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TWO-PARAMETER UNIFORMLY ELLIPTIC STURM–LIOUVILLE PROBLEMS WITH EIGENPARAMETER-DEPENDENT BOUNDARY CONDITIONS

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Abstract We consider the two-parameter Sturm–Liouville system

$$-y_1'' + q_1 y_1 = (\lambda r_{11} + \mu r_{12}) y_1 \quad \text{on } [0,1], \tag{1}$$

with the boundary conditions

$$\frac{y_1'(0)}{y_1(0)} = \cot\alpha_1 \quad \text{and} \quad \frac{y_1'(1)}{y_1(1)} = \frac{a_1\lambda + b_1}{c_1\lambda + d_1},$$

and

$$-y_2'' + q_2 y_2 = (\lambda r_{21} + \mu r_{22}) y_2 \quad \text{on } [0,1], \tag{2}$$

with the boundary conditions

$$\frac{y_2'(0)}{y_2(0)} = \cot\alpha_2 \quad \text{and} \quad \frac{y_2'(1)}{y_2(1)} = \frac{a_2\mu + b_2}{c_2\mu + d_2},$$

subject to the uniform-left-definite and uniform-ellipticity conditions; where q_i and r_{ij} are continuous real valued functions on $[0,1]$, the angle α_i is in $[0,\pi)$ and a_i, b_i, c_i, d_i are real numbers with $\delta_i = a_i d_i - b_i c_i > 0$ and $c_i \neq 0$ for $i,j = 1,2$. Results are given on asymptotics, oscillation of eigenfunctions and location of eigenvalues.

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1. Introduction

The Sturm–Liouville theory associated with the ordinary differential equation

$$-y'' + qy = \lambda r y \quad \text{on } [0,1],$$

with q and r continuous and $r > 0$ subject to the boundary conditions

$$y(0)\cos\alpha = y'(0)\sin\alpha \quad \text{and} \quad y(1)\cos\beta = y'(1)\sin\beta,$$

deals with existence, uniqueness, oscillation of eigenfunctions and completeness. Classical results about these are well known. The study of the above one-parameter equation subject to the parameter-dependent boundary conditions

$$\frac{y'(0)}{y(0)} = \frac{a_0\lambda + b_0}{c_0\lambda + d_0} \quad \text{and} \quad \frac{y'(1)}{y(1)} = \frac{a_1\lambda + b_1}{c_1\lambda + d_1}$$

have been investigated and results about the existence and oscillation theory are known [**6**]; there are also parameter dependence results and asymptotic expansions [**6**]. Klein's oscillation theorem for equations (1) and (2) subject to the fixed boundary conditions

$$y_i(0)\cos\alpha_i = y_i'(0)\sin\alpha_i \quad \text{and} \quad y_i(1)\cos\beta_i = y_i'(1)\sin\beta_i$$

and under the right definiteness condition

$$\det\begin{pmatrix} r_{11}(x) & r_{12}(x) \\ r_{21}(x) & r_{22}(x) \end{pmatrix} > 0 \quad \text{for every } x \in [0,1]$$

states that, for each non-negative integer pair (m, n), there is a unique eigenvalue $(\lambda, \mu) \in \mathbb{R}^2$ and (up to scalar multiples) a unique pair of eigenfunctions (y_1, y_2) such that y_1 has m zeros and y_2 has n zeros in $(0, 1)$. A special case was proved by Klein, and the general one (for continuous coefficients) was proved by Ince [**9**].

Bhattacharyya *et al.* [**1**] started the discussion of (1) and (2) subject to parameter-dependent boundary conditions. Apart from the Sturm–Liouville theory, there are results on asymptotics and location of eigenvalues. The extension of (1) and (2) to several parameters with parameter-independent or parameter-dependent boundary conditions has been discussed by several authors (see, for example, [**2**,**11**] and the references therein). Binding and Browne [**3**,**4**] analysed the abstract problem

$$\left(T_m - \sum_{n=1}^{k} \lambda_n V_{mn}\right) x_m = 0 \quad \text{for } (\lambda_1, \lambda_2, \dots \lambda_k) \in \mathbb{R}^k \text{ and } m = 1, 2, \dots k,$$

under several definiteness conditions and provided an abstract Klein's oscillation theorem. Here the operators T_m are self-adjoint and bounded below with compact resolvent and V_{mn} are bounded and self-adjoint.

In [**1**] the system (1), (2) was studied under the uniform-right-definiteness condition, which is defined in Definition 1.1 below. There it was shown that each eigenvalue has a unique oscillation count (m, n) where m and n are the number of zeros of the corresponding eigenfunctions y_1 and y_2, respectively. In addition, there is an oscillation theorem [**1**, Theorem 4.4] that addresses the extent to which the converse is true.

We begin by stating the definiteness conditions, for which formulation of the problems in terms of Hilbert space operators is essential. In § 2 we prove the oscillation theorem in the uniform-left-definite (ULD) case. As in [**1**], this result depends heavily on the asymptotic nature of the zeroth eigencurves of (1) and (2). In § 3, we remove the ULD assumption and retain only the uniform ellipticity (UE). The emphasis here is on finding the location of the eigenvalues coming out of the intersection of the first and second equation eigencurves. The bounded sets on which these eigenvalues are located arise from the study of the eigencurves of another system which is also explored.

The operator equivalent forms of (1) and (2) are as follows. Let AC be the subspace of $L^2[0,1]$ consisting of absolutely continuous functions. Define linear functionals P_j and Q_j for $j=1,2$ on AC by

$$P_j(y) = b_j y(1) - d_j y'(1), \qquad Q_j(y) = a_j y(1) - c_j y'(1).$$

Consider the Hilbert space $L^2[0,1] \oplus \mathbb{C}$ which has the inner product

$$\langle Y_1, Y_2 \rangle = \int_0^1 y_1 \bar{y}_2 + \alpha\bar{\beta},$$

where

$$Y_1 = \begin{pmatrix} y_1 \\ \alpha \end{pmatrix} \quad \text{and} \quad Y_2 = \begin{pmatrix} y_2 \\ \beta \end{pmatrix}$$

are in $L^2[0,1] \oplus \mathbb{C}$. Now define the unbounded operators T_j for $j=1,2$ and the bounded operators V_{jk} for $j,k=1,2$ on $L^2[0,1] \oplus \mathbb{C}$ by

$$D(T_j) = \left\{ \begin{pmatrix} y \\ -Q_j(y) \end{pmatrix} \in L^2[0,1] \oplus \mathbb{C} : y, y' \in \text{AC}, \right.$$
$$\left. -y'' + q_j y \in L^2[0,1],\ y'(0) = \cot\alpha_j y(0) \right\}$$

$$T_j \begin{pmatrix} y \\ -Q_j(y) \end{pmatrix} = \begin{pmatrix} -y'' + q_j y \\ P_j(y) \end{pmatrix} \quad \text{for} \begin{pmatrix} y \\ -Q_j(y) \end{pmatrix} \in D(T_j) \quad \text{and} \quad V_{jk} \begin{pmatrix} y \\ \alpha \end{pmatrix} = \begin{pmatrix} r_{jk} y \\ \delta_{jk}\alpha \end{pmatrix},$$

where δ_{jk} is the Kronecker delta. Now (1) and (2) are equivalent to

$$(T_j - (\lambda V_{j1} + \mu V_{j2})) \begin{pmatrix} y_j \\ \alpha \end{pmatrix} = 0 \quad \text{for} \begin{pmatrix} y_j \\ \alpha \end{pmatrix} \in D(T_j), \quad j = 1,2.$$

For $Y = (Y_1, Y_2) \in (L^2[0,1] \oplus \mathbb{C}) \times (L^2[0,1] \oplus \mathbb{C})$, we set

$$t_j(Y) = \langle T_j(Y_j), Y_j \rangle, v_{jk}(Y) = \langle V_{jk}(Y_j), Y_j \rangle, \delta_0(Y) = \det[v_{jk}(Y)],$$

and $\delta_{0jk}(Y)$ equal to the cofactor of $v_{jk}(Y)$ in $\delta_0(Y)$. Let U be the unit sphere in $L^2[0,1] \oplus \mathbb{C}$.

Definition 1.1. The basic definiteness assumptions used for the study of multiparameter Sturm–Liouville problems are defined as follows.

(i) Uniform right definiteness (URD): for some $\gamma > 0$ and for each

$$Y = (Y_1, Y_2) \in U \times U, \quad \delta_0(Y) \geqslant \gamma.$$

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(ii) Uniform ellipticity (UE): for some $\gamma > 0$, for each $j, k = 1, 2$, and for each

$$Y = (Y_1, Y_2) \in U \times U, \quad \delta_{0jk}(Y) \geqslant \gamma.$$

(iii) Uniform left definiteness (ULD): UE holds and for some $\gamma > 0$ and for each $j = 1, 2$ and

$$Y = (Y_1, Y_2) \in U \times U \quad \text{with } Y_j \in D(T_j),\ t_j(Y) \geqslant \gamma.$$

2. Uniform left definiteness

In this section we discuss (1) and (2) subject to the ULD condition. Since UE holds, $(-1)^{i+j} r_{ij}(x) > 0$ for $0 \leqslant x \leqslant 1$ and $i, j = 1, 2$ [**2**, Lemma 4.1]. The case when $\delta_0(u) > 0$ for all $u \in U$ is studied in [**1**] and the case when $\delta_0(u) < 0$ for all $u \in U$ is similar. Hence, we shall consider the case when $\delta_0(u)$ changes sign for $u \in U$.

2.1. Eigencurves of the system (1), (2)

Consider (2). If we fix λ and take μ as the parameter, equation (2) is then a one-parameter Sturm–Liouville problem with one boundary condition depending on the parameter. There exist eigenvalues $\mu_{20}(\lambda) < \mu_{21}(\lambda) < \cdots$ with corresponding eigenfunctions $y_{20}, y_{21}, \dots$. Also there exists a natural number N_2 depending on λ such that y_{2n} has n zeros for $n \leqslant N_2$ and $n-1$ zeros for $n > N_2$ in $(0, 1)$, where $\mu_{2N_2} < -d_2/c_2 \leqslant \mu_{2(N_2+1)}$. Moreover, $\mu_{2n}(\lambda)$ are continuous strictly increasing functions of λ [**1**, Lemma 2.1, Theorem 3.1], [**6**, Theorem 3.1]. The graphs of $\mu_{2n}(\lambda)$ for $\lambda \in \mathbb{R}$ are called the second equation eigencurves and are denoted by μ_{2n}. Similarly in (1), by fixing μ and taking λ as the parameter we get eigenvalues $\lambda_{10}(\mu) < \lambda_{11}(\mu) < \cdots$ with eigenfunctions $y_{10}, y_{11}, \dots$. Also there exists a natural number N_1 depending on μ such that y_{1m} has m zeros for $m \leqslant N_1$ and $m-1$ zeros for $m > N_1$ in $(0, 1)$, where $\lambda_{1N_1} < -d_1/c_1 \leqslant \lambda_{1(N_1+1)}$ [**6**, Theorem 3.1]. For every $m = 0, 1, 2 \dots$, the function $\lambda_{1m}(\mu)$ is continuous and strictly increasing in μ. So the inverse of λ_{1m} exists as a function of λ. We call it $\mu_{1m}(\lambda)$. This satisfies $\mu_{10}(\lambda) > \mu_{11}(\lambda) > \mu_{12}(\lambda) > \cdots$. We call the graphs of $\mu_{1m}(\lambda)$ the first equation eigencurves and denote them by μ_{1m}.

The pair (λ, μ) is called an eigenvalue if there exist functions y_1 and y_2 such that (λ, μ, y_1, y_2) satisfies the system. The oscillation count of (λ, μ) is the pair (m, n), $m, n \geqslant 0$, where m and n are the number of zeros of y_1 and y_2, respectively, in $(0, 1)$.

It is well known that, in the uniform left definite case, the first and second equation eigencurves intersect exactly twice. This follows from [**3**, Theorem 3.3] and its subsequent discussion therein. The intersection points are the eigenvalues of the system. Therefore there are countably many eigenvalues for the system. With respect to the point $(-d_1/c_1, -d_2/c_2)$, we consider the following quadrants:

$$Q_1 = \left\{(x, y) : x \geqslant \frac{-d_1}{c_1},\ y \geqslant \frac{-d_2}{c_2}\right\} \quad \text{and} \quad Q_2 = \left\{(x, y) : x < \frac{-d_1}{c_1},\ y \geqslant \frac{-d_2}{c_2}\right\},$$

$$Q_3 = \left\{(x, y) : x < \frac{-d_1}{c_1},\ y < \frac{-d_2}{c_2}\right\} \quad \text{and} \quad Q_4 = \left\{(x, y) : x \geqslant \frac{-d_1}{c_1},\ y < \frac{-d_2}{c_2}\right\}.$$

Let $(m,n)=k$. We denote the two intersection points of μ_{1m} and μ_{2n} as $(\lambda_1^k,\mu)=(\lambda_1^k,\mu_{1m}(\lambda_1^k))$, which is always in Q_3, and $(\lambda_2^k,\mu)=(\lambda_2^k,\mu_{1m}(\lambda_2^k))$, which is in Q_3 for the particular case $m=n=0$, where $\lambda_1^k<\lambda_2^k$.

Lemma 2.1. *The graph of $\mu_{10}(\lambda)$ always lies on the left of the vertical line $\lambda=-d_1/c_1$ and $\lim_{\lambda\to -d_1/c_1}\mu_{10}(\lambda)=\infty$. On the other hand, $\mu_{20}(\lambda)<-d_2/c_2$ for $\lambda\in\mathbb{R}$ and $\lim_{\lambda\to\infty}\mu_{20}(\lambda)=-d_2/c_2$.*

Proof. For any given λ, the value of $\mu_{20}(\lambda)$ is obtained from the point of intersection of the leftmost branch B_0 of $f(\mu)=\cot\theta(1,\lambda,\mu)$ and the hyperbola

$$g(\mu)=\frac{a_2\mu+b_2}{c_2\mu+d_2}$$

(see [**1**, Lemma 2.1] and [**6**, Theorem 3.1]).

The hyperbola $\nu=g(\mu)$ has the horizontal asymptote $\nu=a_2/c_2$ and vertical asymptote $\mu=-d_2/c_2$. Since $\cot\theta(1,\lambda,\mu)$ decreases continuously on B_0, its intersection with the hyperbola must be on the left of $\mu=-d_2/c_2$. It follows that $\mu_{20}(\lambda)<-d_2/c_2$ and, since μ_{20} is increasing, let $\lim_{\lambda\to\infty}\mu_{20}(\lambda)=l$. To show that $l=-d_2/c_2$, it is enough to show that $\lim_{\lambda\to\infty}\cot\theta(1,\lambda,\mu_{20}(\lambda))=\infty$.

Choose $\eta>0$ such that $\eta<\pi-\alpha_2$ and $\eta\leqslant\frac{1}{2}\pi$, where $\theta(0,\lambda,\mu_{20}(\lambda))=\alpha_2\in[0,\pi)$. Consider $S=\{x\in[0,1]:\eta\leqslant\theta(x,\lambda,\mu_{20}(\lambda))\leqslant\pi-\eta\}$. By choosing η small enough we can assure that S is non-empty. Let x_0 be the infimum of S. Choose δ such that $\pi-\eta<\delta\leqslant\pi$. For $x\in S$ and $\lambda>0$, since $\sin\theta\geqslant\sin\eta$, we have

$$\begin{aligned}\theta'(x,\lambda,\mu_{20}(\lambda))&=\cos^2\theta+(\lambda r_{21}+\mu_{20}(\lambda)r_{22}-q_2)\sin^2\theta\\&<1+\Big(\lambda\sup_{x\in[0,1]}r_{21}(x)+l\sup_{x\in[0,1]}r_{22}(x)\Big)\sin^2\eta+\sup_{x\in[0,1]}|q_2(x)|\\&<\frac{\eta-\delta}{1-x_0}\quad\text{for sufficiently large }\lambda.\end{aligned}$$

Note that $\eta-\delta/1-x_0$ is the slope of the line segment h joining the points (x_0,δ) and $(1,\eta)$. Hence, $(\theta-h)'(x)<0$ for $x\in S$. This, together with $(\theta-h)(x_0)<0$, implies that

$$\theta(x,\lambda,\mu_{20}(\lambda))<h(x)\quad\text{for }x\in[x_0,1].\tag{2.1}$$

Let x_1 be the largest number such that $[x_0,x_1]\subset S$. Since θ is continuous in x, such a number exists. From (2.1), we get $x_1\neq 1$. For any $x>x_1$, we have $x\notin S$, since θ is decreasing for all $x\in S$. Therefore, $S=[x_0,x_1]$ and $\theta(x,\lambda,\mu_{20}(\lambda))<\eta$ for $x>x_1$. In particular $\theta(1,\lambda,\mu_{20}(\lambda))<\eta$. Since $\alpha_2=\theta(0,\lambda,\mu_{20}(\lambda))\geqslant 0$ and $\theta'>0$ for $\theta\equiv 0\ (\mathrm{mod}\,\pi)$, we know that $\theta(x,\lambda,\mu_{20}(\lambda))$ cannot be negative for $x\in[0,1]$, for otherwise θ' will have to be negative at the point where θ becomes zero. Hence, $\theta(1,\lambda,\mu_{20}(\lambda))\geqslant 0$. Since $\eta>0$ is arbitrary, we are done.

Proceeding as above using the first differential equation, we get $\lambda_{10}(\mu)<-d_1/c_1$ and $\lim_{\mu\to\infty}\lambda_{10}(\mu)=-d_1/c_1$. Since $\mu_{10}(\lambda)$ is the inverse of $\lambda_{10}(\mu)$, the graph of $\mu_{10}(\lambda)$ lies on the left of $\lambda=-d_1/c_1$ and $\lim_{\lambda\to -d_1/c_1}\mu_{10}(\lambda)=\infty$. □

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Theorem 2.2 (oscillation theorem). *Let*

$$M_1 = \min\{m : (\lambda_2^{(m,0)}, \mu) \in Q_4 \text{ and } (\lambda_2^{(m,1)}, \mu) \in Q_1\}$$

and

$$M_2 = \min\{n : (\lambda_2^{(0,n)}, \mu) \in Q_2 \text{ and } (\lambda_2^{(1,n)}, \mu) \in Q_1\}.$$

With the exceptions below, each oscillation count corresponds to two eigenvalues.

(i) *For $m \geqslant M_1$ and $n \geqslant M_2$, each of the oscillation counts $(m, 0)$ and $(0, n)$ corresponds to exactly three eigenvalues.*

(ii) *For $m < M_1$ and $n < M_2$, the oscillation count $k = (m, n)$ corresponds to at least two eigenvalues and at most five eigenvalues.*

Proof. We find that $\mu_{1m}(\lambda)$ has the oscillation count m when $\lambda < -d_1/c_1$ and $m-1$ when $\lambda \geqslant -d_1/c_1$. $\mu_{2n}(\lambda)$ has the oscillation count n when $\mu_{2n}(\lambda) < -d_2/c_2$ and $n-1$ when $\mu_{2n}(\lambda) \geqslant -d_2/c_2$. Hence the oscillation count of the eigenvalue (λ_i^k, μ) is $(m-1, n-1)$ (respectively, $(m, n-1)$, (m, n) and $(m-1, n)$) if $(\lambda_i^k, \mu) \in Q_1$ (respectively, Q_2, Q_3, Q_4).

Let Γ_i^k, where $k = (m, n)$ denotes the curvilinear cell defined by the vertices

$$(\lambda_i^{(m,n)}, \mu),\ (\lambda_i^{(m+1,n)}, \mu),\ (\lambda_i^{(m+1,n+1)}, \mu) \text{ and } (\lambda_i^{(m,n+1)}, \mu), \quad \text{for } i = 1, 2,$$

and the corresponding eigencurve sections as edges. Note that Γ_1^k, for any $k = (m, n)$, always lies in Q_3. Since the repeated oscillation counts must correspond to the vertices of some cell, a given oscillation count $k = (m, n)$ corresponds to the eigenvalue (λ_1^k, μ) from Γ_1^k and at least one and at most four eigenvalues from Γ_2^k. Hence, the minimum number of occurrences of an oscillation count should be two.

(1) For $m \geqslant M_1$, the oscillation count $(m, 0)$ occurs thrice, once each in Q_3, Q_4 and Q_1, corresponding to $(\lambda_1^{(m,0)}, \mu)$, $(\lambda_2^{(m+1,0)}, \mu)$ and $(\lambda_2^{(m+1,1)}, \mu)$. Similarly when $n \geqslant M_2$, the oscillation count $(0, n)$ corresponds to

$$(\lambda_1^{(0,n)}, \mu) \in Q_3, \quad (\lambda_2^{(0,n+1)}, \mu) \in Q_2 \quad \text{and} \quad (\lambda_2^{(1,n+1)}, \mu) \in Q_1.$$

(2) For $m \geqslant M_1$ and $n \geqslant M_2$, the cell $\Gamma_2^{(m,n)}$ is contained in Q_1; therefore, when $m < M_1$ and $n < M_2$, the oscillation count (m, n) corresponds to at least two eigenvalues and at most five eigenvalues.

□

Remark 2.3. Given an oscillation count, it may be possible that it corresponds to five eigenvalues. It may or may not happen depending on the problem. If it happens there is only one such case. There are finitely many cases where an oscillation count corresponds to four eigenvalues. However, if there is an oscillation count which corresponds to five eigenvalues, then no oscillation count corresponds to four eigenvalues. There are always infinitely many cases where an oscillation count corresponds to three eigenvalues. Similarly, there are always infinitely many cases where an oscillation count corresponds to two eigenvalues. There is no oscillation count which corresponds to one eigenvalue.

Theorem 2.4. *Let $m_1 < m_2 < \cdots < m_k$ be positive integers such that $\mu_{1m_1}, \mu_{1m_2} \dots, \mu_{1m_k}$ intersects the line $\mu = \rho\lambda + c$ ($\rho \leqslant 0$) at $\lambda_1, \lambda_2, \dots, \lambda_k$, respectively. Then $\lambda_1 < \lambda_2 < \cdots < \lambda_k$ and for $\lambda \geqslant \lambda_k$ and $m \leqslant m_k$ we have $\mu_{1m}(\lambda) \geqslant \rho\lambda + c$.*

Proof. For $m_i < m_j$, $1 \leqslant i < j \leqslant k$, since $\mu_{1m_i}(\lambda) > \mu_{1m_j}(\lambda)$, we have $\lambda_i < \lambda_j$. For, if $\lambda_i \geqslant \lambda_j$, then μ_{1m_i} and μ_{1m_j} intersect at some λ_{ij} and $\mu_{1m_i}(\lambda) \leqslant \mu_{1m_j}(\lambda)$ for $\lambda \geqslant \lambda_{ij}$, which is impossible.

Now, for $\lambda \geqslant \lambda_k$ and $m \leqslant m_k$,

$$\mu_{1m}(\lambda) \geqslant \mu_{1m_k}(\lambda) \geqslant \mu_{1m_k}(\lambda_k) = \rho\lambda_k + c \geqslant \rho\lambda + c.$$

□

We state an analogue of the previous theorem for the eigencurves μ_{2n}. It can be proved in a similar way.

Theorem 2.5. *Let $n_1 < n_2 < \cdots < n_k$ be positive integers such that*

$$\mu_{2n_1}, \mu_{2n_2}, \dots, \mu_{2n_k}$$

intersect the line $\mu = \rho\lambda + c$ ($\rho \leqslant 0$) at $\lambda_1, \lambda_2, \dots, \lambda_k$, respectively. Then $\lambda_1 > \lambda_2 > \cdots > \lambda_k$ and, for $\lambda \leqslant \lambda_k$ and $n \leqslant n_k$, we have $\mu_{2n}(\lambda) \leqslant \rho\lambda + c$.

Theorem 2.6. *The functions $\mu_{1m}(\lambda)$ and $\mu_{2n}(\lambda)$ and the eigenfunctions y_{1m} and y_{2n} are analytic in λ*

Proof. Consider the operator equivalent form of (2),

$$(T_2 - (\lambda V_{21} + \mu V_{22})) \begin{pmatrix} y \\ \alpha \end{pmatrix} = 0 \quad \text{for} \quad \begin{pmatrix} y \\ \alpha \end{pmatrix} \in D(T_2).$$

i.e.

$$\begin{pmatrix} \dfrac{-y'' + (q_2 - \lambda r_{21})y}{r_{22}} \\ P_2(y) \end{pmatrix} = \mu \begin{pmatrix} y \\ \alpha \end{pmatrix}.$$

Define the operator $T_\lambda : D(T_2) \to L^2[0,1] \oplus \mathbb{C}$ by

$$T_\lambda \begin{pmatrix} y \\ \alpha \end{pmatrix} = \begin{pmatrix} \dfrac{-y'' + (q_2 - \lambda r_{21})y}{r_{22}} \\ P_2(y) \end{pmatrix}.$$

The linear operator T_λ is a self-adjoint [**2**, Lemma 2.1] holomorphic family of type A [**10**, Chapter VII, § 2:1] defined for λ in any neighbourhood of an interval I of the real

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axis. Let $\beta \in P(T_\lambda)$, the resolvent of T_λ. Then $(T_\lambda - \beta)^{-1} : L^2[0,1] \oplus \mathbb{C} \to D(T_2)$ has the form

$$(T_\lambda - \beta)^{-1} \begin{pmatrix} v \\ \gamma \end{pmatrix} = \begin{pmatrix} Gv \\ -Q_2(y) \end{pmatrix}$$

and is compact [**8**, II, Theorem 6.9]. Here $y = Gv$, where $G : L^2[0,1] \to L^2[0,1]$ is given by

$$(Gv)(x) = \int_0^1 g(x,t)v(t)\,\mathrm{d}t;$$
$$g(x,t) = \begin{cases} c^{-1} y_0(x) y_1(t), & 0 \leqslant x \leqslant t \leqslant 1, \\ c^{-1} y_0(t) y_1(x), & 0 \leqslant t \leqslant x \leqslant 1, \end{cases}$$

where c is the Wronskian of y_0 and y_1, and y_0 is a solution of

$$\frac{-y'' + (q_2 - \lambda r_{21})y}{r_{22}} - \beta y = 0, \quad y'(0) = \cot \alpha_2 y(0),$$

and y_1 is a solution of

$$\frac{-y'' + (q_2 - \lambda r_{21})y}{r_{22}} - \beta y = 0, \quad P_2(y) + \beta Q_2(y) = \gamma.$$

It follows from [**10**, VII, 3:5, Theorem 3.9] that the eigenvalues $\mu_{1m}(\lambda)$ and the eigenfunctions $\binom{y_{1m}}{\alpha}$ of T_λ are analytic in λ. Consequently, y_{1m} is also analytic.

In a similar way, considering the operator equivalent form of (1), we arrive at the conclusion that the eigenvalues $\lambda_{2n}(\mu)$ and the eigenfunctions y_{2n} are analytic. Here we take the operator $T_\mu : D(T_1) \to L^2[0,1] \oplus \mathbb{C}$ to be

$$T_\mu \begin{pmatrix} y \\ \alpha \end{pmatrix} = \begin{pmatrix} \dfrac{-y'' + (q_1 - \mu r_{12})y}{r_{11}} \\ P_1(y) \end{pmatrix}.$$

Being the inverse of λ_{2n}, the function $\mu_{2n}(\lambda)$ is also analytic. □

The expression for the first derivatives of $\mu_{1m}(\lambda)$ and $\mu_{2n}(\lambda)$ with respect to λ are derived in [**1**, Theorem 3.1]. The second derivatives are given below.

Theorem 2.7. *Assume that*

$$\frac{\partial^2 y''}{\partial \lambda^2} = \left(\frac{\partial^2 y}{\partial \lambda^2}\right)'',$$

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where a prime denotes $\partial/\partial x$. The second derivatives of $\mu_{1m}(\lambda)$ and $\mu_{2n}(\lambda)$ with respect to λ are given by

$$\begin{aligned}\frac{\mathrm{d}^2\mu_{1m}(\lambda)}{\mathrm{d}\lambda^2} &= (r_{12}(y_{1m}))^{-1}\biggl\{\frac{2\delta_1}{(c_1\lambda+d_1)^3}\biggl[c_1(y_{1m}(1))^2-(c_1\lambda+d_1)y_{1m}(1)\frac{\partial y_{1m}(1)}{\partial\lambda}\biggr] \\ &\qquad -2\frac{\mathrm{d}\mu_{1m}(\lambda)}{\mathrm{d}\lambda}\int_0^1 r_{12}y_{1m}\frac{\partial y_{1m}}{\partial\lambda}-2\int_0^1 r_{11}y_{1m}\frac{\partial y_{1m}}{\partial\lambda}\biggr\},\end{aligned}$$

$$\begin{aligned}\frac{\mathrm{d}^2\mu_{2n}(\lambda)}{\mathrm{d}\lambda^2} &= \biggl[\frac{\delta_2(y_{2n}(1))^2}{(c_2\mu_{2n}+d_2)^2}+r_{22}(y_{2n})\biggr]^{-1} \\ &\quad\times\biggl\{\frac{2\delta_2}{(c_2\mu_{2n}+d_2)^3}\biggl[c_2(y_{2n}(1))^2\biggl(\frac{\mathrm{d}\mu_{2n}(\lambda)}{\mathrm{d}\lambda}\biggr)^2-(c_2\mu_{2n}+d_2)y_{2n}(1)\frac{\mathrm{d}\mu_{2n}(\lambda)}{\mathrm{d}\lambda}\frac{\partial y_{2n}(1)}{\partial\lambda}\biggr] \\ &\qquad -2\frac{\mathrm{d}\mu_{2n}(\lambda)}{\mathrm{d}\lambda}\int_0^1 r_{22}y_{2n}\frac{\partial y_{2n}}{\partial\lambda}-2\int_0^1 r_{21}y_{2n}\frac{\partial y_{2n}}{\partial\lambda}\biggr\},\end{aligned}$$

where

$$r_{i2}(y_{im}) = \int_0^1 r_{i2}y_{im}^2 \quad \text{for } i=1,2.$$

Proof. Differentiation of (1) twice with respect to λ yields

$$\begin{aligned}-\frac{\partial^2 y''}{\partial\lambda^2}+q_1\frac{\partial^2 y}{\partial\lambda^2} &= \biggl(r_{11}+\frac{\mathrm{d}\mu}{\mathrm{d}\lambda}r_{12}\biggr)\frac{\partial y}{\partial\lambda}+\frac{\mathrm{d}^2\mu}{\mathrm{d}\lambda^2}r_{12}y \\ &\qquad +(\lambda r_{11}+\mu r_{12})\frac{\partial^2 y}{\partial\lambda^2}+\biggl(r_{11}+\frac{\mathrm{d}\mu}{\mathrm{d}\lambda}r_{12}\biggr)\frac{\partial y}{\partial\lambda}. \qquad (2.2)\end{aligned}$$

Multiplying (1) by $\partial^2 y/\partial\lambda^2$, (2.2) by y and subtracting the former from the later, we get

$$-y\frac{\partial^2 y''}{\partial\lambda^2}+y''\frac{\partial^2 y}{\partial\lambda^2} = r_{12}y^2\frac{\mathrm{d}^2\mu}{\mathrm{d}\lambda^2}+2\biggl(r_{11}+\frac{\mathrm{d}\mu}{\mathrm{d}\lambda}r_{12}\biggr)y\frac{\partial y}{\partial\lambda}.$$

Integrating over $[0,1]$, we find that

$$\biggl[-y\frac{\partial}{\partial x}\biggl(\frac{\partial^2 y}{\partial\lambda^2}\biggr)+y'\frac{\partial^2 y}{\partial\lambda^2}\biggr]_0^1 = \frac{\mathrm{d}^2\mu}{\mathrm{d}\lambda^2}\int_0^1 r_{12}y^2+2\frac{\mathrm{d}\mu}{\mathrm{d}\lambda}\int_0^1 r_{12}y\frac{\partial y}{\partial\lambda}+2\int_0^1 r_{11}y\frac{\partial y}{\partial\lambda}. \qquad (2.3)$$

We differentiate the boundary conditions in (1) twice with respect to λ and then use it to solve the left-hand side of (2.3) to get the expression for $\mathrm{d}^2\mu_{1m}(\lambda)/\mathrm{d}\lambda^2$. Using (2) and following similar steps we get

$$\biggl[-y\frac{\partial}{\partial x}\biggl(\frac{\partial^2 y}{\partial\lambda^2}\biggr)+y'\frac{\partial^2 y}{\partial\lambda^2}\biggr]_0^1 = \frac{\mathrm{d}^2\mu}{\mathrm{d}\lambda^2}\int_0^1 r_{22}y^2+2\frac{\mathrm{d}\mu}{\mathrm{d}\lambda}\int_0^1 r_{22}y\frac{\partial y}{\partial\lambda}+2\int_0^1 r_{21}y\frac{\partial y}{\partial\lambda}.$$

Now as in the computation above, we solve the left-hand side using the boundary conditions of (2) to get the required result. □

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3. Uniform ellipticity

From now on we assume only uniform ellipticity for the system (1), (2). The UE condition implies that $(-1)^{i+j}r_{ij}(x) > 0$ for $0 \leqslant x \leqslant 1$ and $i,j = 1,2$ [**2**, Lemma 4.1]. We permit $\delta_0(u)$ to take both positive and negative values for $u \in U$. We also assume that $r_{11}(x_1)r_{22}(x_2) - r_{12}(x_1)r_{21}(x_2)$ for $(x_1,x_2) \in [0,1] \times [0,1]$ is not identically zero and changes sign. The first and second equation eigencurves μ_{1m} and μ_{2n} can be derived exactly as in the uniform left definite case and we are following the same notations. In particular, note that Lemma 2.1 and Theorem 2.6 are valid in this case as well. The intersection points of μ_{1m} and μ_{2n} are the eigenvalues of the system (1), (2).

Lemma 3.1. *The operators T_j for $j = 1,2$ are self-adjoint and bounded below with a compact resolvent.*

Proof. The self-adjointness of T_j follows from [**2**, Lemma 2.1]. For the compactness of the resolvent of T_j, see the proof of Theorem 2.6. Now let us have a look at the eigenvalues of T_j. $T_jY = \mu Y$ implies that

$$-y'' + q_j y = \mu y, \qquad \frac{y'(0)}{y(0)} = \cot\alpha_j, \qquad \frac{y'(1)}{y(1)} = \frac{a_j\mu + b_j}{c_j\mu + d_j}.$$

Theorem 3.1 of [**6**] shows that the system has a countable number of eigenvalues $\mu_j^0 < \mu_j^1 < \mu_j^2 < \cdots$. So the spectrum of the operator T_j is bounded below. The discussion in [**10**, Chapter V, § 3:10, p. 278] concludes that T_j is bounded below. □

Theorem 3.2. *Given $n \geqslant 0$, there exists an integer $N(n) \geqslant 0$ such that the μ_{2n} intersect with μ_{1m} at at least two points if and only if $m \geqslant N(n)$.*

Proof. For $(\lambda,\mu) \in \mathbb{R}^2$, since the operator $T_2 - \lambda V_{21} - \mu V_{22}$ is self-adjoint and bounded below with compact resolvent [**4**, Lemma 1], it has a countable number of eigenvalues $\rho_2^0(\lambda,\mu) \leqslant \rho_2^1(\lambda,\mu) \leqslant \cdots$. For each $n \geqslant 0$ and $\lambda \in \mathbb{R}$, since $-\langle V_{22}(u),u\rangle < 0$ for all $u \in U$, there exists a unique $\mu^{2n}(\lambda)$ such that $\rho_2^n(\lambda,\mu^{2n}(\lambda)) = 0$ and

$$(T_2 - \lambda V_{21} - \mu^{2n}(\lambda)V_{22})\begin{pmatrix} y^{2n} \\ \alpha \end{pmatrix} = 0.$$

Moreover, $\mu^{2n}(\lambda)$ are continuous in λ and $\mu^{20}(\lambda) \leqslant \mu^{21}(\lambda) \leqslant \cdots$ (see [**4**, Theorems 2 and 3], [**5**, Theorem 2.1]). We claim that $\mu^{2n}(\lambda) = \mu_{2n}(\lambda)$. Since $\mu^{20}(\lambda) \leqslant \mu^{21}(\lambda) \leqslant \cdots$ and $\mu_{20}(\lambda) < \mu_{21}(\lambda) < \cdots$, it suffices to show that

$$\{\mu^{2n}(\lambda) : n \geqslant 0\} = \{\mu_{2n}(\lambda) : n \geqslant 0\}.$$

For $\lambda \in \mathbb{R}$, the set $\{\mu_{2n}(\lambda) : n \geqslant 0\}$ forms a complete set of eigenvalues for the equation

$$(T_2 - \lambda V_{21} - \mu V_{22})\begin{pmatrix} y \\ \alpha \end{pmatrix} = 0.$$

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Since $\{\mu^{2n}(\lambda) : n \geqslant 0\}$ are eigenvalues of this equation, we have

$$\{\mu^{2n}(\lambda) : n \geqslant 0\} \subseteq \{\mu_{2n}(\lambda) : n \geqslant 0\}.$$

The eigenvalues $\mu_{2n}(\lambda)$ satisfy the equation

$$(T_2 - \lambda V_{21} - \mu_{2n}(\lambda)V_{22}) \begin{pmatrix} y_{2n} \\ \alpha \end{pmatrix} = 0.$$

Hence $\rho_2^j(\lambda, \mu_{2n}(\lambda)) = 0$ for some $j \geqslant 0$. But $\rho_2^j(\lambda, \mu^{2j}(\lambda)) = 0$. Therefore, $\mu_{2n}(\lambda) = \mu^{2j}(\lambda)$ by the uniqueness of $\mu^{2j}(\lambda)$. Thus, the other inclusion holds.

Now consider $T_1 - \lambda V_{11} - \mu V_{12}$ for $(\lambda, \mu) \in \mathbb{R}^2$. Its eigenvalues can be ordered as $\rho_1^0(\lambda, \mu) \leqslant \rho_1^1(\lambda, \mu) \leqslant \cdots$. For each $m \geqslant 0$ and $\mu \in \mathbb{R}$, since $-\langle V_{11}(u), u\rangle < 0$ for all $u \in U$, there exists a unique $\lambda^{1m}(\mu)$ such that $\rho_1^m(\lambda^{1m}(\mu), \mu) = 0$. Then, by a similar procedure to that above, we can prove that $\lambda^{1m} = \lambda_{1m}$. Since $\lambda_{1m}(\mu)$ is a continuous strictly increasing function of μ [**1**, Theorem 3.1], its inverse $\mu_{1m}(\lambda)$ exists and

$$(T_1 - \lambda V_{11} - \mu_{1m}(\lambda)V_{12}) \begin{pmatrix} y_{1m} \\ \alpha \end{pmatrix} = 0 \quad \text{with } \rho_1^m(\lambda, \mu_{1m}(\lambda)) = 0.$$

Now we are ready to apply corollary 4.2 of [**3**]. We know that

$$(T_2 - \lambda V_{21} - \mu_{2n}(\lambda)V_{22}) \begin{pmatrix} y_{2n} \\ \alpha \end{pmatrix} = 0 \quad \text{for } \lambda \in \mathbb{R} \text{ and } \mu_{2n}(\lambda).$$

It remains to solve the equation

$$(T_1 - \lambda V_{11} - \mu_{2n}(\lambda)V_{12}) \begin{pmatrix} y \\ \alpha \end{pmatrix} = 0 \quad \text{for } \lambda \in \mathbb{R} \text{ and } \mu_{2n}(\lambda).$$

In other words the problem is to find an $m \geqslant 0$ such that $\rho_1^m(\lambda, \mu_{2n}(\lambda)) = 0$ for two values of λ. By [**3**, Corollary 4.2], given $n \geqslant 0$, there exists an integer $N(n) \geqslant 0$ such that

$$(T_1 - \lambda V_{11} - \mu_{2n}(\lambda)V_{12}) \begin{pmatrix} y \\ \alpha \end{pmatrix} = 0$$

for two values of λ, say λ_1 and λ_2, with $\rho_1^m(\lambda_i, \mu_{2n}(\lambda_i)) = 0$ for $i = 1, 2$ if and only if $m \geqslant N(n)$. In this case, since $\rho_1^m(\lambda_i, \mu_{2n}(\lambda_i)) = 0$, we have $\mu_{2n}(\lambda_i) = \mu_{1m}(\lambda_i)$ for $i = 1, 2$. □

A similar argument will give the following result.

Theorem 3.3. *For a given $m \geqslant 0$, there exists an integer $M(m) \geqslant 0$ such that the μ_{1m} intersect with μ_{2n} at at least two points if and only if $n \geqslant M(m)$.*

Corollary 3.4. *The non-negative integers $M(m)$ and $N(n)$ are non-increasing in m and n, respectively, and $M(m_0) = N(n_0) = 0$ for some m_0 and n_0.*

Proof. Let $n < l$. Suppose $N(n) < N(l)$. From the preceding two theorems, we find that the μ_{2n} intersect with μ_{1m} for $m \geqslant N(l)$. Fix m, where $N(n) \leqslant m < N(l)$. Then, since the μ_{1m} intersect with μ_{2k} if and only if $k \geqslant M(m)$, in particular μ_{1m} intersects μ_{2l}, which is a contradiction.

Given $n = 0$, there exists $N(0)$ such that μ_{2n} intersects μ_{1m} if and only if $m \geqslant N(0)$. Now if $m \geqslant N(0)$, then μ_{1m} intersects μ_{20}. So $M(m) = 0$ for $m \geqslant N(0)$. The other assertions are proved in a similar way. □

Thus, for $m \geqslant m_0$, the curves μ_{1m} intersect with all μ_{2n}, where $n \geqslant 0$, and, for $n \geqslant n_0$, the curves μ_{2n} intersect with all μ_{1m}, where $m \geqslant 0$.

The study of the eigencurves of the following equation will enable us to find the location of the intersection points of μ_{1m} and μ_{2n}:

$$\left.\begin{aligned} -y_1'' + (q_1 + Q - \lambda r_{11} - \mu_{2n}(\lambda) r_{12}) y_1 = \Omega Q y_1 \quad \text{on } [0,1],\\ \frac{y_1'(0)}{y_1(0)} = \cot\alpha_1, \quad y_1(1) = 0, \end{aligned}\right\} \tag{3.1}$$

where Q is a positive constant to be suitably chosen, Ω is a real parameter and $(\lambda, \mu_{2n}(\lambda))$, for $n = 0, 1, \dots$, are the eigenpairs of (2). For $\lambda \in \mathbb{R}$ and $\mu_{2n}(\lambda)$, where $n \geqslant 0$ is fixed, the eigenvalues can be ordered as $\Omega^{\mathrm{D}}_{0,n} < \Omega^{\mathrm{D}}_{1,n} < \cdots$ and $\Omega^{\mathrm{D}}_{m,n}(\lambda)$, $m = 0, 1, 2, \dots$, are analytic in λ [**11**, Lemma 3.1]. We now wish to investigate the nature of the eigencurves $\Omega^{\mathrm{D}}_{m,n}$. Our analysis is similar to that of Sleeman [**11**].

First, let us form a differential equation. Multiply (2) by y_2 and integrate over $0 \leqslant x_2 \leqslant 1$. Then substitute the value of $\mu_{2n}(\lambda)$ obtained into (3.1) to get

$$\frac{\mathrm{d}^2 y_1}{\mathrm{d}x_1^2} + (\lambda a(x_1, \lambda) - H_1(x_1, \lambda) + H_2(x_1, \lambda) + Q\Omega - Q) y_1 = 0, \tag{3.2}$$

where

$$a(x_1, \lambda) = \frac{\int_0^1 (r_{11} r_{22} - r_{12} r_{21}) y_{2n}^2 \, \mathrm{d}x_2}{\int_0^1 r_{22} y_{2n}^2 \, \mathrm{d}x_2},$$

$$H_1(x_1, \lambda) = \frac{\int_0^1 q_1 r_{22} y_{2n}^2 \, \mathrm{d}x_2 + \int_0^1 (-r_{12}) q_2 y_{2n}^2 \, \mathrm{d}x_2}{\int_0^1 r_{22} y_{2n}^2 \, \mathrm{d}x_2},$$

$$H_2(x_1, \lambda) = \frac{\int_0^1 (-r_{12}) y_{2n} y_{2n}'' \, \mathrm{d}x_2}{\int_0^1 r_{22} y_{2n}^2 \, \mathrm{d}x_2}.$$

The following asymptotic result of the eigencurve μ_{2n} is useful in providing an estimate for $H_1(x_1, \lambda) - H_2(x_1, \lambda)$. Let

$$K = \inf\left\{\frac{-r_{21}(x)}{r_{22}(x)} : 0 \leqslant x \leqslant 1\right\}.$$

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Then K is finite and $\lim_{\lambda\to\infty}\mu_{2n}(\lambda)/\lambda = K$ for $n > 0$ (see [**1**, Lemma 3.4] and [**2**, Lemma 4.5]). Now, for $\lambda \in \mathbb{R}$ and $\mu_{2n}(\lambda)$, where $n > 0$, we have

$$\begin{aligned} H_1(x_1,\lambda) - H_2(x_1,\lambda) \\ &= q_1(x_1) + \frac{1}{\int_0^1 r_{22} y_{2n}^2} \left[\lambda \int_0^1 -r_{12}(x_1) r_{21} y_{2n}^2 \, \mathrm{d}x_2 - \int_0^1 r_{12}(x_1) r_{22} y_{2n}^2 \, \mathrm{d}x_2\right] \\ &= q_1(x_1) + O(\lambda) \quad \text{as } \lambda \to \infty. \end{aligned}$$

and, for large positive λ and $\mu_{20}(\lambda)$, using Lemma 2.1, we have

$$H_1(x_1,\lambda) - H_2(x_1,\lambda) \leqslant q_1(x_1) + L_1\lambda + L_2\frac{-d_2}{c_2},$$

where L_1 and L_2 are the upper bounds for the respective terms in $H_1 - H_2$.

In the (λ, Ω)-plane we take $\Omega = 0$ as the abscissa and $\lambda = 0$ as the ordinate and introduce the angle ϕ as the angle which a ray through the origin makes with the positive λ-axis.

Define

$$G = -\sup_{(x_1,\lambda)\in[0,1]\times(-\infty,\infty)} \frac{a(x_1,\lambda)}{Q} \quad \text{and} \quad g = -\inf_{(x_1,\lambda)\in[0,1]\times(-\infty,\infty)} \frac{a(x_1,\lambda)}{Q}.$$

Then $G < 0$ and $g > 0$, since $r_{11}r_{22} - r_{12}r_{21}$ changes sign in $[0,1]\times[0,1]$. Let

$$\phi_1 = \tan^{-1} G, \quad \phi_1^* = \tan^{-1} g, \quad \phi_2 = \pi + \phi_1^* \quad \text{and} \quad \phi_2^* = \pi + \phi_1,$$

where the principal branch of the inverse tangent is taken. Clearly, $-\frac{1}{2}\pi < \phi_1 < 0 < \phi_1^* < \frac{1}{2}\pi < \phi_2^* < \pi < \phi_2 < \frac{3}{2}\pi$.

Theorem 3.5. *If $\phi_1^* \leqslant \phi \leqslant \frac{1}{2}\pi$, then in the (λ,Ω)-plane a straight line through the origin with slope $\tan\phi$ cuts each curve $\Omega_{m,n}^{\mathrm{D}}$ at precisely one point $(\lambda(\phi), \Omega_{m,n}^{\mathrm{D}}(\phi))$, for $m = 0,1,2,\dots,$ and $\Omega_{0,n}^{\mathrm{D}}(\phi) < \Omega_{1,n}^{\mathrm{D}}(\phi) < \cdots$ and $\lim_{m\to\infty}\Omega_{m,n}^{\mathrm{D}}(\phi) = \infty$.*

Proof. Consider the case $\phi_1^* \leqslant \phi < \frac{1}{2}\pi$. Since $\tan\phi = \Omega/\lambda$, we have, from (3.2),

$$\frac{\mathrm{d}^2 y_1}{\mathrm{d}x_1^2} + (\lambda Q F_\phi(x_1,\lambda) - H_1(x_1,\lambda) + H_2(x_1,\lambda) - Q) y_1 = 0,$$

where $F_\phi(x_1,\lambda) = \tan\phi + a(x_1,\lambda)/Q$. Since $\tan\phi_2^* \leqslant -a(x_1,\lambda)/Q \leqslant \tan\phi_1^*$, it follows that $F_\phi(x_1,\lambda) \geqslant 0$ and does not vanish identically for all $x_1 \in [0,1]$ and $\phi_1^* \leqslant \phi < \frac{1}{2}\pi$. We introduce the Prüfer transformations:

$$\begin{aligned} y_1(x_1,\lambda,\lambda\tan\phi) &= r(x_1,\lambda,\phi)\sin\theta(x_1,\lambda,\phi), \\ y_1'(x_1,\lambda,\lambda\tan\phi) &= r(x_1,\lambda,\phi)\cos\theta(x_1,\lambda,\phi). \end{aligned}$$

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Then $\theta(x_1, \lambda, \phi)$ is the solution of the initial-value problem

$$\begin{aligned}\theta'(x_1, \lambda, \phi) &= \cos^2\theta(x_1, \lambda, \phi) + [\lambda Q F_\phi(x_1, \lambda) \\ &\qquad - H_1(x_1, \lambda) + H_2(x_1, \lambda) - Q]\sin^2\theta(x_1, \lambda, \phi), \\ \theta(0, \lambda, \phi) &= \alpha_1.\end{aligned}$$

We seek values of λ such that $\theta(1, \lambda, \phi) = m\pi + \pi$. If we take Q to be sufficiently large and positive and argue as in the proof of [**7**, Chapter VIII, Theorem 2.1], we get $\theta(1, 0, \phi) < \pi$.

Claim 3.6. *$\theta(1, \lambda, \phi)$ is a strictly increasing function of λ.*

Differentiate (3.1) with respect to λ, multiply the result by y_1 and substitute [**1**, Theorem 3.1]

$$\frac{\mathrm{d}\mu_{2n}(\lambda)}{\mathrm{d}\lambda} = -\int_0^1 r_{21} y_{2n}^2 \left[\frac{(a_2 d_2 - b_2 c_2)(y_{2n}(1))^2}{(c_2\mu_{2n}(\lambda) + d_2)^2} + \int_0^1 r_{22} y_{2n}^2\right]^{-1}$$

and $\mu_{2n}(\lambda)$ from (3.1). Integration of this with respect to x_1 gives

$$\begin{aligned}&y_1'(1, \lambda, \lambda\tan\phi)\frac{\partial y_1(1, \lambda, \lambda\tan\phi)}{\partial\lambda} - y_1(1, \lambda, \lambda\tan\phi)\frac{\partial y_1'(1, \lambda, \lambda\tan\phi)}{\partial\lambda} \\ &\quad= \left[\frac{(a_2 d_2 - b_2 c_2)(y_{2n}(1))^2}{(c_2\mu_{2n}(\lambda) + d_2)^2} + \int_0^1 r_{22} y_{2n}^2 \,\mathrm{d}x_2\right]^{-1} \\ &\qquad\times \left\{\int_0^1 r_{22} y_{2n}^2 \,\mathrm{d}x_2 \int_0^1 Q F_\phi(x_1, \lambda) y_1^2(x_1, \lambda, \lambda\tan\phi)\,\mathrm{d}x_1 \right. \\ &\qquad\qquad \left. + \frac{(a_2 d_2 - b_2 c_2)(y_{2n}(1))^2}{(c_2\mu_{2n}(\lambda) + d_2)^2}\left[\int_0^1 r_{11} y_1^2 \,\mathrm{d}x_1 + \int_0^1 Q\tan\phi\, y_1^2 \,\mathrm{d}x_1\right]\right\}.\end{aligned}$$

The left-hand side is equal to

$$(r(1, \lambda, \phi))^2 \frac{\mathrm{d}\theta(1, \lambda, \phi)}{\mathrm{d}\lambda}$$

and the right-hand side is positive. Hence $\theta(1, \lambda, \phi)$ is strictly increasing in λ.

Furthermore, $\theta(1, \lambda, \phi) \to \infty$ as $\lambda \to \infty$. This follows on taking Q sufficiently large, using the estimate for $H_1(x_1, \lambda) - H_2(x_1, \lambda)$ and arguing as in the proof of [**7**, VIII, Theorem 2.1]. Thus, the equation $\theta(1, \lambda, \phi) = m\pi + \pi$ for $m = 0, 1, 2\dots$ has a unique solution. Since $\theta(1, 0, \phi) < \pi$, and θ is increasing in λ, these solutions form a strictly increasing sequence of positive numbers which tends to infinity as $m \to \infty$. Hence, the theorem follows in this case.

Let $\phi = \frac{1}{2}\pi$. Clearly, $\Omega_{m,n}^{\mathrm{D}}$ cuts the vertical axis at precisely one point. If $\Omega_{m,n}^{\mathrm{D}}(\phi) \leqslant 0$ then $\Omega_{m,n}^{\mathrm{D}}$ cuts some lines through the origin with slope $\tan\phi'$, $\phi_1^* \leqslant \phi' < \frac{1}{2}\pi$, where $\Omega_{m,n}^{\mathrm{D}}(\phi') \leqslant 0$, which is impossible. □

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Theorem 3.7. *For all $\lambda \in (-\infty, \infty)$ the eigencurve $\varOmega^{\mathrm{D}}_{m,n}$, $m = 0, 1, 2, \dots$, lies in the sector $\phi_1 < \phi < \phi_2$. Furthermore, given any $\epsilon \in (0, \frac{1}{2}\pi)$, there is a positive number $N_{m,n}(\epsilon)$ such that, for $\lambda \geqslant N_{m,n}(\epsilon)$, $\varOmega^{\mathrm{D}}_{m,n}(\lambda)$ lies in the sector $\phi_1 < \phi < \phi_1 + \epsilon$ and, for $\lambda \leqslant -N_{m,n}(\epsilon)$, $\varOmega^{\mathrm{D}}_{m,n}(\lambda)$ lies in the sector $\phi_2 - \epsilon < \phi < \phi_2$.*

Proof. The result follows from [**11**, Theorem 4]. □

Theorem 3.8. *If $\varOmega^* \leqslant 0$, then the line $\varOmega = \varOmega^*$ intersects each curve $\varOmega^{\mathrm{D}}_{m,n}$ at at least two points and at most a finite number of points.*

Proof. For fixed $m \geqslant 0$, we find from Theorem 3.5 that $\varOmega^{\mathrm{D}}_{m,n}(0) > 0$. By choosing $\epsilon > 0$ very small in Theorem 3.7, we arrive at the conclusion that

$$\lim_{\lambda \to \infty} \varOmega^{\mathrm{D}}_{m,n}(\lambda) = \lim_{\lambda \to -\infty} \varOmega^{\mathrm{D}}_{m,n}(\lambda) = -\infty.$$

Hence, $\varOmega^{\mathrm{D}}_{m,n}$ intersects $\varOmega = \varOmega^*$ at at least one point with positive abscissa and at at least one point with negative abscissa. Since $\varOmega^{\mathrm{D}}_{m,n}$ is analytic, there are at most a finite number of points of intersection, with each such point having a non-zero abscissa. □

Let

$$\lambda^-_{m,n} = \min\{\lambda < 0 : \varOmega^{\mathrm{D}}_{m,n} \text{ intersects } \varOmega = 0 \text{ at } \lambda\}$$

and

$$\lambda^+_{m,n} = \max\{\lambda > 0 : \varOmega^{\mathrm{D}}_{m,n} \text{ intersects } \varOmega = 0 \text{ at } \lambda\}.$$

Remark 3.9. Since $\varOmega^{\mathrm{D}}_{m,n}(\lambda)$ is analytic in λ, both the above sets contain only a finite number of elements and, hence, $\lambda^-_{m,n}$ and $\lambda^+_{m,n}$ are finite. Also, note that if $\varOmega^{\mathrm{D}}_{m,n}(\lambda) \geqslant 0$ for some λ, then λ must be in $[\lambda^-_{m,n}, \lambda^+_{m,n}]$. We denote this interval by $S_{m,n}$.

Theorem 3.10. *Given m and n, all the intersection points of μ_{1m} and μ_{2n} are contained in the set $S_{m,n} \cup S_{m-1,n} \cup S_{m-2,n}$.*

Proof. Suppose μ_{1m} and μ_{2n} intersect at λ_1. Consider the equations

$$\left.\begin{aligned} -y'' + (q_1 + Q - \lambda r_{11} - \mu_{2n}(\lambda) r_{12}) y = \varOmega Q y, \qquad \frac{y'(0)}{y(0)} = \cot \alpha_1, \\ \frac{y'(1)}{y(1)} = \frac{a_1 \lambda \varOmega + b_1}{c_1 \lambda \varOmega + d_1}. \end{aligned}\right\} \tag{3.3}$$

For $\lambda \in \mathbb{R}$ and $\mu_{2n}(\lambda)$, the system has eigenvalues $\varOmega_{0,n}(\lambda) < \varOmega_{1,n}(\lambda) < \cdots$. Fix $\lambda = \lambda_1$. Then $\varOmega_{0,n}(\lambda_1) < \varOmega_{1,n}(\lambda_1) < \dots$ and there exists a positive integer $M_1 = M_1(\lambda_1)$, where

$$\varOmega_{M_1,n}(\lambda_1) < \frac{-d_1}{c_1 \lambda_1} \leqslant \varOmega_{M_1+1,n}(\lambda_1),$$

such that the eigenfunction y_l of $\varOmega_{l,n}(\lambda_1)$ has l zeros if $l \leqslant M_1$ and $l - 1$ zeros if $l > M_1$ [**6**, Theorem 3.1]. Since λ_1, $\mu_{1m}(\lambda_1) = \mu_{2n}(\lambda_1)$ and y_{1m} satisfy (1), we have $\varOmega_{l_0,n}(\lambda_1) = 1$ for some $l_0 \geqslant 0$.

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Case 1. $m \leqslant N_1$, where $N_1 = N_1(\mu_{1m}(\lambda_1))$. Then

(1a) if $m < M_1$, then $l_0 = m$;

(1b) if $m = M_1$, then $l_0 = m$ or $m + 1$;

(1c) if $m > M_1$, then $l_0 = m + 1$.

We prove (1a). The proofs of (1b) and (1c) are similar. Let $m < M_1$. Suppose that $l_0 \neq m$. If $l_0 \leqslant M_1$, then y_{l_0} has l_0 zeros. Since the dimension of the eigenspace for $\Omega_{l_0,n}(\lambda_1)$ is one, $y_{l_0} = cy_{1m}$, where c is a constant. Thus, the number of zeros of y_{l_0} and y_{1m} is the same, which is not possible. Similarly, for $l_0 > M_1$, we will find a contradiction.

(1a) $m < M_1$. So $\Omega_{m,n}(\lambda_1) = 1$. By construction

$$\Omega^{\mathrm{D}}_{m-1,n}(\lambda_1) < \Omega_{m,n}(\lambda_1) = 1 < \Omega^{\mathrm{D}}_{m,n}(\lambda_1),$$

where $\Omega^{\mathrm{D}}_{0,n}(\lambda_1) < \Omega^{\mathrm{D}}_{1,n}(\lambda_1) < \cdots$ are the eigenvalues of the Dirichlet problem (3.1) [**6**, Theorem 3.1]. From Remark 3.9 we see that $\lambda_1 \in S_{m,n}$.

(1b) $m = M_1$. In this case $\Omega_{m,n}(\lambda_1) = 1$ or $\Omega_{m+1,n}(\lambda_1) = 1$. It then follows from the inequality

$$\Omega^{\mathrm{D}}_{m-1,n}(\lambda_1) < \Omega_{l_0,n}(\lambda_1) = 1 \leqslant \Omega^{\mathrm{D}}_{m,n}(\lambda_1),$$

where $l_0 = m$ or $m + 1$, that $\lambda_1 \in S_{m,n}$.

(1c) $m > M_1$. Here $\Omega_{m+1,n} = 1$. We also have

$$\Omega^{\mathrm{D}}_{m-1,n}(\lambda_1) < \Omega_{m+1,n}(\lambda_1) = 1 < \Omega^{\mathrm{D}}_{m,n}(\lambda_1).$$

Hence, $\lambda_1 \in S_{m,n}$.

Case 2. $m > N_1$. The following subcases can be proved as in case 1.

(2a) If $m \leqslant M_1$, then $l_0 = m - 1$ such that $\lambda_1 \in S_{m-1,n}$.

(2b) If $m = M_1 + 1$, then $l_0 = m - 1$ or m, and $\lambda_1 \in S_{m-1,n}$.

(2c) If $m > M_1 + 1$, then $l_0 = m - 1$ or m. If $l_0 = m - 1$, then $\lambda_1 \in S_{m-2,n}$. If $l_0 = m$, then $\lambda_1 \in S_{m-1,n}$.

Let λ_2 be another intersection point of μ_{1m} and μ_{2n}. By fixing λ_2 and $\mu_{2n}(\lambda_2)$ in (3.3), the eigenvalues of the equation can be arranged as $\Omega_{0,n}(\lambda_2) < \Omega_{1,n}(\lambda_2) < \cdots$, and there exists a positive integer $M_1(\lambda_2)$ such that the eigenfunction of $\Omega_{l,n}(\lambda_2)$ has l zeros if $l \leqslant M_1(\lambda_2)$ and $l-1$ zeros if $l > M_1(\lambda_2)$. Now, as above, if $m \leqslant N_1$, where $N_1 = N_1(\mu_{1m}(\lambda_2))$, then $\lambda_2 \in S_{m,n}$, and if $m > N_1$, then λ_2 is in $S_{m-1,n}$ or $S_{m-2,n}$. Thus, the theorem follows. □

Corollary 3.11. *The eigencurves μ_{1m} and μ_{2n} intersect at at most a finite number of points.*

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Proof. Suppose that there are infinitely many points of intersection. Then, by Theorem 3.10, these points lie in a bounded set. Since μ_{1m} and μ_{2n} are analytic, $\mu_{1m} \equiv \mu_{2n}$. Therefore, μ_{1m} intersects μ_{1k} for $k \geqslant N(n)$, which is impossible. □

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References

1. T. Bhattacharyya, P. A. Binding and K. Seddighi, Two parameter right definite Sturm–Liouville problems with eigenparameter dependent boundary conditions, *Proc. R. Soc. Edinb.* A **131** (2001), 45–58.
2. T. Bhattacharyya, P. A. Binding and K. Seddighi, Multiparameter Sturm–Liouville problems with eigenparameter dependent boundary conditions, *J. Math. Analysis Applic.* **264** (2001), 560–576.
3. P. Binding, Abstract oscillation theorems for multiparameter eigenvalue problems, *J. Diff. Eqns* **49** (1983), 331–343.
4. P. A. Binding and P. J. Browne, A variational approach to multiparameter eigenvalue problems in Hilbert space, *SIAM J. Math. Analysis* **9** (1978), 1054–1067.
5. P. A. Binding and P. J. Browne, Applications of two parameter spectral theory to symmetric generalised eigenvalue problems, *Applic. Analysis* **29** (1988), 107–142.
6. P. A. Binding, P. J. Browne and K. Seddighi, Sturm–Liouville problems with eigenparameter dependent boundary conditions. *Proc. Edinb. Math. Soc.* **37** (1994), 57–72.
7. E. A. Coddington and N. Levinson, *Theory of ordinary differential equations* (Tata McGraw Hill, New Delhi, 1972).
8. J. B. Conway, *A course in functional analysis*, 2nd edn (Springer, 1990).
9. E. L. Ince, *Ordinary differential equations* (Dover, New York, 1956).
10. T. Kato, *Perturbation theory for linear operators* (Springer, 1966).
11. B. Sleeman, Klein oscillation theorems for multiparameter eigenvalue problems in ordinary differential equations, *Nieuw Arch. Wisk.* **27** (1979), 341–362.