of Bogomolov, Hasssett, Tschinkel, we refer the reader to the survey [Has] by B. Hassett. Among the significant later results, let us mention the paper by E. Amerik and C. Voisin [Am-Vo].

According to the Hartshorne–Frenkel conjecture, proved by S. Mori, a smooth, projective, complex variety whose tangent bundle is ample, is isomorphic to projective space. Over an arbitrary field k of characteristic zero, this implies that a smooth, projective, geometrically

[0]Mathematics Classification Number: 14C25, 14D05, 14D20, 14D21
[0]Keywords: Homogeneous spaces, rational points, completions.
Date: February 11, 2011.

integral k-variety X whose tangent bundle is ample is a Severi-Brauer variety. After a finite extension K/k of the ground field, this variety acquires a rational point and then it k-isomorphic to projective space over K, hence the set $X(K)$ is Zariski dense in X_K. This argument of course has nothing to do with number fields.

One may wonder whether potential density holds more generally for a smooth, projective, geometrically integral k-variety X whose tangent bundle is nef. Such varieties have been studied in particular by Campana, Demailly, Peternell, Schneider. In this note we give a detailed proof of a stability property for potential density (Theorem 2.3 and Corollary 2.5). The result should be more or less obvious to experts. Combined with a conjecture of Campana and Peternell, it predicts potential density for varieties with nef tangent bundle (known in certain cases, see section 3).

1. Known results on homogeneous spaces of linear algebraic groups

The following theorem gathers results of T. A. Springer, J.-C. Douai and M. Borovoi ([Bv]).

Theorem 1.1. *Let k be a field of characteristic zero and let $\overline{k}$ be an algebraic closure of k. Let G/k be a semisimple simply connected group. Let X/k be a homogeneous space of G. Assume that a geometric stabilizer $\overline{H}$ is connected.*

(a) The homogeneous space structure on X defines a k-kernel $\mathcal{L} := (\overline{H}, \kappa)$, and a class $\eta(X)$ in the cohomology set $H^2(k, \mathcal{L})$. This class is neutral if and only if there exists a principal homogeneous space E under G and a G-equivariant map $E \to X$.

(b) Let $\overline{H}^{tor}$ be the maximal toric quotient of $\overline{H}$. The k-kernel $\mathcal{L}$ induces a k-kernel $(\overline{H}^{tor}, \kappa^{tor})$. To the latter is associated a natural k-torus T. There is an induced map of sets

$$H^2(k, \mathcal{L}) \to H^2(k, T).$$

Let $\eta_X^{tor} \in H^2(k, T)$ denote the image of $\eta(X)$.

(c) If L/k is a finite field extension such that $X(L) \neq \emptyset$ then

$$[L : k].\eta_X^{tor} = 0 \in H^2(k, T).$$

(d) If X/k is projective, then $\overline{H}$ is connected, and the associated torus T is a quasitrivial torus.

Proof. For (a), (b), (c), see [Bv] and the review in [CTGP, §5, p. 333–335]. For (d), see [CTGP, Lemma 5.6].

The following theorem combines results of Kneser, Bruhat-Tits (for principal homogeneous spaces of semisimple simply connected groups) and Springer, Douai, Borovoi.

Theorem 1.2. *Let k be a p-adic field. In the situation of Theorem 1.1, the class $\eta(X)$ is neutral if and only if $\eta_X^{tor} = 0 \in H^2(k,T)$. In that case, X has a k-point.*

Proof. [Bv, Thm. 5.5] and [CTGP, Prop. 5.4].

Proposition 1.3. *Let k be a field of characteristic zero and let $\overline{k}$ be an algebraic closure of k. Let X/k be a smooth, projective, geometrically connected variety.*

(a) If $X \times_k \overline{k}$ is a homogeneous space of a linear algebraic group, then there exists a semisimple simply connected group G over k such that X is a homogeneous space of G.

(b)The geometric stabilizers of this action are parabolic groups, in particular they are connected.

(c) If $X(k) \neq \emptyset$, then X is k-birational to projective space.

Proof. Statement (a) is a special case of the following theorem of Demazure. The idea here is to consider the neutral component $G = Aut^0_{X/k}$ of the automorphism group of X over k, which is an adjoint group, and then to take the semisimple cover of that group. For (b), see [Bo, IV.11.6]. For (c), see [Bo, IV.14.21 and V.20.5].

Theorem 1.4. *Let k be a field of characteristic zero and let $\overline{k}$ be an algebraic closure of k. Let $p : X \to Y$ be a smooth, proper k-morphism of smooth, geometrically connected k-varieties.*

(a) (Demazure) If each geometric fibre of p is a homogeneous space of a connected linear algebraic group then the group $G = Aut^0_{X/Y}$ is a semisimple group over Y and $X \to Y$ is a homogeneous space of G. The fibres of G are adjoint groups. There also exists a semisimple group G^{sc} over Y, whose fibres are simply connected semisimple groups, such that X is a homogeneous space of G^{sc}.

(b) In the above situation, there exists a finite Zariski open cover $\{U_i\}_{i\in I}$ of Y and quasifinite, surjective, étale maps $V_i \to U_i$ which factorize as $V_i \to X \times_Y U_i \to U_i$.

(c) There exists an integer d such that for any point $M \in Y$ the torus T_M over the residue field $k(M)$ associated to the homogeneous space $X \times_Y M$ (see Theorem 1.1 (b)) has rank d.

(d) There exists an integer $N > 0$ such that for any field L containing k and any L-point $P \in Y(L)$, the class $\eta^{tor}_{X_P} \in H^2(L, T_P)$ is N-torsion. Here T_P denotes the k-torus associated to the k-variety X_P (fibre of $X \to Y$ at P) viewed as a homogeneous space of G^{sc}_P.

Proof. Statement (a) is [De, Prop. 4]) of Demazure. Statement (b) is a general fact for a smooth, surjective morphism $X \to Y$. For any point $P \in Y(k)$ there exists an i with $P \in U_i$ and a closed point $M \in V_i$ mapping to P. Let $k(M)$ be the residue field of M. Since the set I is finite and for each i the degrees of the fibres of $V_i \to U_i$ are bounded, there exists a fixed integer $N > 0$, independent of P, such that the degree of the field extension $k(M)/k$ divides N. We now use Theorem 1.1, which applies to the present situation in view of Proposition 1.3. The class $\eta \in H^1(k, T_P)$ vanishes in $H^1(k(M), T_P)$. Hence its corestriction $[k(M) : k].\eta$ vanishes in $H^1(k, T_P)$. So does $N.\eta$.

Theorem 1.5. *(Harder) Let k be a number field. Let X/k be a smooth projective homogeneous variety under the action of a connected linear algebraic group. Then the Hasse principle holds for X : if X has points in all completions of k, then it has a point in k.*

Proof. In [Ha], Harder reduces the local-global statement to the Hasse principle for principal homogeneous spaces of semisimple simply connected groups. In this set-up, the local-global principle is due to Eichler, Kneser, Harder, and Chernousov.

2. The theorem

To prove the main theorem, we shall use two further results. The first one is a special case of a standard result in the study of the Hasse principle.

Theorem 2.1. *Let k be a number field. Let $p : X \to Y$ be a smooth, projective morphism of projective, geometrically integral k-varieties. Assume that the fibres of p are homogeneous spaces of connected linear algebraic groups. Then there exists a finite set S of places of k such that for any finite field extension L/k and any place w of L not lying above a place in S the induced map $X(L_w) \to Y(L_w)$ is onto.*

Proof. By Theorem 1.4, there exists a semisimple group G over Y such that X is a homogeneous space of G. By a standard limit argument, which is easy in the present, projective context, (for a more general set up, see EGA IV 8), the whole situation may be spread out over an open set B of the spectrum of the ring of integers of k. Let $\mathbf{X} \to \mathbf{Y}$ and $\mathbf{G}/\mathbf{Y}$ denote the corresponding objects. Let v be a place in B. Let $O_v \subset k_v$ denote the ring of integers in the completion k_v, and let $\mathbb{F}_v$ denote the residue field. Let $P_v \in Y(k_v)$. Since $\mathbf{Y}/B$ is proper, we have $\mathbf{Y}(O_v) = Y(k_v)$, the point P_v may be viewed as a point $\mathbf{P}_v \in \mathbf{Y}(O_v)$. By restriction to $\mathbf{P}_v$ one gets a homogeneous space of the O_v-semisimple group $\mathbf{G} \times_Y \mathbf{P}_v$. One then considers the reduction of all this over the finite field $\mathbb{F}_v$. Any homogeneous space of a connected linear algebraic group over a finite field has a rational point (Lang, Springer, see Serre [S2, Chap. III, §2]). By Hensel's lemma one then lifts such a point to an O_v-point of $\mathbf{X} \times_{\mathbf{Y}} \mathbf{P}_v$. Such a point defines a k_v-point of X whose image is $P_v \in Y(k_v)$. Thus $X(k_v) \to Y(k_v)$ is onto. The same argument works over any finite field extension L of k, with the inverse image of B in the spectrum of the ring of integers of L.

Lemma 2.2. *Let k be a p-adic field. Let T be a quasisplit torus of dimension d. Let $N > 0$ be an integer. If L/k is a field extension whose degree is divisible by $N.d!$ then the restriction map on N-torsion classes*

$$H^2(k, T)[N] \to H^2(L, T)[N]$$

is zero.

Proof. We immediately reduce to the case $T = R_{K/k}\mathbb{G}_m$, where K/k is a field extension of degree $r \leq d$. By a lemma of Faddeev and Shapiro ([S2, Chap. I, §2.5]), the restriction map $H^2(k, T) \to H^2(L, T)$ then reads $Br(K) \to \oplus_i Br(L_i)$, where $L \otimes_k K = \prod_i L_i$ is the decomposition into a finite product of fields. We have the embeddings $k \subset K \subset L_i$

and $k \subset L \subset L_i$. By assumption, $N.d!$ divides $[L : k]$, which divides $[L_i : k] = [K : k][L_i : K] = r[L_i : K]$. It follows that N divides $[L_i : K]$. But the map of Brauer groups of local fields $Br(K) \to Br(L_i)$ reads as multiplication by $[L_i : K]$ on $\mathbf{Q}/\mathbf{Z}$ ([S1, Chap. XIII, §3, Prop. 7 p. 201]. Hence on N-torsion it is zero.

Theorem 2.3. *Let k be a number field. Let $p : X \to Y$ be a smooth, proper morphism of geometrically integral varieties. Assume that the geometric fibres of p are homogeneous spaces of connected linear algebraic groups. Then there exists a finite field extension L/k such that $Y(k) \subset Y(L)$ lies in the image of $X(L) \to Y(L)$. If $Y(k)$ is Zariski dense in Y, then for L as above, $X(L)$ is Zariski dense in X_L.*

Proof. By Theorem 2.1, there exists a finite set S of places of k, which we assume to contain all archimedean places, such that for any finite field extension L/k and any place w of L not lying above a place in S the induced map $X(L_w) \to Y(L_w)$ is onto. By Theorem 1.4, there exists an integer $d > 0$ and an integer $N > 0$, which we may choose even, such that for any field L containing k, and any point $M \in Y(L)$, the torus T_M over L associated to the homogeneous space X_M defined by the fibre at M is a quasitrivial torus over field L, of dimension d, and the class $\eta^{tor}(X_M) \in H^2(L, T_M)$ is annihilated by N. For each finite place $v \in S$ let us pick a field extension F^v/k_v of degree $N.d!$. For each archimedean place v of k let F^v/k_v be a separable extension of k_v of degree $N.d!$, hence even, which breaks up as the product of copies of the complex field. By weak approximation for the field k and Krasner's lemma [S1, Chap. II, §2, Exercice 2, p. 40], there exists a field extension L/k of degree $N.d!$ such that for each $v \in S$, there is an isomorphism $L \otimes_k k_v \simeq F^v$. In particular, for each finite place v of k in S, there is just one place w of L above v.

Let now $P \in Y(k)$ be an arbitrary point, let $T = T_P$ be the k-torus of dimension d associated to the homogeneous space X_P and let $\eta = \eta^{tor}(T) \in H^2(k, T)$ be the associated class. This class is annihilated by N. At any place v of k not in S, the fibre X_P has a k_v-point, hence $\eta_v = 0 \in H^2(k_v, T)$. If w is a place of L over a place of S, Lemma 2.2 and the choice of the extension L/k imply that the image of η in $H^2(L_w, T)$ vanishes. Thus $\eta_L \in H^2(L, T)$ vanishes over each completion of L. By theorem 1.2 this implies that $X_P \otimes_k L$ has points in all completions of L. By Theorem 1.5 this implies that the L-variety $X_P \otimes_k L$ has an L-point, and then that $X_P \otimes_k L$ is L-birational to projective space over L, in particular L-points are Zariski dense on X_L. This completes the proof of the theorem.

Remark 2.4. In the more general context of integral points, a special case of the above theorem (family of Severi–Brauer varieties) was remarked some time ago by the first named author [HT, Thm. 2.8]. One could certainly also write down an integral points version of Theorem 2.3.

Corollary 2.5. *Let k be a number field. Let A be abelian variety over k. Let $p : X \to A$ be a smooth, proper morphism of geometrically integral varieties. Assume that the geometric fibres of p are homogeneous spaces of connected linear algebraic groups. Then there exists a finite field extension K/k such that $X(K)$ is Zariski dense in X_K.*

Proof. Since potential density holds for abelian varieties ([Has, Prop. 4.2]), this is an immediate consequence of Theorem 2.3.

3. Varieties with nef tangent bundles : the conjecture of Campana and Peternell

In this section we discuss potential density of rational points for smooth, projective, geometrically integral varieties over number field, under the assumption that their tangent bundle is numerically effective (nef). By definition, this means that the line bundle $\mathcal{L} := \mathcal{O}_{PT(X)}(1)$ on the projectivized tangent bundle $PT(X)$, is numerically effective, i.e. $\mathcal{L}.C \geq 0$, for any curve C on $PT(X)$.

Recall that a smooth, projective variety X is a Fano variety if the anticanonical line bundle $-K_X$ is ample. The following theorem was conjectured by Campana and Peternell and proved by them in dimension at most 3 [Ca-Pe, Theorem, p.169].

Theorem 3.1. *(Demailly–Peternell–Schneider)* [DPS, Main Theorem, p. 296] *Let k be an algebraically closed field of characteristic zero. Let X be a smooth, projective, connected variety with nef tangent bundle. Then there exists a finite étale connected cover $X' \to X$ such that for any k-point of X' the associated Albanese map $X' \to A$ to the Albanese* *variety of X' (which is an abelian variety) is a smooth, projective morphism whose fibres are Fano varieties with nef tangent bundles.*

Campana and Peternell put forward the following conjecture.

Conjecture 3.2. [Ca-Pe, Conjecture 11.1, p. 185] *Over an algebraically closed field of characteristic zero, a Fano variety with nef tangent bundle is a projective homogeneous variety of a linear algebraic group, i.e. it is of the shape G/P for G a connected linear algebraic group and P a parabolic subgroup.*

A variant is formulated by J-M. Hwang [Hw, Conjecture 4.1, p. 622]: this should be the case as soon as all rational curves on X are free.

The Campana-Peternell conjecture 3.2 was proved by Campana and Peternell in dimension up to 3 and by J.-M. Hwang in dimension 4. It has also been proved for higher dimensional Fano, when the Betti numbers satisfy $b_2 = b_4 = 1$, and the variety of minimal rational tangents at a general point is one-dimensional [Mk, Main Theorem, p. 2641], [Hw, Theorem 4.3, p. 623]. See [Hw, section 4] for a discussion and references. See also a related recent work [Bi-Br]. In these various cases, the following theorem therefore applies.

Theorem 3.3. *Suppose X is a smooth projective variety with a nef tangent bundle, defined over a number field. Under Conjecture 3.2 on Fano varieties, potential density holds for X.*

Proof. Combine Theorem 3.1 (which descends from an algebraic closure of k to some finite extension of k), Conjecture 3.2 and Corollary 2.5.

Question 3.4. *Let k be an algebraically closed field. Let $X \to Y$ be a smooth, projective family of homogenous spaces of connected linear algebraic groups. Does there exist a finite étale map $Z \to Y$ with Z connected such that $X \times_Y Z \to Z$ admits a rational section?*

Since potential density is inherited by finite étale covers (Chevalley–Weil, cf. [Has, Prop. 3.4]), an affirmative answer to the question would lead to an alternate proof of Theorem 2.3.

Over an algebraically closed field, a connected, finite étale cover of an abelian variety may be given the structure of an abelian variety. If for

Y is an abelian variety the above question had an affirmative answer, this would give an alternate, less arithmetic proof for Corollary 2.5 and therefore for Theorem 3.3.

In the special case where $X \to Y$ is a Severi-Brauer scheme over an abelian variety Y, the answer to the above question is in the affirmative (see the proof of [Ca-Pe, Lemma 7.4 (1)]).

Acknowledgement: This work was discussed at the Université Paris-Sud, Orsay, during Nov-14-Dec 14 2010. The second named author thanks Laurent Clozel for the kind invitation and the Département de Mathématiques, for their hospitality and support from the ARCUS programme.

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