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The Hodge conjecture for certain moduli varieties

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Abstract. For smooth projective varieties X over $\mathbb{C}$, the Hodge Conjecture states that every rational Cohomology class of type (p,p) comes from an algebraic cycle. In this paper, we prove the Hodge conjecture for some moduli spaces of vector bundles on compact Riemann surfaces of genus 2 and 3.

Keywords. Chow groups; Abel-Jacobi maps; moduli spaces; normal functions; Hecke correspondences.

Introduction

For smooth projective varieties X over $\mathbb{C}$, the field of complex numbers, the Hodge conjecture states that every rational cohomology class of type (p,p) comes from an algebraic cycle. More precisely, consider the Hodge decomposition

$$H^i(X,\mathbb{C}) = \sum_{p+q=i} H^{p,q}(X).$$

Let $C^p(X)$ denote the Chow group of algebraic cycles of codimension p on X, modulo rational equivalence. Then one has the 'class map'

$$\lambda_X^p : C^p(X)\otimes\mathbb{Q}\to H^{2p}(X,\mathbb{Q})\cap H^{p,p}(X).$$

Then the Hodge (p,p) conjecture states that λ_X^p is surjective.

Let C be an irreducible smooth projective curve if genus $g\geqslant 2$, and let $M(n,\xi)$ be the moduli space of stable vector bundles V on C, of rank n, $\det V\simeq\xi$, ξ a line bundle of degree d such that $(n,d)=1$. The aim of this paper is to prove the Hodge (p,p) conjecture in the case when $g=2$, $n=3$ ($\dim M(3,\xi)=8$). In the case when $n=2$, $g=2,3,4$, the Hodge conjecture can be proved by elementary means which we indicate at the end of the paper.

The case we consider is of interest, as it gives a non-trivial family of examples where the general method of normal functions is used to prove the conjecture. Geometric descriptions given in [T] in the rank 2 case lead to elementary proofs of the Hodge conjecture. In the rank 3 case, any such description does not give elementary proofs of the Hodge conjecture. (cf. Remark 4.3, 4.4)

The Poincaré–Lefschetz theory of normal functions was generalized and developed by Griffiths and Zucker and had the proof of the Hodge (p,p) conjecture as a primary goal. In this paper we give a natural construction of a smooth projective variety and a proper generically finite morphism onto the moduli of rank n, degree $(ng-n)$ bundles which plays the role of the Lefschetz pencil in the context of normal functions. From the remarks of Zucker (cf. [Z-2], pp. 266) all the known examples where normal functions have been used to prove the Hodge conjecture, more elementary methods have been successful (cf. [M], [Z-2], and [Sh] for a full survey of the Hodge conjecture); however, in the present case this seems unlikely.

In § 1, we recall some general facts. Section 2, contains a theorem giving a criterion for a variational Hodge (p,p) conjecture to hold under some stringent conditions. In § 3, we give a pencil type construction in the context of moduli. Section 4, gives the proof of the conjecture for $M(3,\xi)$.

Some notations. Let X be a smooth projective variety defined over $\mathbb{C}$ the field of complex numbers. We state at the outset that our base field is $\mathbb{C}$. Let $C^p(X)$ denote the Chow group of cycles of codimension p modulo rational equivalence and $A^p(X) \subset C^p(X)$ the subgroup of cycle classes algebraically equivalent to zero.

1. Preliminaries

Lemma 1.1. (*cf.* [Z-1] *A*.2) *Let X and Y be smooth projective varieties, $f: X \to Y$ be a proper generically finite surjection. If the Hodge (p,p) conjecture is true for X, then it is true for Y.*

Proof. We note that, $f_* f^* =$ multiplication by d, both on cycles and cohomology, where $d = [k(X):k(Y)]$. Therefore, if $\gamma \in H^{p,p}(Y,\mathbb{Q})$, $f^*\gamma \in H^{p,p}(X,\mathbb{Q})$; so if $f^*\gamma$ is a rational cycle Z, then

$$d\gamma = (f_* f^* \gamma) = f_* Z$$

implying γ is a rational cycle $1/d(f_* z)$ on Y.

Lemma 1.2. *Let E be a vector bundle of rank $r = e+1$, and let $P = P(E)$. Let $f: P \to X$ be the associated projective bundle. Then the Hodge (p,p) conjecture is true for X if and only if it is true for P.*

Proof. Let h be the relative ample class $\mathcal{O}_P(1)$, and $\hat{h} = c_1(\mathcal{O}_P(1))$. Then we have the well-known decompositions of the Chow groups and cohomology groups of P, and we have the diagram:

$$\begin{array}{ccccccc} C^p(P) & = f^*C^p(X) & \oplus & hf^*C^{p-1}(X) & \oplus \cdots \oplus & h^e f^* C^{p-e}(X) \\ \downarrow \lambda_P^p & \downarrow \lambda_X^p & & \downarrow \lambda_X^{p-1} & & \downarrow \lambda_X^{p-e} \\ H^{2p}(P) & = f^*H^{2p}(X) & \oplus & \hat{h} f^* H^{2p-2}(X) & \oplus \cdots \oplus & \hat{h}^e f^* H^{2p-2e}(X) \end{array}$$

From this diagram, the proof follows easily, noting the fact that f^* is an injection both on cycles and cohomology.

Lemma 1.3. *Let X be a smooth projective variety, $Y \hookrightarrow X$ a smooth closed subvariety of codimension r; let $U \hookrightarrow X$ be $X - Y$, i (resp. j) the inclusion of Y (resp. U) in X. Then we have the following commutative diagram:*

$$\begin{array}{ccccc} C^{q-r}(Y) & \xrightarrow{i_*} & C^q(X) & \xrightarrow{j_*} & C^q(U) \to 0 \\ \downarrow \lambda_Y & & \downarrow \lambda_X & & \downarrow \\ H^{2q-2r}(Y) & \xrightarrow{\text{Gysin}} & H^{2q}(X) & \longrightarrow & H^{2q}(U) \end{array}$$

Proof. This follows from the existence of the Gysin map i_* which is functorial with respect to the class map λ. (cf. J Milne, Etale Cohomology, Proposition 9.3, Ch. VI).

DEFINITION 1.4

Let $J^p(X)$ be the pth Griffiths-intermediate Jacobian of X based on $H^{2p-1}(X)$ ([G], [Z-2]) and let

$$\theta^p: A^p(X) \to J^p(X)$$

be the Abel–Jacobi map on codimension p-cycles algebraically equivalent to 0.

We say, X has the Abel–Jacobi property for p, if θ^p is surjective.

Lemma 1.5. *Let X be a smooth projective variety and E a vector bundle of rank $r = e + 1$ on X. Let $P = \mathbb{P}(E)$ be the associated projective bundle and $f: P \to X$ the projection. Then $A^i(X)$ has the Abel–Jacobi property for all i if and only if $A^p(P)$ has it for all p.*

Proof. Let $\theta^p: A^p(X) \to J^p(X)$ be the Abel–Jacobi map. Then by assumption, θ^p is surjective for all p. By the standard decomposition theorems for Chow groups, and cohomology of a projective bundle we have

$$H^{2p-1}(P) \simeq f^*H^{2p-1}(X) \oplus \sum_{i=1}^{e} f^*H^{2p-2i-1}(X)\hat{h}^i \qquad (*)$$

$$\hat{h} = c_1(\mathcal{O}_P(1)).$$

We note that this decomposition is true for cohomology with integer coefficients, further, since $\hat{h}$ is of type (1, 1), the isomorphism $(*)$ preserves the Hodge decomposition. Hence, the complex structure on the Griffiths Jacobian on P is canonically isomorphic to the one induced by $(*)$. Therefore, one has

$$J^p(P) \simeq J^p(X) \oplus J^{p-1}(X) \oplus \cdots \oplus J^{p-e}(X).$$

Further, one has a similar decomposition for the Chow groups

$$A^p(P) \simeq A^p(X) \oplus A^{p-1}(X) \oplus \cdots \oplus A^{p-e}(X).$$

Combining this with the functoriality of the Abel–Jacobi maps, we get

$$\theta^p: A^p(P) \to J^p(P)$$

is surjective, since it is so in all the terms in the decomposition. The proof of the converse is similar.

2. Normal functions

Let $f: X \to S$ be a proper smooth morphism, with X, a smooth projective variety, and S a non-singular complex curve. In this section, we prove a theorem which under some very strong assumptions on the fibres of f give the Hodge (p, p) conjecture for X. The basic ideas in this theorem come from the work of Griffiths and Zucker ([Z-1], [Z-2], [Z-3], [Z-4]).

Theorem 2.1. *Let $f: X \to S$ be as above. Let $X_s = f^{-1}(s) \forall s \in S$. Suppose that the following conditions hold:*

(*a*) *Hodge (p, p) and Hodge $(p-1, p-1)$ are true for $X_s \forall s \in S$.*
(*b*) *X_s has the Abel–Jacobi property in codimension p, i.e. the map*

$$\theta^p: A^p(X_s) \to J^p(X_s)$$

is surjective $\forall s \in S$.

Then Hodge (p,p) holds for X.

Proof. Consider the Leray filtration $\{L^p\}$ on $H^*(X)$ associated to the morphism f. Since the spectral sequence degenerates (cf. [G]), we have:

$$L^0 \supset L^1 \supset L^2.$$

We need the following description of the Leray filtration from ([Z-3], pp. 194):

$$L^1 = \ker\{H^{2p}(X) \to H^{2p}(X_s)\}$$
$$L^2 = \ker\{H^{2p}(X) \to H^{2p}(X - X_s)\}$$
$$= \operatorname{Im}\{H^{2p-2}(X_s) \xrightarrow{\text{Gysin}} H^{2p}(X)\} \quad \text{(cf. Lemma 1.3)}$$

for any $s \in S$, and

$$L^0/L^1 \simeq H^0(S, R^{2p}f_*\mathbb{Q}).$$

We need to handle the (p,p) classes in the rational cohomology of X, which come from the various parts of the Leray filtration.

The *primitive class* i.e. the (p,p) classes lying in L^1 can be dealt with as follows:

(i) Observe firstly that $L^1/L^2 \simeq H^1(S, R^{2p-1}f_*\mathbb{Q})$. Integral (p,p) classes in L^1/L^2, thus arise as cohomology classes of normal functions i.e. holomorphic sections of the intermediate Jacobian bundle, $J^p(X_s) \to S$. This is a consequence of Theorem 2.13 of [Z-4]. Our assumption (b) then ensures by [Z-1], that this normal function comes from a relative algebraic cycle on X.

(ii) (p,p) classes which lie in L^2: Note that

$$L^2 = \operatorname{Im}\{H^{2p-2}(X_s) \xrightarrow{\text{Gysin}} H^{2p}(X)\}$$

and by assumption (a) and Lemma 1.3 of §1, since Hodge $(p-1, p-1)$ holds for X_s, (p,p) classes in L^2 come from algebraic cycles.

Now for the remaining classes, in L^0/L^1, let γ be a (p,p) class in $H^{2p}(X)$, which restricts to non-zero classes γ_s on X_s for all $s \in S$. Let $\sum_{X/S}^d$ denote the Chow variety (or reduced Hilbert scheme) of relative codimension p cycles of degree d on X. By the theory of Hilbert schemes, for some $d \gg 0$, the natural morphism

$$\phi_d: \sum_{X/S}^d \to S$$

is a surjection. Hence for all $\lambda \geqslant 1$, $\sum_{X/S}^{\lambda d} \to S$ is surjective.

Let $V^{\lambda d}$ be the non-empty open subset of S for all $\lambda \geqslant 1$, such that

$$\phi_{\lambda d}: \phi_{\lambda d}^{-1}(V^{\lambda d}) \to V^{\lambda d}$$

is *flat*. (Such a non-empty $V^{\lambda d}$ exists since ϕ_α is a proper surjective morphism.) By a Baire argument, it is easy to see that $\bigcap_{\lambda \geqslant 1} V^{\lambda d} \neq \phi$; choose an $s \in \bigcap_{\lambda \geqslant 1} V^{\lambda d}$ and fix this s. Consider $\gamma|_{X_s} = \gamma_s$; then by (a) of Theorem 2.1, since Hodge (p,p) is true for X_s, express $\gamma_s = \alpha_s - \beta_s$, where α_s and β_s are effective codim p-cycles on X_s of degree l and m respectively. Since we are interested only in rational cohomology, we may assume, without loss of generality that l and m are multiples of d.

Therefore, by choice $s \in V^l \cap V^m$, and $\alpha_s \in \phi_l^{-1}(s)$. Since ϕ_l is flat over V^l, all irreducible components of $\phi_l^{-1}(V^l)$ dominate V^l (S being a smooth curve). Choose an irreducible component of $\phi_l^{-1}(V^l)$ which contains α_s. Then it is easy to see, (by choosing a curve C through α_s and taking its closure in $\sum_{x/S}^l$), that we get a curve S' and a finite morphism $S' \to S$, such that (we could assume S' is also smooth without loss of generality by going to the normalization if need be).

$$\begin{array}{ccc} \Sigma' & \to & \Sigma^l \\ \downarrow & & \downarrow \\ S' & \to & S \end{array}$$

and there is a section for Σ' over S', which passes through α_s. That is if

$$\begin{array}{ccc} X' & \xrightarrow{\mu} & X \\ \downarrow & & \downarrow \\ S' & \longrightarrow & S \end{array}$$

then, there exists an effective codimension p-cycle α of degree l on X', such that $\alpha|_{X_s} = \alpha_s$; where $s' \to s$. We can similarly get a β of deg m over another finite extension, and we can therefore get T, a smooth curve, with a finite morphism

$$T \to S,$$

such that

$$\begin{array}{ccc} Y & \xrightarrow{\mu} & X \\ \downarrow & & \downarrow \\ T & \longrightarrow & S \\ & t \longmapsto s & \end{array}$$

and α and β give codimension p-cycles on Y of degree l and m respectively, s.t.

$$\alpha|_{Y_t} = \alpha_s, \quad \beta|_{Y_t} = \beta_s. \qquad (*)$$

Thus,

$$\varepsilon = [\mu^*\gamma - (\alpha - \beta)] \in H^{2p}(Y, \mathbb{Q})$$

is a cohomology class which (by $(*)$) lies in

$$\ker (H^{2p}(Y) \to H^{2p}(Y_t)).$$

Hence ε is a *primitive cohomology* class on Y; observe that fibres of $Y \to T$ are the same as those of $X \to S$, and hence the hypotheses of Theorem 2.1, hold for the fibres of $Y \to T$ as well. So by the first part of our proof, ε comes from a codimension p-algebraic cycle ε' on Y. i.e.

$$\mu^*\gamma - (\alpha - \beta) = \varepsilon' \Rightarrow \mu^*\gamma = \varepsilon' + (\alpha - \beta)$$

is algebraic. Since $\mu: Y \to X$ is a proper finite surjection, by Lemma 1.1, it follows that γ itself is algebraic.

3. A pencil-type construction for moduli

In the discussion that follows, we describe a pencil-type construction in the context of moduli spaces of vector bundles. We remark that, in general, the geometry of a general hyperplane section in the moduli space is not very transparent and so the usual theory of normal functions and Lefschetz pencil cannot be applied in this setting. We begin by proving a lemma which is essential in the construction.

Lemma 3.1. *Let W be a stable vector bundle of rank 2 and degree 3. Let V be a non-split extension*

$$0 \to \mathcal{O} \to V \xrightarrow{v} W \to 0.$$

Then V is semi-stable.

Proof. This is an elementary consequence of Propositions 4.3, 4.4. and 4.6 of [N-S]. To see this, suppose that V is not semistable, then by Proposition 4.6 there exists an F, stable of rank $\leqslant 2$ such that

$$\mu(F) \geqslant \mu(V) = 1$$

and a non-zero element $f \in \mathrm{Hom}(F, V)$. Thus $\mu(F) \geqslant \mu(W)$. Thus $v \circ f \in \mathrm{Hom}(F, W)$. If $v \circ f$ is zero, f must factor through $\mathcal{O}$ which gives an immediate contradiction. If $v \circ f$ is non-zero, by Proposition 4.4 of [N-S], if W_1 is the subbundle of W generated by $\mathrm{Im}(v \circ f)$ then $\mu(W_1) \geqslant \mu(W)$.

Since W is stable, it implies $W_1 \simeq W$ and $v \circ f$ is an isomorphism, which gives a splitting for v, q.e.d.

Let $M_L = M(3, L)$, be the moduli space of semi-stable bundles of rank 3, deg $3g - 3$, $\wedge^n V \simeq L$, g being the genus of C, i.e. $\deg(L) = 3g - 3$, $g = 2$.

Consider the Θ-divisor in M_L which is defined as follows:

$$\Theta = \{V \in M_L | h^0(V) > 0\}.$$

More generally, we can define for all $\xi \in J^0(C)$, the divisor

$$\Theta_\xi = \{V \in M_L | h^0(V \otimes \xi) > 0\}.$$

Let $\mathscr{W}_\xi$ be the universal family on $C \times M(2, \xi \otimes L)$ and consider the bundle of extensions given by

$$P_\xi = \mathbb{P}(R^1 p_* \mathscr{W}_\xi^*),$$

where $p: C \times M(2, \xi \otimes L) \to M(2, \xi \otimes L)$. Observe that, if $W \in M(2, \xi \otimes L)$, then the points of P_ξ lying above W are given by non-split extensions

$$0 \to \mathcal{O} \to V \to W \to 0. \qquad (1)$$

By Lemma 3.1, we see that bundles V obtained above are semistable. Thus we can define a morphism

$$\phi_\xi: P_\xi \to M(3, L)$$
$$V \mapsto V \otimes (\xi^{1/3})^*.$$

Note that since $\det V \simeq \xi \otimes L$, $\det(V \otimes (\xi^{1/3})^*) = L$. Also this map is well-defined since P_ξ parameterizes a universal family and $M(3, L)$ has the coarse moduli property.

It is easy to see that $\mathrm{Im}\, \phi_\xi \subset \Theta_\eta$ (when $\eta = \xi^{1/3}$). Further, by ([S] Theorem IV, 2.1), the component of $\mathrm{Im}\, \phi_\xi$ in θ_η is of codimension at least 2 (in general for rank n it is $n - 1$)

and therefore contains a non-empty open subset of Θ_η, hence by the properness of ϕ_ξ,

$$\operatorname{Im} \phi_\xi = \Theta_\eta$$

(in fact by [S], ϕ_ξ is birational).

The above construction of P_ξ can be globalized as follows:

Let $M(2,3)$ be the moduli space of vector bundles of rank 2 and degree 3. Let $\mathscr{W} \to C \times M(2,3)$ be the universal family. Define $P = \mathbb{P}(R^1 p_* \mathscr{W}^*)$. Then the morphism ϕ_ξ globalizes to give:

$$\phi: P \to M(3,L)$$

(the ambiguity of 'cube roots' can be resolved to by pulling back P by the following diagram:

$$\begin{array}{ccc} P' & \longrightarrow & P \\ {\scriptstyle L^{-1}\otimes \det}\downarrow & & \downarrow{\scriptstyle \det \otimes L^{-1}} \\ J & \longrightarrow & J \end{array}$$

$$\eta \mapsto \eta^3$$

(so in fact, ϕ is well-defined as a morphism ϕ: $P' \to M(3,L)$). Define P_C by the following base-change diagram:

$$\begin{array}{ccc} P_C & \to & P' \\ \pi\downarrow & & \downarrow \\ C & \hookrightarrow & J \end{array}$$

where $C \hookrightarrow J$ by mapping a base point x_0 to the fixed degree 3 line bundle L. (Note that C is in fact connected). Then ϕ induces a morphism

$$\phi: P_C \to M(3,L) = M_L.$$

We claim that ϕ is surjective. This is not hard to see since $\operatorname{Im} \phi$ contains the Θ-divisor; further, one can easily get a point in $M_L - \Theta$ in $\operatorname{Im} \phi$. Now surjectivity follows from the fact that P_C and M_L are irreducible and ϕ is a proper morphism, such that $\operatorname{Im} \phi$ properly contains a divisor.

Since

$$\dim P_C = \dim \Theta + \dim C = \dim M_L,$$

ϕ gives a generically finite proper surjection.

Remark 3.2. We remark that the above construction can be done for all ranks by using the construction of desingularization of the Θ-divisor in [RV]. Our variety P can be related to their $\tilde{\Theta}$ but we would not go into it here.

4. Proof of the Hodge conjecture for $M(3, \eta)$

In this section we complete the proof of the Hodge (p,p) conjecture for $M(3,\eta)$, where $\deg \eta = 1$ or 2, $g = 2$. The strategy is to relate the geometry of $M(3,\eta)$ and $M(3,L)$ by the Hecke correspondence (cf. [B]).

PROPOSITION 4.1

Let $M_L = M(3, L)$, $\deg L = 3g - 3$. Let $g = 2$ and consider the moduli space P_C constructed in §3. Then Hodge (p, p) is true for P_C for all p.

Proof. By Theorem 2.1, it is enough to prove the properties (a) and (b) in its statement for $\pi^{-1}(y)$ for all $y \in C$, where

$$\pi : P_C \to C.$$

By §3, $\pi^{-1}(y)$'s are the moduli spaces P_ξ. Since P_ξ is a projective bundle on $M(2, \xi \otimes L)$ associated to a vector bundle, to prove (a) and (b) of Theorem 2.1 for P_ξ, it is enough to check them for $M(2, \xi \otimes L)$ because of Lemma 1.2 and Lemma 1.5. Since $M(2, \xi \otimes L)$ is a 3-fold, the Hodge conjecture follows from the Lefschetz (1, 1) theorem. That $A^2(M(2, \xi \otimes L))$ has the Abel–Jacobi property follows from ([B-M] pp. 78) since $M(2, \xi \otimes L)$ is a rational 3-fold.

We could also prove the above Proposition for P_C more directly by using the following fact:

By Thaddeus [T], (cf. also [N]), we could, consider the variety obtained by blowing up the curve C embedded in a suitable projective space of extensions. It corresponds to the variety M_1 in [T]. Denote this by $M'(2, \xi \otimes L)$. Then, when $g = 2$, it is easy to see that

$$M'(2, \xi \otimes L) \to M(2, \xi \otimes L)$$

is a birational morphism. Since $M'(2, \xi \otimes L)$ also parameterizes family of vector bundles (in fact a family of pairs!), we have a variety P'_ξ, a projective bundle associated to a vector bundle on $M'(2, \xi \otimes L)$ and a birational morphism

$$P'_\xi \to P_\xi.$$

Properties (a) and (b) of Theorem 2.1 are fairly simple for P'_ξ. Now construct globally the variety P'_C such that

$$\begin{array}{ccc} P'_C & \to & P_C \\ \downarrow & & \downarrow \\ C & \to & C \end{array}$$

Observe that by Theorem 2.1, Hodge (p, p) is true for P'_C. Since $P'_C \to P_C$ is a generically finite surjection, Hodge (p, p) for P_C follows from Hodge (p, p) for P'_C, by Lemma 1.1.

Theorem 4.2. *The Hodge (p, p) conjecture is true for $M(3, \eta)$, where $\deg \eta = 1$ and 2. $(g = 2)$.*

Proof. We prove it for $\deg \eta = d = 1$. Proof for $d = 2$ follows along identical lines.

Let P_x be the moduli space of parabolic stable bundles, (V, Δ), V of rank 3, $\deg 3g - 3 = 3$, $\det V \simeq L$, with parabolic structure Δ at $x \in C$ given by

$$0 \neq F^2 V_x \subset V_x.$$

$F^2 V_x$ a subspace of dim 1, and weights taken sufficiently small (cf. [B], ...). Then, we have the Hecke correspondence

$$\begin{array}{ccccc} & & P_x & & \\ & \swarrow^{\psi} & & \searrow^{h} & \\ M(3,\eta) & & & & M_L \end{array}$$

where η is a line bundle of $\deg \eta = 3g - 5 = 1$. The morphisms ψ and h are given by

$$\psi(V, \Delta) = W,$$

$$h(V, \Delta) = V$$

where W is obtained from the following exact sequence.

$$0 \to W \to V \to T \to 0,$$

T being a torsion sheaf of height 2 given by

$$T = \begin{cases} V_x/F^2V_x & \text{at } x \\ 0 & \text{elsewhere.} \end{cases}$$

Then it is known that ψ is a projective bundle associated to a vector bundle on $M(3, \eta)$ (cf. [B]) and the map h (in $(*)$ above) is generically a projective bundle over the stable points of M_L. Therefore by Lemma 1.2, it is enough to prove the theorem for P_x.

Now P_C by construction parameterizes a universal family $\mathscr{V} \to C \times P_C$. By the definition of ϕ and h, it is easy to see that $\tilde{P}_x \simeq \mathbb{P}(\mathscr{V}_x^*)$, where $\mathscr{V}_x$ is the bundle on P_C obtained by restriction of $\mathscr{V}$ to $x \times P_C$, and $\mathscr{V}_x^*$ its dual. Thus by the coarse moduli property of parabolic bundles for P_x, we have a morphism $\tilde{\phi}$: $\tilde{P}_x \to P_x$ and the following commutative diagram:

$$\begin{array}{ccc} \tilde{P}_x & \xrightarrow{\tilde{\phi}} & P_x \\ {\scriptstyle \tilde{h}}\downarrow & & \downarrow{\scriptstyle h} \\ P_C & \xrightarrow{\phi} & M_L \end{array}$$

By Proposition 4.1, Hodge (p, p) is true for P_C and hence by Lemma 1.2, it is true for $\tilde{P}_x$. Thus by Lemma 1.1, since $\tilde{\phi}$ is a generically finite surjection, Hodge (p, p) is true for P_x for all p, which proves the theorem.

To prove it when $\deg \eta = 2$, we modify the parabolic structure by giving $F^2V_x \subset V_x$, as a subspace of dim 2 and the rest of the argument is similar.

Remark 4.3. (The Hodge (p, p) conjecture for rank 2 moduli when $g = 3, 4$).

In these cases when rank is 2, there is a geometrical picture due to Thaddeus (cf. [T]); in his notation, if $d > 2g - 2$, d being the degree, then the moduli space of stable pairs P_i, $i = (d-1)/2$, dominates $M(2, \xi)$ $d(\xi) = d$. Further, when $d = 2g - 1$, P_i, $i = (d-1)/2$, has the property that

$$\phi : P_i \to M(2, \xi)$$

is a birational surjection. Thus, in the case when $g = 3$, (resp. 4) $d = 5$ (resp. 7), the index, $i = 2$ (resp. 3).

Now, the variety P_2 (resp. P_3) is obtained by a sequence of blow-ups and blow-downs where the centres are smooth and Hodge conjecture is easily verified by using the 'formule-clef' which expresses the Chow ring (resp. cohomology) of the blow-up in terms of the Chow ring (resp. cohomology) of the base and the centre of the blow-up. Then by Lemma 1.1, using ϕ, Hodge (p, p) follows for $M(2, \xi)$. When $g = 5$, the centres blown-up are projective bundles over S^4C, the 4th symmetric power of C and hence Hodge (p, p) would follow, once it is known for S^nC, $n \geqslant 4$.

Remark 4.4. In the rank 3 case, even when $g = 2$, the centres of blow-ups in any attempt at such descriptions seem much more complicated, vis-a-vis the Hodge conjecture. Also, it is not clear if the centres are smooth in the first place. Our proof, which is inductive, uses the simple nature of the geometry of rank 2 moduli spaces.

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References

[B] Balaji V, Intermediate Jacobian of some moduli spaces of vector bundles on curves, *Am. J. Math.* **112** (1990) 611–630

[B-M] Bloch S and Murre J P, On the Chow group of certain types of Fano threefolds, *Compos. Math.* **39** (1979) 47–105

[G] Griffiths P, Periods of integrals on algebraic manifolds III, *Publ. Math. I.H.E.S.* **38** (1970) 125–180

[M] Murre J P, On the Hodge conjecture for unirational fourfolds, *Indagationes Math.* **80** (1977) 230–232

[N] Newstead P E, Stable bundles of rank 2 and odd degree on a curve of genus 2, *Topology* **7** (1968) 205–215

[N-S] Narasimhan M S and Seshadri C S, Stable and unitary vector bundles on a compact Riemann surface, *Ann. Math.* Vol. 82, **3** (1965) 540–567

[R-V] Raghavendra N and Vishwanath P A, Moduli of pairs and generalized theta divisors, *Tohoku Math. J.* **46** (1994) 321–340

[SH] Shioda T, What is known about the Hodge conjecture?, *Advanced Studies in Pure Mathematics – I*, (1983) pp. 55–68

[S] Sundaram N, Special divisors and vector bundles, *Tohoku Math. J.* (1987) pp. 175–213

[T] Thaddeus M, Stable pairs, linear systems and the Verlinde formula, *Invent. Math.* **117** (1994) 317–353

[Z-1] Zucker S, The Hodge conjecture for cubic fourfolds, *Compos. Math.* **34** (1977) 199–209

[Z-2] Zucker S, *Intermediate Jacobians and normal functions*, *Ann. Math. Stud.* (1984) (Princeton Univ. Press, New Jersey) No. 106

[Z-3] Zucker S, Generalized intermediate Jacobians and the theorem on normal functions, *Invent. Math.* **33** (1976) 185–222

[Z-4] Zucker S, Hodge theory with degenerating coefficients: L_2 cohomology in the Poincaré metric, *Ann. Math.* **109** (1979) 415–476