On Digital Differentiators, Hilbert Transformers, and Half-Band Low-Pass Filters

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Abstract—**Interrelationships between the digital differentiator (DD), the digital Hilbert transformer (DHT), and the half-band low-pass filter (1/2-LPF) have been brought out. A number of important properties, confirming the close proximity of these filters, are highlighted. Theoretical results have been substantiated by transforming minimax relative error DD's to equiripple DHT's and equiripple 1/2-LPF's.**

I. Introduction

Digital differentiators (DD), digital Hilbert transformers (DHT), and half-band low-pass filters (1/2-LPF) are closely related to each other, but their interrelationships do not appear to have been recorded in the literature in completeness. A few characteristics for FIR-DD's and FIR-DHT's were brought out by Rabiner and Schafer [1], [2]. Jackson [3] suggested a procedure, using rotation in the Z plane, for conversion of impulse response of a DHT to that of a 1/2-LPF. Crochiere and Rabiner [4] derived an explicit formula for computation of impulse response for an ideal DHT from that of an ideal 1/2-LPF. The purpose of this paper is to present a complete picture of the interrelationships and to show how the design of one member of the family can be transformed to that of another, with particular emphasis of minimax relative error designs [1]. The notations $h(n)$ and $H(w)$ will be used for impulse response and frequency response, respectively, and the subscripts D, H, and L will be used to indicate DD, DHT, and 1/2-LPF, respectively. Furthermore, a tilde above the symbol h or H will be used to mean the ideal case, while absence of a tilde will mean a realizable approximation.

II. Interrelationships

A. Relations Connecting the Impulse Responses

For the ideal case,

$$\tilde{H}_L(w) = \begin{cases} 1, & -\pi/2 \le w \le \pi/2 \\ 0, & \text{elsewhere} \end{cases} \tag{1a}$$

$$\tilde{h}_L(n) = \begin{cases} 1/2, & n = 0 \\ [1/(\pi n)] \sin(n\pi/2), & n \ne 0 \end{cases} \tag{1b}$$

$$\tilde{H}_H(w) = \begin{cases} -j, & 0 \le w \le \pi \\ +j, & -\pi \le w \le 0 \end{cases} \tag{2a}$$

$$\tilde{h}_H(n) = \begin{cases} 0, & n = 0 \\ [2/(\pi n)] \sin^2(n\pi/2), & n \ne 0 \end{cases} \tag{2b}$$

$$\tilde{H}_D(w) = jw, \qquad -\pi \le w \le \pi \tag{3a}$$

$$\tilde{h}_D(n) = \begin{cases} 0, & n = 0 \\ (1/n)\cos n\pi, & n \ne 0. \end{cases} \tag{3b}$$

Fig. 1 shows plots of these responses where the ordinates are appropriately normalized so as to make the interrelationships obvious. A close look at the impulse responses reveals the following main features:

(i) $\tilde{h}_D(n)$ and $\tilde{h}_H(n)$ have odd symmetry whereas $\tilde{h}_L(n)$ has even symmetry about $n = 0$. Consequently, $\tilde{h}_D(0) = \tilde{h}_H(0) = 0$.

(ii) $\tilde{h}_L(n) = \tilde{h}_H(n) = 0$ for even values of n $(n \ne 0)$.

From these observations, explicit relations connecting the impulse responses are easily derived and are given in Table I. In this, $\delta(n)$ represents the unit sample and the symbol $S(n)$ has been used, for brevity, to denote sin $(n\pi/2)$. This will be useful when one wishes to transform a practical design of one member of the family to another; the appropriate off-diagonal entry in Table I is then to be used with tilde removed.

B. Relations Between $H_D(w)$ and $H_H(w)$

A little reflection will show that $\tilde{H}_D(w)/(jw)$ or $d(\tilde{H}_D(w)/j)/dw$ can be used interchangeably since both equal to unity. For the ideal cases, we can write

$$\tilde{H}_H(w) = \begin{cases} -\tilde{H}'_D(w), & 0 \le w \le \pi \\ +\tilde{H}'_D(w), & -\pi \le w \le 0 \end{cases} \tag{4}$$

where prime stands for differentiation with respect to w.

It is obvious that $\tilde{H}_D(w)/j$ represents an ideal all pass filter. The ideal DD and the ideal DHT, like the ideal LPF are noncausal and, therefore, cannot be realized exactly; to implement these, some approximation is necessary. Assume noncausal, finite impulse response of length N (assumed to be odd) for each of these filters. Note that these impulse responses can be made causal by adding a delay of at least $(N - 1)/2$ samples. Let $H_D(w)$ be a minimax relative error approximation of $\tilde{H}_D(w)$ [1]. Then the relative error $r_D(w)$ is an equiripple error function for

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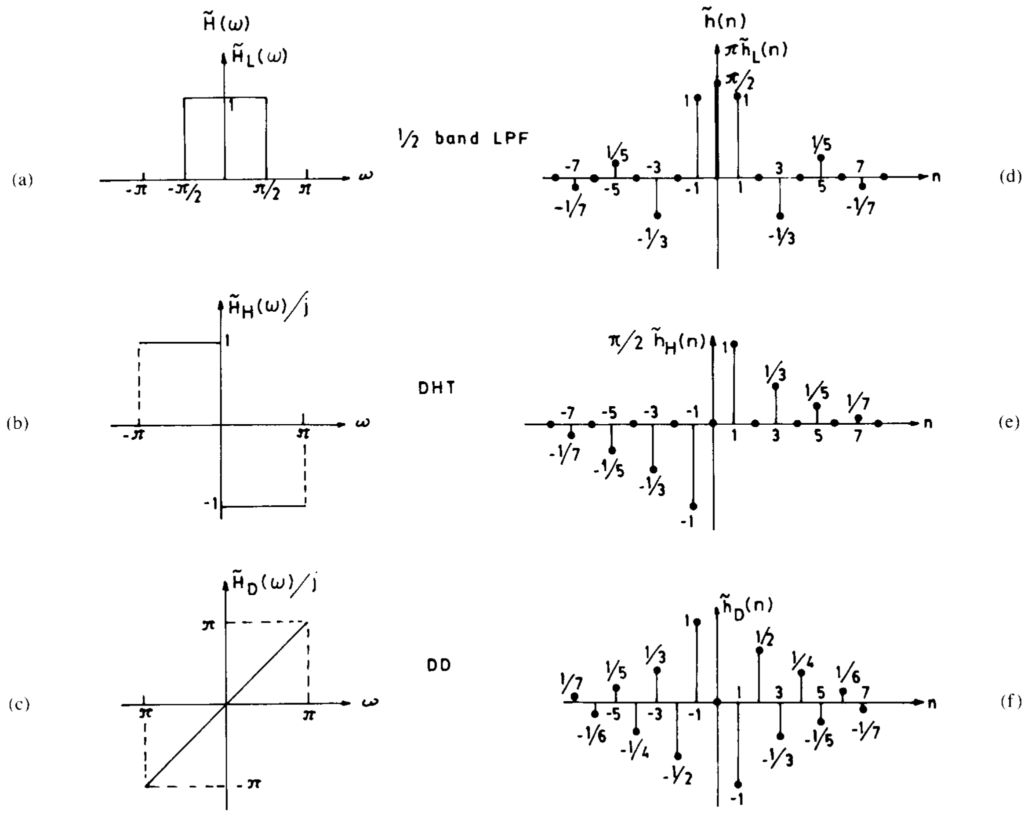

Fig. 1. (a), (b), (c) Frequency response of ideal half-band low-pass filter, digital Hilbert transformer, and digital differentiator, respectively. (d), (e), (f) Impulse response of ideal half-band low-pass filter, digital Hilbert transformer, and digital differentiator, respectively, shown up to $n = \pm 7$.

TABLE I
RELATIONS CONNECTING THE IMPULSE RESPONSES $\tilde{h}_L(n)$, $\tilde{h}_H(n)$, and $\tilde{h}_D(n)$

Conversion from ↓ to →	1/2-LPF	DHT	DD
1/2-LPF	$\tilde{h}_L(n) = \begin{cases} \frac{1}{2}, & n=0 \\ \frac{1}{\pi n} S(n), & n \neq 0 \end{cases}$	$\tilde{h}_H(n) = 2S(n) \cdot \tilde{h}_L(n)$	$\tilde{h}_D(n) = \begin{cases} 0, & n=0 \\ \frac{1-2n\pi \tilde{h}_L(n) \cdot S(n)}{n}, & n \neq 0 \end{cases}$
DHT	$\tilde{h}_L(n) = \frac{1}{2}[\delta(n) + S(n)\tilde{h}_H(n)]$	$\tilde{h}_H(n) = \begin{cases} 0, & n=0 \\ \frac{2}{\pi n} S^2(n), & n \neq 0 \end{cases}$	$\tilde{h}_D(n) = \begin{cases} 0, & n=0 \\ \frac{1-n\pi \tilde{h}_H(n)}{n}, & n \neq 0 \end{cases}$
DD	$\tilde{h}_L(n) = \frac{1}{2}\delta(n) - \frac{1}{\pi} S(n)\tilde{h}_D(n)$	$\tilde{h}_H(n) = \frac{-2}{\pi} S^2(n)\, \tilde{h}_D(n)$	$\tilde{h}_D(n) = \begin{cases} 0, & n=0 \\ \frac{1}{n} S(2n+1), & n \neq 0 \end{cases}$

Notes: 1. $S(n) \triangleq \sin\left(\frac{n\pi}{2}\right)$

2. In the above relations, zero delay impulse responses (i.e. non-causal) have been assumed for simplicity. For causal sequences with (N-1)/2 samples delay, replace n by $\left(n - \frac{N-1}{2}\right)$ in the above relations.

the DD and is given by

$$r_D(w) \triangleq [H_D(w)/j - w]/w = H_D(w)/(jw) - 1, \quad -\pi \le w \le \pi. \tag{5}$$

At the extremal frequencies w_i we have constant ripple peaks. Let

$$|r_D(w_i)| \triangleq K \text{ (a constant)}, \quad \pi \le w_i \le \pi. \tag{6}$$

Obviously the frequencies w_i's correspond to the solutions of $d\, r_D(w)/dw = 0$. Differentiating (5) with respect to w and equating to zero, we get

$$w_i = H_D(w_i)/H_D'(w_i), \quad -\pi \le w_i \le \pi. \tag{7}$$

From (5) and (7), we obtain

$$r_D(w_i) = H_D'(w_i)/j - 1, \quad -\pi \le w_i \le \pi. \tag{8a}$$

Dropping the subscript i from w_i, we have the equiripple error function, $r_D(w)$, given by

$$r_D(w) = H_D'(w)/j - 1, \quad -\pi \le w \le \pi. \tag{8b}$$

Equations (6) and (8b) show that $H_D'(w)/j$ represents the frequency response of an equiripple, all-pass filter, as expected, the ripple extrema occurring at frequencies $w = w_i$, and the magnitude of the ripple peaks being equal to K. Thus, if we choose the frequency response $H_H(w)$ such that

$$H_H(w) = \begin{cases} -H_D'(w), & 0 \le w \le \pi \\ +H_D'(w), & -\pi \le w \le 0 \end{cases} \tag{9}$$

then, clearly, $H_H(w)$ represents an equiripple DHT. The ripple contents $r_H(w)$ in $H_H(w)$ are defined by

$$r_H(w) \triangleq \begin{cases} H_H(w)/j + 1, & 0 \le w \le \pi \\ H_H(w)/j - 1, & -\pi \le w \le 0 \end{cases} \tag{10}$$

which, using (9), becomes

$$r_H(w) = \begin{cases} -H_D'(w)/j + 1, & 0 \le w \le \pi \\ +H_D'(w)/j - 1, & -\pi \le w \le 0 \end{cases} \tag{11a}$$

$$= \begin{cases} -r_D(w), & 0 \le w \le \pi \\ +r_D(w), & -\pi \le w \le 0 \end{cases} \quad [\text{due to (8b)}]. \tag{11b}$$

Using (6) and (11b), we obtain

$$|r_H(w_i)| = |r_D(w_i)| = K, \quad -\pi \le w_i \le \pi. \tag{12}$$

Thus, given a minimax relative error DD, $H_D(w)$, we can always transform it to an equiripple DHT, $H_H(w)$, using the transformation (9); the ripple peaks of $H_D(w)$ and $H_H(w)$ occur at the same respective frequencies with equal magnitude K.

C. Relations Between $H_H(w)$ and $H_L(w)$ and Between $H_L(w)$ and $H_D(w)$

Taking N samples (symmetrical about $n = 0$), from Fig. 1(e), the transfer function of an approximation $H_H(z)$ of a causal, FIR DHT can be written as

$$H_H(z) = z^{-(N-1)/2} \sum_{i=-n}^{n} h(i)\, z^{-i}, \quad n = (N-1)/2 \tag{13a}$$

$$= -\frac{2}{\pi} z^{-(N-1)/2} \left[(z - z^{-1}) + \frac{1}{3}(z^3 - z^{-3}) + \frac{1}{5}(z^5 - z^{-5}) + \cdots + \frac{1}{n}(z^n - z^{-n}) \right] \tag{13b}$$

which gives the frequency response

$$H_H(w) = \frac{-4j}{\pi} \left[\sin w + \frac{1}{3}\sin 3w + \frac{1}{5}\sin 5w + \cdots + \frac{1}{n}\sin nw \right], \quad -\pi \le w \le \pi; \quad n = (N-1)/2 \tag{14a}$$

$$= \frac{-4j}{\pi} \sum_{i=1}^{n} \frac{1}{i} \sin i\, w, \quad i \text{ odd}; \quad -\pi \le w \le \pi. \tag{14b}$$

Similarly, taking N samples in Fig. 1(d), the frequency response of approximation, $H_L(w)$, of causal, FIR 1/2-LPF is given by

$$H_L(w) = \frac{1}{2} + \frac{2}{\pi} \left[\cos w - \frac{1}{3}\cos 3w + \frac{1}{5}\cos 5w - \cdots + \frac{1}{n}\cos nw \right], \quad -\pi \le w \le \pi; \quad n = (N-1)/2. \tag{15}$$

With a little manipulation, (15) can be written as

$$H_L(w) = \begin{cases} \frac{1}{2} + \frac{2}{\pi} \left[\sin(w + \pi/2) + \frac{1}{3}\sin 3(w + \pi/2) + \frac{1}{5}\sin 5(w + \pi/2) + \cdots + \frac{1}{n} \cdot \sin n(w + \pi/2) \right], & -\pi/2 \le w \le \pi/2 \\ \frac{1}{2} - \frac{2}{\pi} \left[\sin(w - \pi/2) + \frac{1}{3}\sin 3(w - \pi/2) + \sin\frac{1}{5} 5(w - \pi/2) + \cdots + \frac{1}{n} \cdot \sin n(w - \pi/2) \right], & -\pi \le w \le -\pi/2; \quad \pi/2 \le w \le \pi \end{cases} \tag{16a}$$

TABLE II
VARIOUS PARAMETERS FOR THE DHT'S AND 1/2-LPF'S DERIVED, THROUGH TRANSFORMATION, FROM MINIMAX RELATIVE ERROR DD'S [1]

S.No.	Parameter	DD	DHT	1/2-LPF
1.	Ideal Frequency Response	$\tilde{H}_D(w)=jw,\ -\pi\le w\le\pi$ (3a)	$\tilde{H}_H(w)=\begin{cases}-j, & 0\le w\le\pi\\ +j, & -\pi\le w\le 0\end{cases}$ (2a)	$\tilde{H}_L(w)=\begin{cases}1, & -\pi/2\le w\le\pi/2\\ 0, & \text{elsewhere}\end{cases}$ (1a)
2.	Frequncy Response of derived filters	$H_D(w)=j\sum_{i=1}^{n} d_i \sin iw,\ -\pi\le w\le\pi$; d_i correspond to minimax rel.error DDs [1] (given)	$H_H(w)=\begin{cases}-H'_D(w), & 0\le w\le\pi\\ +H'_D(w), & -\pi\le w\le 0\end{cases}$ (9)	$H_L(w)=\begin{cases}\frac{1}{2}[1-jH'_D(w+\pi/2)], & -\pi/2\le w\le\pi/2\\ \frac{1}{2}[1+jH'_D(w-\pi/2)], & \text{elsewhere}\end{cases}$ (18)
3.	Ripple signals	$r_D(w)\triangleq[H_D(w)/jw-1],\ -\pi\le w\le\pi$ (5)	$r_H(w)=\begin{cases}-r_D(w), & 0\le w\le\pi\\ +r_D(w), & -\pi\le w\le 0\end{cases}$ (11b)	$r_L(w)=\begin{cases}\frac{1}{2}r_D(w+\pi/2), & -\pi/2\le w\le\pi/2\\ \frac{-1}{2}r_D(w-\pi/2), & \text{elsewhere}\end{cases}$ (20)
4.	Ripple extremal frequencies	at $w=w_i$; w_i are solutions of $dr_D(w)/dw=0$	at $w=w_i$ (12)	at $w=w_i\pm\pi/2$ (20)
5.	Ripple peaks magnitude	$\lvert r_D(w_i)\rvert \triangleq K$ (6)	K (12)	$K/2$ (20)

$$
=\begin{cases}\dfrac{1}{2}+\dfrac{2}{\pi}\displaystyle\sum_{i=1}^{n}\dfrac{1}{i}\sin i(w+\pi/2),\ i\ \text{odd},\\ \quad -\pi/2\le w\le\pi/2\\ \dfrac{1}{2}-\dfrac{2}{\pi}\displaystyle\sum_{i=1}^{n}\dfrac{1}{i}\sin i(w-\pi/2),\ i\ \text{odd, elsewhere.}\end{cases}
\tag{16b}
$$

From (14b) and (16b), we obtain

$$
H_L(w)=\begin{cases}\frac{1}{2}[1+jH_H(w+\pi/2)], & -\pi/2\le w\le\pi/2\\ \frac{1}{2}[1-jH_H(w-\pi/2)], & \text{elsewhere.}\end{cases}
\tag{17}
$$

It is easy to show that $H_H(w+\pi/2)$, $-\pi/2\le w\le\pi/2$ and also $H_H(w-\pi/2)$, $-\pi\le w\le-\pi/2$; $\pi/2\le w\le\pi$ are both equivalent to $H_H(w')$, $0\le w'\le\pi$. Hence, from (9) and (17), we get

$$
H_L(w)=\begin{cases}\frac{1}{2}[1-jH'_D(w+\pi/2)], & -\pi/2\le w\le\pi/2\\ \frac{1}{2}[1+jH'_D(w-\pi/2)], & \text{elsewhere}\end{cases}
\tag{18}
$$

which is the relation connecting DD and the 1/2-LPF. Using (8b) and (18), we can also write

$$
H_L(w)=\begin{cases}1+\frac{1}{2}r_D(w+\pi/2), & -\pi/2\le w\le\pi/2\\ -\frac{1}{2}r_D(w-\pi/2), & \text{elsewhere.}\end{cases}
\tag{19}
$$

Clearly, (19) shows that $H_L(w)$ is the frequency response of equiripple 1/2-LPF with ripple contents $r_L(w)$ given by

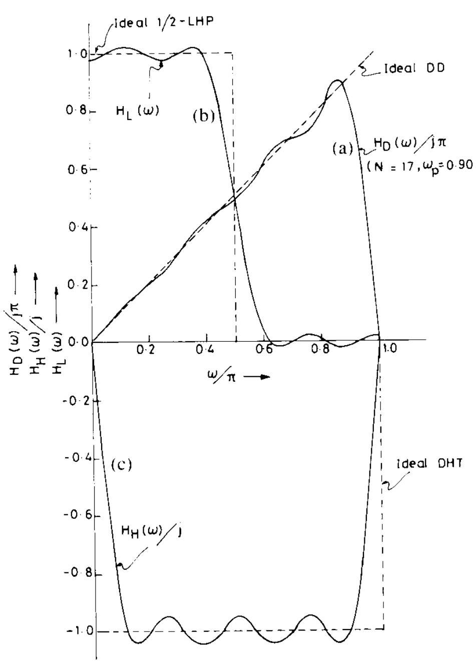

Fig. 2. Frequency responses for (a) Given minimax relative error DD (F_p = 0.45, N = 17). (b) Transformed 1/2-LPF. (c) Transformed DHT.

$$
r_L(w)=\begin{cases}\frac{1}{2}r_D(w+\pi/2), & -\pi/2\le w\le\pi/2\\ -\frac{1}{2}r_D(w-\pi/2), & \text{elsewhere.}\end{cases}
\tag{20}
$$

The ripple peaks of $H_L(w)$ are of magnitude $K/2$ and occur at frequencies $w=w_i\pm\pi/2$.

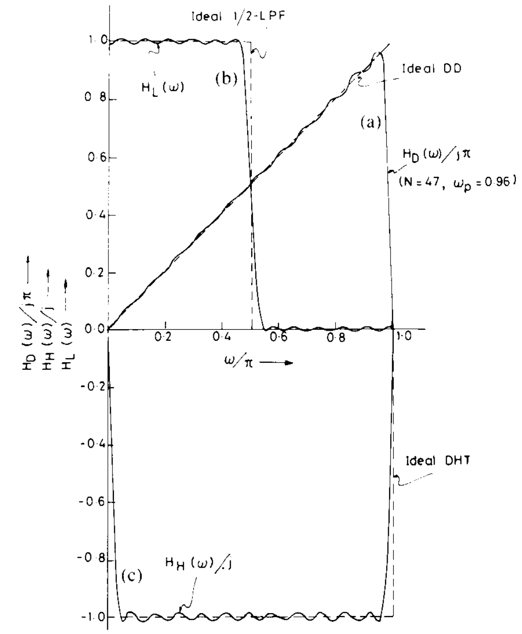

Fig. 3. Frequency responses for (a) Given minimax relative error DDL ($F_p = 0.48$, $N = 47$). (b) Transformed 1/2-LPF. (c) Transformed DHT.

Table II gives the various transformational relations for frequency responses, the ripple signals, their extremal frequencies together with the magnitude of the ripple peaks. It establishes the proximity of minimix relative error DDs to equiripple DHT's and equiripple 1/2-LPF's.

III. Performance

Making use of the relations of Table I and taking data for minimax relative error DD's from [1], for $F_p = 0.45$, $N = 17$ and $F_p = 0.48$, $N = 47$, the corresponding DHT's and 1/2-LPF's were derived. Their frequency responses are shown in Figs. 2 and 3, respectively. The results agree with the theoretical values with regard to the equiripple nature, the ripple magnitudes and the extremal frequencies, etc.

IV. Conclusion

It has been shown that the minimax relative error digital differentiators can easily be transformed to equiripple digital Hilbert transformers and equiripple half-band low-pass filters. Relations connecting their impulse responses and also their frequency responses have been brought out. The precise frequencies of ripple extrema and the magnitudes of their peaks are also shown to be simply related to the corresponding values for the minimax relative error differentiators. The proposed relations portray the picture in totality and give a clear insight into the connection between the subject filters. These can also affect considerable savings in the memory space in digital processing systems since the coefficients for Hilbert transformers are seen to be readily obtainable from those of the digital differentiators, obviating the necessity for their separate tabulation as done in [1] and [2]. It is concluded that the digital differentiators, the digital Hilbert transformers, and the digital half-band low-pass filters belong to a very closely knit family of filters.

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