SOME THOUGHTS ON ANDO'S THEOREM AND PARROTT'S EXAMPLE

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Abstract. In this short note, we present an elementary proof of Ando's theorem within a restricted class $\mathcal{P}$ of homomorphisms modeled after Parrott's example. We also show by explicit estimation that the cb-norm of the contractive homomorphism ρ of the tri-disc algebra, induced by the commuting triple of Parrott, exceeds 1. Indeed, we construct a polynomial P with matrix coefficients with the property $\|\rho(P)\| > \|P\|_\infty$. In particular, we show that there are contractive homomorphisms of the tri-disc algebra which are not even 2-contractive.

1. Introduction

Let Ω be a bounded, open and connected subset of $\mathbb{C}^m$. The algebra of continuous functions on the closure $\bar{\Omega}$ of Ω which are holomorphic on Ω is denoted by $\mathcal{A}(\Omega)$. It is a Banach algebra with respect to the supremum norm on Ω. If Ω is a polynomially convex domain then $\mathcal{A}(\Omega)$ is the closure of the polynomials with respect to the supremum norm. Let $\mathcal{M}_k$ be the C^* - algebra of $k \times k$ matrices over the complex scalars $\mathbb{C}$. For $(\!(f_{ij})\!)$ in $\mathcal{A}(\Omega) \otimes \mathcal{M}_k$, define the norm

$$\|(\!(f_{ij})\!)\| = \sup\{\|(\!(f_{ij}(z))\!)\|_{\mathrm{op}} : z \in \Omega\}.$$

Clearly, $\mathcal{A}(\Omega) \otimes \mathcal{M}_k$ is a Banach algebra with respect to this norm. Let $\mathcal{H}$ be a separable Hilbert space and $\mathcal{B}(\mathcal{H})$ be the C^* - algebra of bounded linear operators on $\mathcal{H}$. Finally, let $\rho : \mathcal{A}(\Omega) \to \mathcal{B}(\mathcal{H})$ be an algebra homomorphism.

Recall that the homomorphism ρ is said to dilate if there exists a $*$ - homomorphism $\tilde{\rho}$ of the algebra $C(\partial\Omega)$ of continuous functions on the Silov boundary of the domain Ω into the bounded linear operators on a Hilbert space $\mathcal{K} \supseteq \mathcal{H}$ such that

$$P_{\mathcal{H}}\tilde{\rho}(f)|_{\mathcal{H}} = \rho(f), \; f \in \mathcal{A}(\Omega),$$

where $P_{\mathcal{H}}$ is the orthogonal projection on to the Hilbert space $\mathcal{H}$. The $*$ - homomorphism $\tilde{\rho}$ is called a dilation of ρ. Clearly, if the homomorphism ρ dilates then it is contractive, that is, $\|\rho(f)\| \le \|f\|_\infty$. Therefore, it is natural to ask if every contractive homomorphism dilates.

For each $k = 1, 2, \ldots$, there is an induced homomorphism

$$\rho^{(k)} \stackrel{\text{def}}{=} \rho \otimes I_k : \mathcal{A}(\Omega) \otimes \mathcal{M}_k \to \mathcal{B}(\mathcal{H}) \otimes \mathcal{M}_k \simeq \mathcal{B}(\mathcal{H} \otimes \mathbb{C}^k).$$

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The supremum of the non-decreasing sequence $\{\|\rho^{(k)}\|\}_{k\geq 1}$ is called the "cb norm" of the homomorphism ρ and is denoted by $\|\rho\|_{\rm cb}$. Arveson [2, 3] has shown that the existence of a dilation for $\rho : \mathcal{A}(\Omega) \to \mathcal{B}(\mathcal{H})$ is equivalent to $\|\rho\|_{\rm cb} \leq 1$.

If Ω is the unit disc then the von Neumann's inequality says that a homomorphism ρ of the disc algebra $\mathcal{A}(\mathbb{D})$ is contractive if and only if $\rho(\text{id})$ is contractive for the identity function $\text{id} \in \mathcal{A}(\mathbb{D})$. It was proved by B. Sz.-Nagy that if ρ is a homomorphism of the disc algebra and $\rho(\text{id})$ is contractive then it dilates to a $*$ - homomorphism of the C^* - algebra $C(\mathbb{T})$. It is easy to deduce von Neumann's inequality from this result together with the spectral theorem for $*$ - homomorphisms.

Once we fix a domain Ω, we may ask what are the contractive homomorphisms, which among these are completely contractive? In particular, one may ask if contractive homomorphisms are necessarily completely contractive. We find that for the disc algebra, contractive homomorphisms are induced by contraction operators (von Neumann's inequality), these contractive homomorphisms always dilate (Sz.-Nagy's theorem) and therefore they are completely contractive. In the case of the bi-disc, the well-known dilation theorem of Ando [1] says that a homomorphism of the bi-disc algebra $\mathcal{A}(\mathbb{D}^2)$, induced by a pair of contractions, dilates. (In particular, any commuting pair of contractions induces a contractive homomorphism of the bi-disc algebra.) This shows that the case of the bi-disc is no different than that of the disc. However, Parrott [11] produced examples of three contractions such that the induced homomorphism of the tri-disc algebra is contractive but does not dilate! Also, Varopoulos [14] has shown that a triple of contractions does not necessarily induce a contractive homomorphism of the tri-disc algebra.

Now we consider a class $\mathcal{P}$ of homomorphisms defined from $\mathcal{A}(\Omega)$ into $\mathcal{B}(\mathbb{C}^{p+q})$ which are modeled after the examples due to Parrott - although, as we explain below, some of these homomorphisms appear naturally as localizations of Cowen-Douglas operators.

Given any $\boldsymbol{w} \in \Omega$ and any tuple $\mathbf{V} = (V_1, \ldots, V_m)$ of $p \times q$ matrices, the operator tuple

$$\mathbf{N} = \left(\begin{pmatrix} w_1 I_p & V_1 \\ 0 & w_1 I_q \end{pmatrix}, \ldots, \begin{pmatrix} w_m I_p & V_m \\ 0 & w_m I_q \end{pmatrix} \right) \tag{1.1}$$

induces (via the usual functional calculus) a homomorphism $\rho_{\boldsymbol{w},\mathbf{V}}$ of the algebra $\mathcal{A}(\Omega)$ into $\mathcal{M}_{p+q} \stackrel{\rm def}{=} \mathcal{B}(\mathbb{C}^{p+q})$ defined by

$$\rho_{\boldsymbol{w},\mathbf{V}}(f) = \begin{pmatrix} f(\boldsymbol{w}) I_p & \langle \nabla f(\boldsymbol{w}), \mathbf{V} \rangle \\ 0 & f(\boldsymbol{w}) I_q \end{pmatrix}.$$

Here $\langle \nabla f(\boldsymbol{w}), \mathbf{V} \rangle$ stands for $V_1 \frac{\partial f}{\partial z_1}(\boldsymbol{w}) + \cdots + V_m \frac{\partial f}{\partial z_m}(\boldsymbol{w})$. Let $\mathcal{P}$ denote the class of homomorphisms $\{\rho_{\boldsymbol{w},\mathbf{V}} : w \in \Omega \text{ and } \mathbf{V} \in \mathcal{M}_{p+q} \otimes \mathbb{C}^m\}$.

As pointed out in the beginning, for each $k = 1, 2, \ldots$, there is an induced homomorphism

$$\rho^{(k)}_{\boldsymbol{w},\mathbf{V}} := \rho_{\boldsymbol{w},\mathbf{V}} \otimes I_k : \mathcal{A}(\Omega) \otimes \mathcal{M}_k \to \mathcal{M}_{p+q} \otimes \mathcal{M}_k \simeq \mathcal{M}_{(p+q)k}.$$

It is easy to verify that $\rho^{(k)}_{\boldsymbol{w},\mathbf{V}}$ is unitarily equivalent (via a fixed unitary which happens to be a permutation matrix) to the map

$$F \mapsto \begin{pmatrix} F(\boldsymbol{w}) \otimes I_p & \langle DF(\boldsymbol{w}), \mathbf{V} \rangle \\ 0 & F(\boldsymbol{w}) \otimes I_q \end{pmatrix}, \ F \in \mathcal{A}(\Omega) \otimes \mathcal{M}_k.$$

Here $\langle DF(\boldsymbol{w}), \mathbf{V}\rangle = \frac{\partial F}{\partial z_1}(\boldsymbol{w})\otimes V_1 + \cdots + \frac{\partial F}{\partial z_m}(\boldsymbol{w})\otimes V_m$. Using a bi-holomorphic automorphism of the unit ball $(\mathcal{M}_k)_1$ of $\mathcal{M}_k$ which takes $F(\boldsymbol{w})$ to 0, one can prove the following theorem (cf. [7, Lemma 3.3] and [13]).

THEOREM 1.1. *If k is any positive integer, then $\|\rho^{(k)}_{\boldsymbol{w},\mathbf{V}}\| \le 1$ if and only if $\|\langle DF(\boldsymbol{w}), \mathbf{V}\rangle\| \le 1$ for all $F \in \mathcal{A}(\Omega)\otimes(\mathcal{M}_k)_1$ satisfying $F(\boldsymbol{w}) = 0$.*

Let $\mathrm{Hol}(\Omega_1, \Omega_2)$ be the set of all holomorphic functions f from Ω_1 into Ω_2. It is easy to verify that $\{\nabla f(\boldsymbol{w}) : f \in \mathrm{Hol}(\Omega, \mathbb{D}), f(\boldsymbol{w}) = 0\} \subseteq \mathbb{C}^m$ is a unit ball with respect to some norm. The dual of this norm, called the Carathéodory norm, is denoted by $C_{\boldsymbol{w},\Omega}$. Thus the homomorphism $\rho_{\boldsymbol{w},\mathbf{V}}$ is contractive if and only if the linear map $L_{\boldsymbol{w},\mathbf{V}} : (\mathbb{C}^m, C^*_{\boldsymbol{w},\Omega}) \to (\mathcal{M}_n, \mathrm{op})$ defined by $L_{\boldsymbol{w},\mathbf{V}} : (z_1, \ldots, z_m) \mapsto z_1V_1 + \cdots + z_mV_m$ is contractive. We record the particular case of $\Omega = \mathbb{D}^m$ and $w = 0$ as a separate Lemma.

LEMMA 1.2. *Every contractive homomorphism of $\mathcal{A}(\mathbb{D}^m)$ which is in the class $\mathcal{P}$ is induced by an m-tuple of contractions.*

It is not much harder to verify that $\{DF(\boldsymbol{w}) : F \in \mathrm{Hol}(\Omega, \big(\mathcal{M}_n\big)_1), F(\boldsymbol{w}) = 0\} \subseteq \mathcal{M}_n \otimes \mathbb{C}^m$ is a unit ball with respect to some norm. In analogy with the case of $k = 1$, we let $C^{(k)}_{\boldsymbol{w},\Omega}$ denote the norm of the dual space. It is clear that $DF(\boldsymbol{w})$ maps the unit ball in $\mathbb{C}^m$ with respect to the Carathéodory norm contractively into the unit ball of $\mathcal{M}_n$ with respect to the operator norm. However, the question of whether any such linear contraction is $DF(\boldsymbol{w})$ for some F in $\mathrm{Hol}(\Omega, \mathcal{M}_n)$ was first raised in Paulsen [12]. If Ω is a unit ball with respect to some norm, say $\|\cdot\|$, and $\boldsymbol{w} = 0$ then an easy application of the Schwarz lemma shows that $C_{0,\Omega} = \|\cdot\|$. In this case, it is easy to see that the answer to the question of Paulsen is yes. There are examples to show that the answer is no in general - even when Ω is a unit ball with respect to some norm but $\boldsymbol{w} \ne 0$. Now, the homomorphism $\rho^{(k)}_{\boldsymbol{w},\mathbf{V}}$, for any integer $k > 1$, is contractive if and only if the linear map $L^{(k)}_{\boldsymbol{w},\mathbf{V}} : (\mathbb{C}^m \otimes \mathcal{M}_n, C^{(k)*}_{\boldsymbol{w},\Omega}) \to (\mathcal{M}_n \otimes \mathcal{M}_k, \mathrm{op})$ defined by $L^{(k)}_{\boldsymbol{w},\mathbf{V}} : (\Lambda_1, \ldots, \Lambda_m) \mapsto \Lambda_1 \otimes V_1 + \cdots + \Lambda_m \otimes V_m$ is contractive. The fact that contractive homomorphisms from the class $\mathcal{P}$, with $p = q = 1$, are completely contractive (cf. [7] and [8]) plays a significant role in proving that Caratheodory metric coincides with the Kobayashi metric for such domains (cf. [15]).

We now point out that many of the homomorphisms $\rho_{\boldsymbol{w},\mathbf{V}}$ are induced by localization of a certain tuple $\mathbf{T}$ of commuting bounded linear operators in the Cowen-Douglas class $B_k(\Omega)$ (cf. [5]). Such tuples satisfy (among other things) :

$$\dim \cap_{j=1}^m \ker(T_j - w_j) = k$$

for some positive integer k and for each $\boldsymbol{w} = (w_1, \ldots, w_m) \in \Omega$. Adjoints of multiplication operators on a functional Hilbert space consisting of holomorphic functions defined on Ω and taking values in $\mathbb{C}^k$ provide an abundance of such operator tuples (cf. [6]). The study of such operators was initiated in [5] by Cowen and Douglas using a slightly different language. If $\gamma^{(\ell)}_{\boldsymbol{w}}$ is a joint eigenvector for the operator tuple for $1 \le \ell \le k$ then differentiating the relation $(T_j - w_j)(\gamma^{(\ell)}_{\boldsymbol{w}}) = 0$, it is easy to see that $(T_\ell - w_\ell)^2\partial_j\gamma^{(\ell)}_{\boldsymbol{w}} = 0$ for $1 \le j \le m$. Thus the dimension of $\ker(T_\ell - w_\ell)^2$ is $(m+1)k$. Then it is easily verified that the localizations $N(w_1) = (T_1 - w_1)|_{\cap_{j=1}^k \ker(T_j - w_j)^2}, \ldots, N(w_m) = (T_m - w_m)|_{\cap_{j=1}^k \ker(T_j - w_j)^2}$ map $(\cap_{j=1}^m \ker(T_j - w_j))^\perp$ to $\cap_{j=1}^m \ker(T_j - w_j)$. Hence their matricial representation is of the form (1.1). This operator tuple therefore defines a homomorphism which is in the class $\mathcal{P}$ (with $p = k,\ q = mk$). It is possible to build the operator tuple $\mathbf{T}$ (up to unitary equivalence) from the localizations $\{N(\boldsymbol{w}) = (N(w_1), \dots, N(w_m)) : \boldsymbol{w} \in \Omega\}$. For a precise statement, we refer the reader to [5]. Therefore, it is likely that a better understanding of these localizations will play a significant role in answering questions about the operator tuple $\mathbf{T}$. In particular, one may ask under what circumstance a dilation of the tuple $\mathbf{T}$ may be built out of given dilations of the localizations $N(\boldsymbol{w}),\ w \in \Omega$.

In the next section, we study properties of the homomorphism $\rho_{\boldsymbol{w},\mathbf{V}} : \mathcal{A}(\Omega) \to \mathcal{M}_n$ in the class $\mathcal{P}$. From now on, we make the simplifying assumption $p = q$.

2. The homomorphism $\rho_{\boldsymbol{w},\mathbf{V}}$

Suppose Ω is the unit ball with respect to some norm. If Ω is also homogeneous, that is, if for each $\boldsymbol{w}$ in Ω there is a bi-holomorphic automorphism φ of Ω which maps $\boldsymbol{w}$ to 0 then

$$\{G \in \mathrm{Hol}(\Omega, (\mathcal{M}_n)_1) : G(\boldsymbol{w}) = 0\} = \{F \circ \varphi : F \in \mathrm{Hol}(\Omega, (\mathcal{M}_n)_1),\ F(0) = 0\}.$$

The chain rule then implies that $D\varphi(\boldsymbol{w})$ is unitary with respect to the two norms $C^{(k)}_{\boldsymbol{w},\Omega}$ and $C^{(k)}_{0,\Omega}$. Consequently, it is easy to see that $\rho^{(k)}_{\boldsymbol{w},\mathbf{V}}$ is contractive if and only if $\rho^{(k)}_{0,\mathbf{V}} \cdot D\varphi(\boldsymbol{w})$ is contractive. Therefore, it is enough to consider only the case $\boldsymbol{w} = 0$.

The first lemma is obvious and we omit the proof. If $V_i = V_{i1} \oplus V_{i2}$ for all $i = 1, \dots, m$, where V_{i1} are $\ell \times \ell$ and V_{i2} are $(n-\ell) \times (n-\ell)$, let $\mathbf{V}_{(\mathbf{j})} = (V_{1j}, \dots, V_{mj})$ for $j = 1, 2$.

Lemma 2.1. $\|\rho_{\boldsymbol{w},\mathbf{V}}\| = \max\{\|\rho_{\boldsymbol{w},\mathbf{V}_{(\mathbf{j})}}\| : j = 1, 2\}$ *and* $\|\rho_{\boldsymbol{w},\mathbf{V}}\|_{cb} = \max\{\|\rho_{\boldsymbol{w},\mathbf{V}_{(\mathbf{j})}}\|_{cb} : j = 1, 2\}$.

Given an $n \times n$ matrix V and a positive integer k, let $V^{\otimes k} \in \mathcal{B}((\mathbb{C}^n)^{\otimes k})$ be the k-fold tensor product $V \otimes \cdots \otimes V$. Given a tuple $\mathbf{V} = (V_1, \dots, V_m)$, we denote the tuple $(V_1^{\otimes k}, \dots, V_m^{\otimes k})$ by $\mathbf{V}^{\otimes \mathbf{k}}$. When $\boldsymbol{w} = 0$, we shall denote $\rho_{\boldsymbol{w},\mathbf{V}}$ by $\rho_{\mathbf{V}}$ for the sake of brevity.

Lemma 2.2. *Let $\rho_{\boldsymbol{w},\mathbf{V}}$ defined on $\mathcal{A}(\mathbb{D}^m)$ be completely contractive and k be a positive integer. Then $\rho_{\boldsymbol{w},\mathbf{V}^{\otimes \mathbf{k}}}$ is completely contractive.*

Proof We first show that $\rho_{\boldsymbol{w},\mathbf{V}^{\otimes \mathbf{2}}}$ is completely contractive. The vector $\boldsymbol{w}$ can be taken to be 0 without loss of generality. Let $X_1, \dots, X_m$ be such that $\|\lambda_1 X_1 + \cdots + \lambda_m X_m\| \le 1$ for any choice of $(\lambda_1, \dots, \lambda_m) \in \mathbb{D}^m$. We have to show that $\|X_1 \otimes V_1 \otimes V_1 + \cdots + X_m \otimes V_m \otimes V_m\| \le 1$. Let $\tilde{X}_i = z_i X_i$ for $(z_1, \dots, z_m) \in \mathbb{D}^m$ and $i = 1, \dots, m$. Then

$$\|\lambda_1 \tilde{X}_1 + \cdots + \lambda_m \tilde{X}_m\| = \|\lambda_1 z_1 X_1 + \cdots + \lambda_m z_m X_m\| \le 1,$$

because $|\lambda_i z_i| = |\lambda_i z_i| \le 1$. Since $\rho_{\mathbf{V}}$ is completely contractive, we have $\|\tilde{X}_1 \otimes V_1 + \cdots + \tilde{X}_m \otimes V_m\| \le 1$. So the tuple $(X_1 \otimes V_1, \dots, X_m \otimes V_m)$ is such that $\|z_1(X_1 \otimes V_1) + \cdots + z_m(X_m \otimes V_M)\| \le 1$ for any $(z_1, \dots, z_m)$ with $|z_i| \le 1$. Using the complete contractivity of $\rho_{\mathbf{V}}$ again, we have $\|X_1 \otimes V_1 \otimes V_1 + \cdots + X_m \otimes V_m \otimes V_m\| \le 1$. This, by our original assumption on $X_1, \dots, X_m$ means that $\rho_{\mathbf{V}^{\otimes \mathbf{2}}}$ is completely contractive. The rest follows by induction. □

Given two tuples $\mathbf{V} = (V_1, \dots, V_m)$ and $\mathbf{W} = (W_1, \dots, W_m)$, let $\mathbf{V} \otimes \mathbf{W}$ denote the tuple $(V_1 \otimes W_1, \dots, V_m \otimes W_m)$. A generalization of the lemma above is the following whose proof is similar and hence we omit.

Lemma 2.3. *If $\rho_{\boldsymbol{w},\mathbf{V}}$ and $\rho_{\boldsymbol{w},\mathbf{W}}$ defined on $\mathcal{A}(\mathbb{D}^m)$ are completely contractive, then $\rho_{\boldsymbol{w},\mathbf{V}\otimes\mathbf{W}}$ is completely contractive.*

The next Lemma, on the one hand, provides a natural proof of Ando's theorem within the class $\mathcal{P}$. On the other hand, it also shows that if there were contractive homomorphisms in

$\mathcal{P}$ of the tri-disc algebra which do not dilate then there must be one which is induced by a triple (I, U, V), where U, V are non-commuting unitaries. Therefore, in looking for Parrott-like examples within the class $\mathcal{P}$, it is natural to restrict to tuples $\mathbf{V}$ of unitaries (as Parrott actually did without any apparent justification).

LEMMA 2.4. *If there is a contractive homomorphism $\rho_{\mathbf{V}}$ which is not completely contractive, then there is a contractive homomorphism $\rho_{\mathbf{U}}$ with $U_1, \ldots, U_m$ unitaries which is not completely contractive.*

PROOF If $F : \mathbb{D}^m \to \mathcal{M}_k$ is a holomorphic function with $F(0) = 0$, then $\frac{\partial F}{\partial z_1}(0), \ldots, \frac{\partial F}{\partial z_m}(0)$ satisfy

$$\|z_1 \frac{\partial F}{\partial z_1}(0) + \cdots + z_m \frac{\partial F}{\partial z_m}(0)\| \leq 1 \text{ for all } (z_1, \ldots, z_m) \text{ satisfying } |z_1| \leq 1, \ldots, |z_m| \leq 1$$

and conversely given any $X_1, \ldots, X_m \in \mathcal{B}(\mathcal{H})$ satisfying

$$\|z_1 X_1 + \cdots + z_m X_m\| \leq 1 \text{ for all } (z_1, \ldots, z_m) \text{ with } |z_1| \leq 1, \ldots, |z_m| \leq 1,$$

the contractive function $F : \mathbf{D}^m \to \mathcal{M}_k$ given by $F(z_1, \ldots, z_m) = z_1 X_1 + \cdots + z_m X_m$ vanishes at 0 and has $X_1, \ldots, X_m$ as its first partial derivatives with respect to $z_1, \ldots, z_m$ respectively. Thus,

$$\begin{aligned} \|\rho_{\mathbf{V}}\|_{cb} &= \sup\{\|\rho_{\mathbf{V}}(F)\| : \|F\| \leq 1\} \\ &= \sup\{\|X_1 \otimes V_1 + \cdots + X_m \otimes V_m\| : \|z_1 X_1 + \cdots + z_m X_m\| \leq 1\}. \end{aligned}$$

Now if there is a contractive homomorphism $\rho_{\mathbf{V}}$ which is not completely contractive, then $\sup\{\|\rho_{\mathbf{V}}\|_{cb} : V_1, \ldots, V_m \text{ are contractions}\} > 1$. But since the set of extreme points of the convex set consisting of contractions, is the set of unitaries, this supremum is attained for a unitary tuple $\mathbf{U} = (U_1, \ldots, U_m)$. □

We end this section by proving that contractivity implies complete contractivity for the homomorphisms $\rho_{\mathbf{V}}$ on the bi-disc algebra. Of course, this is Ando's theorem [1] for homomorphisms in the class $\mathcal{P}$.

COROLLARY 2.5. *If Ω is the bi-disc and $\rho_{\mathbf{V}}$ is contractive then $\rho_{\mathbf{V}}$ is completely contractive.*

PROOF The computations of the above lemma show that

$$\begin{aligned} &\sup\{\|\rho_{\mathbf{V}}\|_{cb} : V_1, \ldots, V_m \text{ are contractions}\} \\ &= \sup\{\|\rho_{\mathbf{U}}\| : U_1, \ldots, U_m \text{ are unitaries}\} \\ &= \sup\{\|X_1 \otimes U_1 + \cdots + X_m \otimes U_m\| : \|z_1 X_1 + \cdots + z_m X_m\| \leq 1, U_1, \ldots, U_m \text{ are unitaries}\}. \end{aligned}$$

In the case $m = 2$, this last quantity is

$$\sup\{\|X_1 \otimes U_1 + X_2 \otimes U_2\| : \|z_1 X_1 + z_2 X_2\| \leq 1, U_1, U_2 \text{ are unitaries}\}.$$

Since norm is invariant under multiplication by a unitary, multiplying by $I \otimes U_2^*$ from left and putting $W = U_2^* U_1$, we get

$$\begin{aligned} \|\rho_{\mathbf{V}}\|_{cb} &\leq \sup\{\|X_1 \otimes I + X_2 \otimes W\| : \|z_1 X_1 + z_2 X_2\| \leq 1, W \text{ unitary}\} \\ &= \sup\{\max \|X_1 + \lambda_2 X_2\| : \|z_1 X_1 + z_2 X_2\| \leq 1, \lambda_1, \lambda_2 \text{ are eigenvalues of } W\} \leq 1. \end{aligned}$$

That completes the proof. □

3. The tri-disc algebra

In this section, for the case $\Omega = \mathbb{D}^3$, we characterize the homomorphisms $\rho_{\mathbf{V}}$ induced by a triple of $n \times n$ unitaries $\mathbf{V} = (V_1, V_2, V_3)$. We first make a simplification by putting $U = V_1^* V_2$ and $V = V_1^* V_3$. For the rest of the article, ρ will denote the homomorphism induced by the vector $(0,0,0)$ and the triple (I, U, V) on $\mathcal{A}(\mathbb{D}^3)$. Since the operator norm is unitarily invariant, we have $\|\rho_{\mathbf{V}}^{(k)}\| = \|\rho^{(k)}\|$ for all $k = 1, 2, \ldots$. Thus $\rho_{\mathbf{V}}^{(k)}$ is contractive (respectively completely contractive) if and only if ρ is so. Moreover, $UV = VU$ if $\mathbf{V}$ is a commuting tuple. Thus without loss of generality, we shall henceforth be concerned with ρ and show, in this section that the homomorphism ρ is completely contractive if and only if U commutes with V.

Lemma 3.1. *If $UV = VU$, then ρ is completely contractive.*

Proof Since $UV = VU$, there is a unitary W such that $WUW^* = D_1 = \operatorname{diag}(\lambda_1, \ldots, \lambda_n)$ and $D_2 = \operatorname{diag}(\mu_1, \ldots, \mu_n)$, where $\lambda_1, \ldots, \lambda_n$ and $\mu_1, \ldots, \mu_n$ are eigenvalues of U and V respectively. For any positive integer k, let $X, Y, Z \in \mathcal{M}_k$ satisfy

$$\|\alpha X + \beta Y + \gamma Z\| \leq 1 \tag{3.1}$$

for any scalars α, β, γ of modulus at most 1. Thus,

$$\begin{aligned} \|I_n \otimes X + U \otimes Y + V \otimes Z\| &= \|I_n \otimes X + D_1 \otimes Y + D_2 \otimes Z\| \\ &= \| \oplus_{i=1}^n (X + \lambda_i Y + \mu_i Z)\| \\ &= \max_{1 \leq i \leq n} \|X + \lambda_i Y + \mu Z\| \leq 1, \end{aligned}$$

by (3.1). Now $\|\rho\|_{cb} = \sup\{\|I_n \otimes X + U \otimes Y + V \otimes Z\|\}$, where the supremum is over all positive integers k and all X, Y, Z in $\mathcal{M}_k$ satisfying (3.1). So $\|\rho\|_{cb} \leq 1$. □

Theorem 3.7 is the converse to this Lemma. We need the following lemmas to complete the proof of this Theorem.

Lemma 3.2. *The unitaries U and V have a joint eigenvector of joint eigenvalue* $(1,1)$ *if and only if $\|I + U + V\| = 3$.*

Proof If there is a joint eigenvector of joint eigenvalue $(1,1)$, then obviously $\|I+U+V\| = 3$. Conversely, suppose $\|I+U+V\| = 3$. Choose a unit vector x in $\mathbb{C}^n$ satisfying $\|(I+U+V)x\| = 3$. Then there are nine terms in the expansion of $\|x + Ux + Vx\|^2$, each of modulus at most 1. Since their sum is 9, modulus of each term is equal to 1. By the condition of equality in Cauchy-Schwarz inequality, we have $Ux = cx$ and $Vx = dx$ for some scalars c, d of modulus 1. But $|1 + c + d| = \|x + Ux + Vx\| = 3$. Hence $c = d = 1$. Thus $Ux = Vx = x$. □

Lemma 3.3. *If U and V are any two $n \times n$ unitary matrices, the matrices $U^{\otimes n}$ and $V^{\otimes n}$ have a common eigenvector.*

Proof Let $\lambda_1, \ldots, \lambda_n$ be the eigenvalues of U and $x_1, \ldots, x_n$ be the corresponding orthonormal eigenbasis for $\mathbb{C}^n$, i.e., $Ux_i = \lambda_i x_i$ for all $i = 1, \ldots, n$. Let $\mu_1, \ldots \mu_n$ be the eigenvalues of V and $y_1, \ldots, y_n$ the orthonormal eigenbasis corresponding to V. For any σ in the permutation group S_n, let $x_\sigma \stackrel{\text{def}}{=} x_{\sigma(1)} \otimes \cdots \otimes x_{\sigma(n)}$ and

$$x_1 \wedge \cdots \wedge x_n \stackrel{\text{def}}{=} \sum_{\sigma \in S_n} \epsilon_\sigma x_\sigma$$

where ϵ_σ is ± 1, depending on whether σ is an even or an odd permutation. Similarly define y_σ and $y_1 \wedge \cdots \wedge y_n$. Then $x_1 \wedge \cdots \wedge x_n = e^{i\theta} y_1 \wedge \cdots \wedge y_n$ for some $\theta \in [0, 2\pi)$. Hence $x_1 \wedge \cdots \wedge x_n$ is a common eigenvector of $U^{\otimes n}$ and $V^{\otimes n}$ with eigenvalues $\prod \lambda_i$ and $\prod \mu_i$ respectively. □

THEOREM 3.4. *If U and V are two $n\times n$ unitary matrices, which do not have a joint eigenvector of joint eigenvalue $(1,1)$, then ρ is not completely contractive.*

PROOF First note that Lemma 3.3 implies that the two matrices $U^{\otimes n}\otimes(U^*)^{\otimes n}$ and $V^{\otimes n}\otimes(V^*)^{\otimes n}$ have a joint eigenvector with joint eigenvalue $(1,1)$. Let ρ be completely contractive. Then it follows from Lemma 2.2 and Lemma 2.3 that the homomorphism, say η, determined by the triple $(I, U^{\otimes n}\otimes(U^*)^{\otimes(n-1)}, V^{\otimes n}\otimes(V^*)^{\otimes(n-1)})$ is completely contractive. Consider the function

$$f(z_1,z_2,z_3)=z_1I_n+z_2U^*+z_3V^*.$$

Since U and V do not have a common eigenvector, by Lemma 3.2, we have $\|f\|<3$. But $\|\eta(f)\|=\|I+U^{\otimes n}\otimes(U^*)^{\otimes n}+V^{\otimes n}\otimes(V^*)^{\otimes n}\|=3$ by the observation above. That is a contradiction. □

REMARK 3.5. An inductive argument for the proof of the theorem above can also be given as follows. First note that if η is the homomorphism determined by the triple (I,U^*,V^*), then ρ is completely contractive if and only if η is so. If $f_1(z_1,z_2,z_3)=z_1I+z_2U^*+z_3V^*$, then $\|f_1\|<3$ by Lemma 3.2, and so $\|\rho(f_1)\|=\|I\otimes I+U\otimes U^*+V\otimes V^*\|\le 3$. If $\|\rho(f_1)\|>\|f_1\|$, then ρ is not n-contractive. Otherwise, take $f_2(z_1,z_2,z_3)=z_1I\otimes I+z_2U\otimes U^*+z_3V\otimes V^*$ and observe that $\|f_2\|<3$. If $\|\rho(f_2)\|>\|f_2\|$, then ρ is not n^2-contractive. Otherwise, take $f_3(z_1,z_2,z_3)=z_1I\otimes I\otimes I+z_2U\otimes U\otimes U^*+z_3V\otimes V\otimes V^*$. Then $\|f_3\|<3$. If $\|\eta(f_3)\|>\|f_3\|$, then η is not n^3-contractive, and so ρ is not n^3-contractive. Otherwise continue with this procedure using η and ρ alternatively. This procedure will stop, by Lemma 3.3, since $\|\eta(f_{2n-1})\|=3$.

LEMMA 3.6. *If U and V are two $n\times n$ non-commuting unitary matrices, then there is an $n\times n$ unitary matrix W and an integer k between 0 and $n-2$ such that*

(a) $WUW^*=I_k\oplus\tilde U$ and $WVW^*=I_k\oplus\tilde V$,

(b) $\|I_{n-k}+\tilde U+\tilde V\|<3$.

PROOF If $\|I_n+U+V\|<3$ to start with, then $k=0$ and $W=I_n$. If $\|I_n+U+V\|=3$, then by Lemma 3.2, we choose $x_1\in\mathbb{C}^2$ such that $\|x_1\|=1$ and $Ux_1=Vx_1=x_1$. Since U and V are unitaries, the subspace $\mathcal{L}_1\stackrel{\text{def}}{=}\text{span}\,\{x_1\}^\perp$ is invariant under them. Either $\|I_{n-1}+U_{\mathcal{L}_1}+V_{\mathcal{L}_1}\|<3$ or we repeat the process till we reach a k such that for the invariant subspace $\mathcal{L}\stackrel{\text{def}}{=}\text{span}\,\{x_1,\dots,x_k\}^\perp$, we have $\|I_{n-k}+U|_{\mathcal{L}}+V|_{\mathcal{L}}\|<3$. Choosing an orthonormal basis $\{y_1,\dots,y_{n-k}\}$ for $\mathcal{L}$, put W to be the unitary with column $x_1,\dots,x_k,y_1,\dots,y_{n-k}$. □

THEOREM 3.7. *If U and V are two $n\times n$ non-commuting unitary matrices, then ρ is not completely contractive.*

PROOF Let $\tilde U$ and $\tilde V$ be as in Lemma 3.6. If $\tilde\rho$ is the homomorphism determined on $\mathcal{A}(\mathbb{D})$ by the tuple $(I_{n-k},\tilde U,\tilde V)$, then we saw in Lemma 3.4 that $\|\tilde\rho\|_{cb}>1$. But then Lemma 2.1 implies that $\|\rho\|_{cb}>1$. □

4. THE 2-CONTRACTIVITY

In this section, we show that if U and V are 2×2 non-commuting unitary matrices, then the induced homomorphism ρ on the tri-disc algebra is not even 2-contractive.

LEMMA 4.1. *If $U,V\in\mathcal{M}_2$ and $UV\neq VU$, then $\|I_2+U+V\|<3$.*

PROOF If $\|I_2+U+V\|=3$, then by Lemma 3.2, there is a unit joint eigenvector x with joint eigenvalue $(1,1)$. Choose $y\perp x$ with $\|y\|=1$. Then y is an eigenvector of both U and V. So in the basis $\{x,y\}$, the unitaries U and V are diagonal. So they commute. □.

LEMMA 4.2. *If* $U, V \in \mathcal{M}_2$, *then* $\|I_2 \otimes I_2 + U \otimes U^* + V \otimes V^*\| = 3$.

PROOF Let

$$\begin{aligned} \mathcal{L}_1 &= \text{span}\,\{x \otimes x : x \in \mathbb{C}^2 \text{ and } x \text{ is an eigenvector of } U\} \text{ and} \\ \mathcal{L}_2 &= \text{span}\,\{x \otimes x : x \in \mathbb{C}^2 \text{ and } x \text{ is an eigenvector of } V\}. \end{aligned}$$

Then $\dim\mathcal{L}_1 = \dim\mathcal{L}_2 = 2$. Since both $\mathcal{L}_1$ and $\mathcal{L}_2$ are subsets of the symmetric tensor product of $\mathbb{C}^2$ with itself which has dimension 3, they have non-trivial intersection. Choose $y \in \mathcal{L}_1 \cap \mathcal{L}_2$ with $\|y\| = 1$. Note that $(U \otimes U^*)y = (V \otimes V^*)y = y$. So $\|(I_2 \otimes I_2 + U^* \otimes U + V \otimes V^*)y\| = 3$. $\square$

THEOREM 4.3. *If* U *and* V *are* 2×2 *non-commuting unitaries, then the homomorphism* ρ *on* $\mathcal{A}(\mathbb{D}^3)$ *is contractive, but not 2-contractive.*

PROOF The homomorphism ρ determined by I, U, V is always contractive. Let $f : \mathbb{D}^3 \to \mathcal{M}_2$ be defined by $f(z_1, z_2, z_3) = z_1 I + z_2 U^* + z_3 V^*$. Then $\|f\| < 3$, but $\|\rho(f)\| = 3$. First note that $\|f\| < 3$ by Lemma 3.2. Then note that $\|\rho(f)\| = 3$ by Lemma 4.2. Hence the proof. $\square$

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