IMRN International Mathematics Research Notices
Volume 2006, Article ID 80641, Pages 1–16

Universal Families on Moduli Spaces of Principal Bundles on Curves

V. Balaji, I. Biswas, D. S. Nagaraj, and P. E. Newstead

1 Introduction

Let X be a compact connected Riemann surface of genus at least three. Let $\mathcal{M}_X(n, d)$ denote the moduli space of all stable vector bundles over X of rank n and degree d, which is a smooth irreducible quasiprojective variety defined over $\mathbb{C}$. A vector bundle $\mathcal{E}$ over $X \times \mathcal{M}_X(n, d)$ is called *universal* if for every point $m \in \mathcal{M}_X(n, d)$, the restriction of $\mathcal{E}$ to $X \times \{m\}$ is in the isomorphism class of holomorphic vector bundles over X defined by m. A well-known theorem says that there is a universal vector bundle over $X \times \mathcal{M}_X(n, d)$ if and only if d is coprime to n (see [13] for existence in the coprime case, [8] for nonexistence in the noncoprime case, and [7] for a topological version of nonexistence in the case $d = 0$).

Let H be a connected semisimple linear algebraic group defined over the field of complex numbers. Ramanathan extended the notion of (semi-)stability to principal H-bundles and constructed moduli spaces for stable principal H-bundles over X [9–11]. The construction works for any given topological type, yielding a moduli space which is an irreducible quasiprojective variety defined over $\mathbb{C}$. We are concerned here with the case of topologically trivial stable principal H-bundles. Let $\mathcal{M}_X(H)$ denote the moduli space of topologically trivial stable principal H-bundles over X.

Let $Z(H) \subset H$ be the centre. For any H-bundle E_H, the group $Z(H)$ is contained in the automorphism group $\mathrm{Aut}(E_H)$. Let

$$\mathcal{M}'_X(H) \subset \mathcal{M}_X(H) \tag{1.1}$$

Received 5 September 2005; Revised 27 November 2005; Accepted 8 January 2006
Communicated by Simon K. Donaldson

be the subvariety consisting of all H-bundles E_H over X with the property $\mathrm{Aut}(E_H) \cong Z(H)$. It is known that $\mathcal{M}'_X(H)$ is a dense Zariski open subset contained in the smooth locus of $\mathcal{M}_X(H)$.

A principal H-bundle E over $X \times \mathcal{M}'_X(H)$ will be called a *universal bundle* if for every point $m \in \mathcal{M}_X(n, d)$, the restriction of $\mathcal{E}$ to $X \times \{m\}$ is in the isomorphism class of stable H-bundles over X defined by the point m of the moduli space.

The following theorem is the main result proved here.

Theorem 1.1. There is a universal H-bundle over $X \times \mathcal{M}'_X(H)$ if and only if $Z(H) = e$. □

Remark 1.2. Nonexistence for $H = SL(n, \mathbb{C})$ was previously known [8, 7], as also was existence for $H = PGL(n, \mathbb{C})$.

2 Existence of universal bundle

We begin this section by recalling very briefly certain facts from [2]. Unless otherwise specified, all bundles and sections considered will be algebraic.

Remark 2.1. Let E be a principal G-bundle over X, where G is a reductive linear algebraic group defined over $\mathbb{C}$ and $H \subset G$ is a Zariski closed semisimple subgroup. For any variety Y equipped with an action of G, the fibre bundle $(E \times Y)/G$ over X associated to E will be denoted by $E(Y)$.

(1) There is a natural action of the group $\mathrm{Aut}_G E$, defined by all automorphisms of E over the identity map of X that commute with the action of G, on $\Gamma(X, E(G/H))$ (the space of all holomorphic sections of the fibre bundle $E(G/H) = E/H$ over X) and the orbits correspond to the equivalence classes of H-reductions of E with two reductions being equivalent if the corresponding principal H-bundles are isomorphic.

(2) Let $G = GL(n, \mathbb{C})$ and let $\phi : H \hookrightarrow G$ be a faithful representation of the semisimple group H. Let Q denote the open subset of semistable principal G-bundles (or equivalently of topologically trivial semistable vector bundles of rank n) of the usual "Quot scheme," and let $Q(\phi)$ be the "Quot scheme" which parametrises pairs of the form (E', s), where E' is a principal G-bundle and s is a reduction of structure group of E' to H. Then $Q(\phi)$ is in a sense a "relative Quot scheme." As is clear from the definition and the notation, this scheme is dependent on the choice of the inclusion $\phi : H \hookrightarrow G$ (for details, see [2, 10, 11]).

One also has a *tautological* sheaf on $X \times Q$ which in fact is a vector bundle. We denote by $\mathcal{E}$ the *associated tautological* principal G-bundle on $X \times Q$.

Recall that the moduli space of principal G-bundles ($G = GL(n, \mathbb{C})$) is realised as a good quotient of Q by the action of a reductive group $\mathcal{G}$. We may also assume that the group $\mathcal{G}$ is with trivial centre (see, e.g., [4]).

Remark 2.2. It is immediate that the action of $\mathcal{G}$ on Q lifts to an action on $Q(\phi)$, where $\mathcal{G}$, Q, and $Q(\phi)$ are defined in Remark 2.1.

We have a morphism

$$\psi : Q(\phi) \longrightarrow Q, \tag{2.1}$$

which sends any H-bundle E' to the $GL(n, \mathbb{C})$-bundle obtained by extending the structure group of E' using the homomorphism ϕ. In fact, ψ is a $\mathcal{G}$-equivariant *affine morphism* (see [2]).

Continuing with the notation in the above two remarks, consider the $\mathcal{G}$-action on $Q(\phi)$ (defined in Remark 2.1(2)) with the linearisation induced by the affine $\mathcal{G}$-morphism ψ in (2.1). Since a good quotient of Q by $\mathcal{G}$ exists and since ψ is an affine $\mathcal{G}$-equivariant map, a good quotient $Q(\phi)//\mathcal{G}$ exists (see [11, Lemma 5.1]).

Moreover by the universal property of categorical quotients, the canonical morphism

$$\overline{\psi} : Q(\phi)//\mathcal{G} \longrightarrow Q//\mathcal{G} \tag{2.2}$$

given by ψ is also *affine*.

Theorem 2.3. Let $\mathcal{M}_X(H)$ denote the scheme $Q(\phi)//\mathcal{G}$ (see (2.2)). Then this scheme is the coarse moduli scheme of semistable H-bundles. Further, the scheme $\mathcal{M}_X(H)$ is projective, and if $H \hookrightarrow GL(V)$ is a faithful representation, then the canonical morphism

$$\overline{\psi} : \mathcal{M}_X(H) \longrightarrow \mathcal{M}_X\big(GL(V)\big) = Q//\mathcal{G} \tag{2.3}$$

is finite. □

Let $Q(\phi)^s$ be the open subscheme of $Q(\phi)$ consisting of stable H-bundles.

Lemma 2.4. Let $Q'_H \subset Q(\phi)^s$ be the subset parametrising all stable H-bundles whose automorphism group is $Z(H)$. Then the action of $\mathcal{G}$ on the subset Q'_H is free, and furthermore, the quotient morphism $Q'_H \to Q'_H/\mathcal{G}$ is a principal $\mathcal{G}$-bundle. In fact, $Q'_H/\mathcal{G}$ is precisely the Zariski open subset $\mathcal{M}'_X(H)$ (the variety $\mathcal{M}'_X(H)$ is defined in the introduction).

□

Proof. For any point $E_H \in Q(\phi)$, the isotropy subgroup of E_H for the action of $\mathcal{G}$ coincides with $\mathrm{Aut}(E_H)/Z(H)$. This can be seen as follows: firstly, the point E_H is a pair (E, s), where $E \in Q$ and s is a reduction of structure group to H of the G-bundle E. It is well known for the action of $\mathcal{G}$ on Q that the isotropy at E is precisely the group $\mathrm{Aut}(E)/Z(G)$. From this, it is easy to see that the isotropy of E_H for the action of $\mathcal{G}$ on $Q(\phi)$ is the group $\mathrm{Aut}(E, s)/Z(H) = \mathrm{Aut}(E_H)/Z(H)$.

Hence it follows that the action of $\mathcal{G}$ on the open subset Q'_H is free, and the proof of the lemma is complete. ■

We remark that, at least when H is of adjoint type, Q'_H is a nonempty open subset of $Q(\phi)^s$. Openness is easy and can be seen, for example, from [5, Theorem II.6(ii)]. Nonemptiness follows from Proposition 2.6 below.

We prove a proposition on semisimple groups, possibly known to experts but which we could not locate in any standard text.

Proposition 2.5. Let H be a semisimple algebraic group. Then H has a faithful irreducible representation $\phi : H \to GL(V)$ if and only if the centre of H is cyclic. □

Proof. One way the implication is easy, namely, suppose that a faithful irreducible representation exists, then the centre $Z(H)$ of H is cyclic. To see this, first note that under the representation ϕ, the centre $Z(H)$ maps to a subgroup which commutes with all elements of $\phi(H)$. Since ϕ is irreducible, this implies by Schur's lemma that $\phi(Z(H)) \subset Z(GL(V))$, where $Z(GL(V))$ is the centre of $GL(V)$. Observe further that since H is semisimple, we have $\phi(H) \subset SL(V)$. Hence, $Z(H) \subset Z(SL(V))$ and is therefore cyclic.

The contrapositive statement is harder to prove. We proceed as follows. Let H' be the simply connected cover of H and let $\overline{H}$ be the associated adjoint group, namely, $H/Z(H)$. Let Λ (resp., Λ_R) be the weight lattice (resp., root lattice). In other words, by the Borel-Weil theorem $\Lambda = \mathcal{X}(B)$ and $\Lambda_R = \mathcal{X}(\overline{B})$, where B, $\overline{B}$ are fixed Borel subgroups of H, $\overline{H}$. Then, from the exact sequence

$$e \longrightarrow Z(H) \longrightarrow H \longrightarrow \overline{H} \longrightarrow e \tag{2.4}$$

we see that $\Lambda/\Lambda_R \simeq \mathcal{X}(Z(H))$. In other words, the quotient group Λ/Λ_R is cyclic of order m.

Let $\overline{\lambda}$ be a generator of the cyclic group Λ/Λ_R. Then, by an action of the Weyl group we may assume that the coset representative $\lambda \in \Lambda$ is actually a dominant weight.

Suppose that the root lattice has the following decomposition (corresponding to the simple components of $\overline{H}$):

$$\Lambda_R = \bigoplus_{i=1}^{\ell} \Lambda^i. \tag{2.5}$$

Then, by possibly adding dominant weights from the Λ^i, we may assume that $(m \cdot \lambda) \in \Lambda_R$ has all its direct sum components $\lambda_i \neq 0$, $i \in [1, \ell]$, where λ is as above.

For this choice of $\lambda \in \Lambda$ let V_λ be the corresponding H-module given by

$$\phi_\lambda : H \longrightarrow GL\left(V_\lambda\right). \tag{2.6}$$

Then one knows that ϕ_λ is an irreducible representation of H.

We claim that the representation ϕ_λ is even *faithful*. Suppose that this is not the case. Let $K_\lambda := \text{kernel}(\phi_\lambda) \neq e$.

Firstly, $K_\lambda \subset Z(H)$. To see this, let K' be the inverse image of K_λ in the simply connected cover H' of H. Then the choice of λ so made that its simple components are nonzero in fact forces the following. Suppose that $H' = H_1 \times \cdots \times H_\ell$ is the decomposition of H' into its almost simple factors. (We recall that a semisimple algebraic group is called almost simple if the quotient of it by its centre is simple.) Then the normal subgroup K' in its decomposition in H' is such that $K_i := K' \bigcap H_i$ are proper normal subgroups of H_i. In particular, $K_i \subset Z(H_i)$ for all $i \in [1, \ell]$. This implies that $K' \subset Z(H')$ and hence $K_\lambda \subset Z(H)$.

Note that the dominant character λ is nontrivial on the generator of the centre $Z(H)$ because $\mathcal{X}(Z(H)) = \Lambda/\Lambda_R$. Now $K_\lambda \subset Z(H)$ and $Z(H)$ cyclic implies that λ is nontrivial on the generator of K_λ as well. This contradicts the fact that $K_\lambda = \text{kernel}(\phi_\lambda)$. This proves the claim. Therefore, the proof of the proposition is complete. ■

If H is of adjoint type, then by **Proposition 2.5** we can choose the inclusion $\phi : H \hookrightarrow G$ in **Remark 2.1**(2) to be an irreducible representation. Henceforth, ϕ will be assumed to be irreducible.

Let E be a stable H-bundle of trivial topological type. Recall that one can realise E from a unique, up to an inner conjugation, irreducible representation of $\pi_1(X)$ in a maximal compact subgroup of H (see [9]). For notational convenience, we will always suppress the base point in the notation of fundamental group. Denote by $M(E)$ the Zariski closure of the image of $\pi_1(X)$ in H.

Proposition 2.6. Let H be of adjoint type and let ϕ be a faithful irreducible representation of H in V (see **Proposition 2.5**). Let $U \subset Q(\phi)^s$ be the subset defined as follows:

$$U = \left\{E \in Q(\phi)^s \mid M(E) = H\right\}. \tag{2.7}$$

Then U is nonempty and is contained in the subset Q'_H of stable H-bundles which have trivial automorphism group. (Since H is of adjoint type, $Z(H) = e$.) □

Proof. Since ϕ is irreducible, using [12, Lemma 2.1] we conclude that there is an irreducible representation

$$\rho : \pi_1(X) \longrightarrow H \tag{2.8}$$

which has the property that the composition $\phi \circ \rho : \pi_1(X) \to G := GL(V)$ continues to remain irreducible. This implies that $M(E_\rho) = H$ by the construction in [12, Lemma 2.1] and hence $E_\rho \in U$, that is, U is nonempty.

Let $E \in U$. Then by the definition of U, there exists a representation ρ of $\pi_1(X)$ in H such that $E \simeq E_\rho$, where E_ρ is the flat principal H-bundle given by ρ. Observe that E_ρ is a stable H-bundle and the associated G-bundle is also stable. Hence all the automorphisms of this associated G-bundle lie in $Z(G)$. Since H is of adjoint type, it follows that the H-bundle E_ρ has no nontrivial automorphisms. Hence it follows that $U \subset Q'_H$. ■

We have the following theorem on existence of universal families.

Theorem 2.7. Let H be a group of adjoint type and $\mathcal{M}'_X(H)$ the Zariski open subset of $\mathcal{M}_X(H)$ defined in the introduction. Then there exists a universal family of principal H-bundles on $X \times \mathcal{M}'_X(H)$. □

Proof. We first observe that the variety $\mathcal{M}'_X(H)$ is precisely the image of Q'_H under the quotient map for the action of $\mathcal{G}$. We recall that Q'_H is nonempty by Proposition 2.6.

Since $Z(H) = e$, it follows that $H \subset \overline{G} := G/Z(G)$, where $G = GL(V)$ is as in Remark 2.1(2).

Consider the tautological principal G-bundle $\mathcal{E}$ on $X \times Q^s$, and let $\overline{\mathcal{E}}$ be the corresponding $\overline{G}$-bundle obtained by extending the structure group. Then it is well known that the adjoint universal bundle $\overline{\mathcal{E}}$ descends to the quotient $\mathcal{M}_X(G)^s$ (see [4]).

We follow the same strategy for $\mathcal{M}'_X(H)$ as well. Consider the pulled-back $\overline{G}$-bundle $(\mathrm{Id}_X \times \psi)^*\overline{\mathcal{E}}$ on $X \times Q'_H$, where ψ is the map in (2.1) and $\overline{G} := G/Z(G)$. The action of $\mathcal{G}$ on Q'_H is free by Lemma 2.4. Therefore, the quotient $Q'_H \to \mathcal{M}'_X(H)$ is a principal $\mathcal{G}$-bundle. Further, the action of $\mathcal{G}$ lifts to the tautological bundle $(\mathrm{Id}_X \times \psi)^*\overline{\mathcal{E}}$. In particular, the principal $\overline{G}$-bundle $(\mathrm{Id}_X \times \psi)^*\overline{\mathcal{E}}$ descends to a principal $\overline{G}$-bundle over $X \times \mathcal{M}'_X(H)$.

Let us denote this descended $\overline{G}$-bundle over $X \times \mathcal{M}'_X(H)$ by $\overline{\mathcal{E}}_0$.

Let $\pi : (\mathrm{Id}_X \times \psi)^*\mathcal{E}(G/H) \to (\mathrm{Id}_X \times \psi)^*\overline{\mathcal{E}}(\overline{G}/H)$ be the natural map induced by the projection $G/H \to \overline{G}/H$, where ψ is the map in (2.1). We note that the universal H-bundle over $X \times U$, where U is defined in Proposition 2.6, is a reduction of structure group of the pulled-back G-bundle $(\mathrm{Id}_X \times \psi)^*\mathcal{E}$. Let

$$\sigma : X \times U \longrightarrow \left(\mathrm{Id}_X \times \psi\right)^*\mathcal{E}(G/H) \tag{2.9}$$

be the section giving this reduction of structure group. Then the composition $\pi \circ \sigma$ is a section of $(\mathrm{Id}_X \times \psi)^* \overline{\mathcal{E}}(\overline{G}/H)$ over $X \times U$.

Since H is semisimple, a lemma of Chevalley says that there is a $\overline{G}$-module W and an element $w \in W$ such that H is precisely the isotropy subgroup for w (see [3, page 89, Theorem 5.1]). Therefore, $\overline{G}/H$ is identified with the closed $\overline{G}$-orbit in W defined by w. Then we see that $\overline{\mathcal{E}}(\overline{G}/H) \hookrightarrow \overline{\mathcal{E}}(W)$. We may therefore view the section

$$\pi \circ \sigma : X \times U \longrightarrow \left(\mathrm{Id}_X \times \psi\right)^* \overline{\mathcal{E}}\left(\overline{G}/H\right) \tag{2.10}$$

as a section of the vector bundle $(\mathrm{Id}_X \times \psi)^* \overline{\mathcal{E}}(W)$ over $X \times U$.

Since $(\mathrm{Id}_X \times \psi)^* \overline{\mathcal{E}}$ descends to the $\overline{G}$-bundle $\overline{\mathcal{E}}_0$ over $X \times \mathcal{M}'_X(H)$, it follows that the associated vector bundle $(\mathrm{Id}_X \times \psi)^* \overline{\mathcal{E}}(W)$ also descends to $X \times \mathcal{M}'_X(H)$. Clearly this vector bundle is nothing but the associated vector bundle $\overline{\mathcal{E}}_0(W)$, associated to the $\overline{G}$-bundle $\overline{\mathcal{E}}_0$ on $X \times \mathcal{M}'_X(H)$ for the $\overline{G}$-module W.

Since each point of $\mathcal{M}'_X(H)$ represents an isomorphism class of stable H-bundles, it follows that, set theoretically, the reduction section $\pi \circ \sigma \in \Gamma((\mathrm{Id}_X \times \psi)^* \overline{\mathcal{E}}(W))$ descends to a section on $X \times \mathcal{M}'_X(H)$ of the descended vector bundle $\overline{\mathcal{E}}_0(W)$.

We now appeal to [6, Proposition 4.1], which implies that the section $\pi \circ \sigma$ in fact descends to give a holomorphic section of $\overline{\mathcal{E}}_0(W)$ over $X \times \mathcal{M}'_X(H)$.

Again set theoretically, the image of this section of $\overline{\mathcal{E}}_0(W)$ lies in $\overline{\mathcal{E}}_0(\overline{G}/H) \subset \overline{\mathcal{E}}_0(W)$. As before, from [6, Proposition 4.1] it follows that $\pi \circ \sigma$ in (2.10) gives a reduction of structure group to H of the descended $\overline{G}$-bundle $\overline{\mathcal{E}}_0$. The H-bundle over $X \times \mathcal{M}'_X(H)$ obtained this way is the required universal H-bundle. This completes the proof of the theorem. ∎

3 Nonexistence of universal bundle

Let H be a complex semisimple linear algebraic group and let $K \subset H$ be a maximal compact subgroup. The Lie algebra of K will be denoted by $\mathfrak{k}$. A homomorphism

$$\rho : \pi_1(X) \longrightarrow K \tag{3.1}$$

is called *irreducible* if no nonzero vector in $\mathfrak{k}$ is fixed by the adjoint action of the subgroup $\rho(\pi_1(X)) \subset K$ on $\mathfrak{k}$. Let $\mathrm{Hom}^{\mathrm{irr}}(\pi_1(X), K)$ denote the space of all irreducible homomorphisms such that the corresponding K-bundle is topologically trivial. So any homomorphism in $\mathrm{Hom}^{\mathrm{irr}}(\pi_1(X), K)$ is induced by a homomorphism from $\pi_1(X)$ to the universal cover of K.

Let g denote the genus of X. Assume that $g \geq 3$. If we choose a basis

$$\{a_1, \ldots, a_g, b_1, \ldots, b_g\} \subset \pi_1(X) \tag{3.2}$$

such that

$$\pi_1(X) = \left\langle a_1, \ldots, a_g, b_1, \ldots, b_g : \prod_{i=1}^{n} a_i b_i a_i^{-1} b_i^{-1} \right\rangle, \tag{3.3}$$

then $\mathrm{Hom}^{\mathrm{irr}}(\pi_1(X), K)$ gets identified with a real analytic subspace of K^{2g}.

For any $\rho \in \mathrm{Hom}^{\mathrm{irr}}(\pi_1(X), K)$, the principal H-bundle obtained by extending the structure group of the principal K-bundle given by ρ is stable. A theorem of Ramanathan says that all topologically trivial stable principal H-bundles arise in this way, that is, the space of all equivalence classes of irreducible homomorphisms of $\pi_1(X)$ to K is in bijective correspondence with the space of all stable H-bundles over X [9, Theorem 7.1]. More precisely (see the proof of [9, Theorem 7.1]), the real analytic space underlying the moduli space $\mathcal{M}_X(H)$ parametrising topologically trivial stable H-bundles is analytically isomorphic to the quotient space

$$R\big(\pi_1(X), H\big) := \mathrm{Hom}^{\mathrm{irr}}\big(\pi_1(X), K\big)/K, \tag{3.4}$$

for the action constructed using the conjugation action of K on itself. Let

$$q : \mathrm{Hom}^{\mathrm{irr}}\big(\pi_1(X), K\big) \longrightarrow R\big(\pi_1(X), H\big) \cong \mathcal{M}_X(H) \tag{3.5}$$

be the quotient map. The open subset $\mathcal{M}'_X(H) \subset \mathcal{M}_X(H)$ is a smooth submanifold and the restriction of the map in (3.5),

$$q\,|_{q^{-1}(\mathcal{M}'_X(H))} : q^{-1}\big(\mathcal{M}'_X(H)\big) \longrightarrow \mathcal{M}'_X(H), \tag{3.6}$$

is a smooth principal K/Z-bundle, where Z is the centre of K. Note that Z coincides with the centre of H (as H is semisimple, its centre is a finite group).

If $\mathcal{U}_H$ is a universal H-bundle over $X \times \mathcal{M}'_X(H)$, then we have a reduction of structure group of $\mathcal{U}_H$ to K which is constructed using the correspondence established in [9] between stable H-bundles and irreducible flat K-bundles. Let

$$\mathcal{U}_K \longrightarrow X \times \mathcal{M}'_X(H) \tag{3.7}$$

be the smooth principal K-bundle obtained from $\mathcal{U}_H$ this way. Consider the principal K/Z-bundle over $X \times \mathcal{M}'_X(H)$, where $Z \subset K$ is the centre, obtained by extending the structure group of $\mathcal{U}_K$ using the natural projection of K to K/Z. The restriction of this K/Z-bundle to $p \times \mathcal{M}'_X(H)$, where p is the base point in X used for defining $\pi_1(X)$, is identified with the K/Z-bundle in (3.6). To see this first note that the universal K-bundle over $X \times \mathrm{Hom}(\pi_1(X), K)$ is obtained as a quotient by the action of $\pi_1(X)$ on $\widetilde{X} \times \mathrm{Hom}(\pi_1(X), K) \times K$, where $\widetilde{X}$ is the pointed universal cover of X for the base point p; the action of $z \in \pi_1(X)$ sends any (α, β, γ) to $(\alpha, z, \beta, \beta(z)^{-1}\gamma)$. From this it follows that the restriction of this universal bundle to $p \times \mathrm{Hom}(\pi_1(X), K)$ is canonically trivialized. The above-mentioned identification is constructed using this trivialization.

Our aim is to show that no K-bundle over $\mathcal{M}'_X(H)$ exists that produces the K/Z-bundle in (3.6) by extension of structure group, provided the centre Z is nontrivial.

Let F_3 denote the free group on three generators. Fix a surjective homomorphism

$$f : \pi_1(X) \longrightarrow F_3 \tag{3.8}$$

that sends a_i, $1 \le i \le 3$, to the ith generator of F_3 and sends a_i, $4 \le i \le g$, and b_i, $1 \le i \le g$, to the identity element, where a_j, b_j are as in (3.3) (recall that $g \ge 3$).

Set

$$R(F_3, H) := \mathrm{Hom}^{\mathrm{irr}}(F_3, K)/K \tag{3.9}$$

to be the equivalence classes of irreducible representations. Note that $\mathrm{Hom}(F_3, K) \simeq K^3$, and under this identification the action

$$\mu : \mathrm{Hom}(F_3, K) \times K \longrightarrow \mathrm{Hom}(F_3, K) \tag{3.10}$$

given by $\mu(\rho, A) = A^{-1}\rho A$ corresponds to the simultaneous diagonal conjugation action of K on the three factors.

Since F_3 is a free group and K is connected, any homomorphism ρ from F_3 to K can be deformed to the trivial homomorphism. This implies that the principal H-bundle corresponding to the homomorphism

$$\rho \circ f : \pi_1(X) \longrightarrow K \tag{3.11}$$

is topologically trivial, where f is defined in (3.8). Therefore, we have an embedding

$$R(F_3, H) \longrightarrow R(\pi_1(X), H) \cong \mathcal{M}_X(H) \tag{3.12}$$

that sends any ρ to $\rho \circ f$. Let $R'(F_3, H) \subset R(F_3, H)$ be the inverse image of the open subset $\mathcal{M}'_X(H)$ under the above map. By Proposition 2.6 it follows that $R'(F_3, H)$ is a *nonempty* open subset of $R(F_3, H)$.

Fix a point $p \in X$, which will also be the base point for the fundamental group. Consider the restriction of the principal K-bundle $\mathcal{U}_K$ in (3.7) to $p \times \mathcal{M}'_X(H) \hookrightarrow X \times \mathcal{M}'_X(H)$ and denote it by $\mathcal{U}_{K,p}$.

Let

$$\gamma : p \times R'(F_3, H) \longrightarrow p \times \mathcal{M}'_X(H) \tag{3.13}$$

be the map given by the embedding $R'(F_3, H) \to \mathcal{M}'_X(H)$ constructed in (3.12). Taking the pullback of the above-defined principal K-bundle $\mathcal{U}_{K,p}$ over $p \times \mathcal{M}'_X(H)$ under the morphism γ in (3.13)

$$\begin{array}{ccc} \widetilde{M} & & \mathcal{U}_{K,p} \\ \downarrow & & \downarrow \\ p \times R'(F_3, H) & \xrightarrow{\gamma} & p \times \mathcal{M}'_X(H) \end{array} \tag{3.14}$$

we obtain a principal K-bundle $\widetilde{M} := \gamma^*\mathcal{U}_{K,p}$ on $R'(F_3, H)$ whose associated K/Z-bundle (as before, Z is the centre of K) is precisely the K/Z-bundle $q^{-1}(R'(F_3, H)) \to R'(F_3, H)$ constructed in (3.6). In particular, $q^{-1}(R'(F_3, H)) \simeq \widetilde{M}/Z$.

Let $\mathrm{Hom}^{\mathrm{irr}}(F_3, K)' \subset \mathrm{Hom}^{\mathrm{irr}}(F_3, K)$ be the inverse image of $R'(F_3, H)$ under the quotient map in (3.9). Let

$$q_0 : \mathrm{Hom}^{\mathrm{irr}}(F_3, K)' \longrightarrow R'(F_3, H) \tag{3.15}$$

be the restriction of the quotient map in (3.9). So q_0 is the quotient for the conjugation action of K. This map q_0 defines a principal K/Z-bundle over $R'(F_3, H)$ which is evidently identified with the K/Z-bundle $\widetilde{M}/Z \to R'(F_3, H)$ obtained from (3.14).

We will prove (by contradiction) that such a K-bundle $\widetilde{M}$ does not exist. In other words, we will show that there is no principal K-bundle over $R'(F_3, H)$ whose extension of structure group is the K/Z-bundle in (3.15).

Assume the contrary, then we get the following diagram of topological spaces and morphisms:

$$\begin{array}{ccccc} Z & \longrightarrow & K & \longrightarrow & K/Z \\ \| & & \downarrow & & \downarrow \\ Z & \longrightarrow & \widetilde{M} & \longrightarrow & q_0^{-1}(R'(F_3, H)) \\ & & \downarrow & & \downarrow \\ & & R'(F_3, H) & = & R'(F_3, H) \end{array} \tag{3.16}$$

We will need a couple of results. The proof of the nonexistence of the bundle $\widetilde{M}$ will be completed after establishing Proposition 3.2.

Lemma 3.1. There exist finitely many compact differentiable manifolds N_i and differentiable maps

$$f_i : N_i \longrightarrow K^3, \tag{3.17}$$

$1 \leq i \leq d$, such that if $N := \bigcup_{i=1}^{d} f_i(N_i)$, then

(1) the complement $K^3 \setminus N$ is contained in $q_0^{-1}(R'(F_3, H))$, where q_0 is the projection in (3.15),

(2) $\dim K^3 - \dim N_i \geq 4$ for all $i \in [1, d]$. □

Proof. Take any homomorphism $\rho : \pi_1(X) \to K$. Let E_ρ denote the corresponding polystable principal H-bundle over X [9]. The automorphism group of the polystable H-bundle E_ρ coincides with the centraliser of $\rho(\pi_1(X))$ in H. Hence if $\rho(\pi_1(X))$ is dense in K, then the automorphism group of E_ρ is the centre $Z(H) \subset H$. Therefore, if the topological closure $\overline{\rho(\pi_1(X))}$ of $\rho(\pi_1(X))$ is K and E_ρ is topologically trivial, then $E_\rho \in \mathcal{M}'_X(H)$.

Now assume that the centraliser $C(\rho) \subset H$ of $\rho(\pi_1(X))$ in H properly contains $Z(H)$. Since the complexification of $\overline{\rho(\pi_1(X))}$ is reductive and the centraliser of a reductive group is reductive, we conclude that $C(\rho)$ is reductive. Take a semisimple element $z \in (H \setminus Z(H)) \bigcap C(\rho)$ (since $C(\rho)$ is reductive and larger than $Z(H)$, such an element exists). Let $C_z \subset K$ be the centraliser of z in K. We have $\overline{\rho(\pi_1(X))} \subset C_z$ and C_z is a proper subgroup of K as $z \notin Z(H)$. Also, C_z contains a maximal torus of K.

Fix a maximal torus $T \subset K$. (Since any two maximal tori are conjugate, any subgroup C_z of the above type would contain T after an inner conjugation.) Consider all proper Lie subgroups M of K satisfying the following two conditions:

(i) $T \subseteq M$,

(ii) there exists a semisimple element $z \in H \setminus Z(H)$ such that M is the centraliser of z in K.

Let $\mathcal{S}$ denote this collection of Lie subgroups of K.

The connected ones among $\mathcal{S}$ are precisely the maximal compact subgroups of Levi subgroups of proper parabolic subgroups of H containing T. Note that if P is a proper parabolic subgroup of H, then $P \bigcap K$ is a maximal compact subgroup of a Levi subgroup of P. All the connected ones among $\mathcal{S}$ arise as $P \bigcap K$ for some proper parabolic subgroup $P \subset H$ containing T.

This collection $\mathcal{S}$ is a finite set. To see this, we first note that there are only finitely many parabolic subgroups of H that contain T. For a proper parabolic subgroup $P \subset H$ containing T, there are only finitely many Lie subgroups of K that have $P \bigcap K$ as the connected component containing the identity element. Thus $\mathcal{S}$ is a finite set.

Let $M_1, \ldots, M_d$ be the subgroups of K that occur in $\mathcal{S}$.

We will show that the codimension of each M_i in K is at least two. It suffices to show that for a maximal proper parabolic subgroup P of G containing T, the codimension of $M := P \bigcap K$ in K is at least two.

To prove that the codimension of $M = P \bigcap K$ in K is at least two, let $\mathfrak{k}$ be the Lie algebra of K and let $\mathfrak{h}$ be the Lie algebra of T (the Cartan subalgebra). Let

$$\mathfrak{k} = \mathfrak{h} + \sum_{\alpha \text{ a root}} \mathfrak{k}^{\alpha} \tag{3.18}$$

be the root space decomposition of $\mathfrak{k}$. Let $\mathfrak{m}$ be a Lie algebra of M which is a Lie subalgebra of $\mathfrak{k}$ containing $\mathfrak{h}$. Then we have the decomposition

$$\mathfrak{m} = \mathfrak{h} + \sum_{\alpha \text{ a root}} \mathfrak{m}^{\alpha} \tag{3.19}$$

for $\mathfrak{m}$ where each $\mathfrak{m}^{\alpha}$ is irreducible for $\mathfrak{h}$, and hence has to coincide with one of the $\mathfrak{k}^{\alpha}$. Then $\operatorname{codim}_{\mathfrak{k}}(\mathfrak{m}) \geq 2$ since if α is a root for the subalgebra $\mathfrak{m}$, then so is $-\alpha$ (see [1, page 83, Corollary 4.15]).

Consequently, the codimension of M in K is at least two. Thus the codimension of each M_i, $i \in [1, d]$, in K is at least two.

For each $i \in [1, d]$, let

$$\overline{f}_i : K \times M_i^3 \longrightarrow K^3 \tag{3.20}$$

be the map defined by $(x, (y_1, y_2, y_3)) \mapsto (xy_1x^{-1}, xy_wx^{-1}, xy_3x^{-1})$. Consider the free action of M_i on $K \times M_i^3$ defined by

$$z \cdot \left(x, \left(y_1, y_2, y_3\right)\right) = \left(xz^{-1}, \left(zy_1z^{-1}, zy_2z^{-1}, zy_3z^{-1}\right)\right), \tag{3.21}$$

where $x \in K$ and z, y_1, y_2, $y_3 \in M_i$. The map f_i in (3.20) clearly factors through the quotient

$$\frac{K \times M_i{}^t}{M_i} \tag{3.22}$$

for the above action. Therefore, we have

$$f_i : N_i := \frac{K \times M_i{}^t}{M_i} \longrightarrow K^t \tag{3.23}$$

induced by f_i.

To prove part (1) of the lemma, we recall the earlier remark that for any homomorphism $\rho' \in q_0^{-1}(R'(F_3, H))$ (the map q_0 is defined in (3.15)), the automorphism group of the principal H-bundle corresponding to $\rho' \circ f$ (the homomorphism f is defined in (3.8)) coincides with $Z(H)$ if the image of ρ' is dense in K. From the properties of the collection $\{M_1, \ldots, M_d\}$ we conclude that

$$N := \bigcup_{i=1}^{d} f_i\left(N_i\right) \supseteq q_0^{-1}\left(R'\left(F_3, H\right)\right)^c, \tag{3.24}$$

where f_i are defined in (3.23) and

$$q_0^{-1}\left(R'\left(F_3, H\right)\right)^c \subset K^3 \tag{3.25}$$

is the complement of $q_0^{-1}(R'(F_3, H))$ in K^3.

Therefore, proof of part (1) is complete. To prove part (2), we note that

$$\dim N_i = \dim K + 2 \cdot \dim M_i \tag{3.26}$$

for all $i \in [1, d]$. It was shown earlier that $\dim M_i \leq \dim K - 2$. Therefore,

$$\dim K^3 - \dim N_i = 3 \cdot \dim K - \dim N_i \geq 3 \cdot \dim K - 3 \cdot \dim K + 4 = 4. \tag{3.27}$$

This completes the proof of the lemma. ■

Proposition 3.2. Consider the K/Z-principal bundle $q_0^{-1}(R'(F_3, H)) \to R'(F_3, H)$, where q_0 is the projection in (3.15). The induced homomorphism on fundamental groups

$$\pi_1(K/Z) \longrightarrow \pi_1\left(q^{-1}\left(R'(F_3, H)\right)\right) \tag{3.28}$$

obtained from the homotopy exact sequence is trivial. □

Proof. Let $x_1, x_2, x_3 \in K$ be regular elements; we recall that $x \in K$ is called regular if the centralizer $C(x) = \{y \in K \mid yx = xy\}$ is a maximal torus in K. Since the set of regular elements is dense in K, and $q_0^{-1}(R'(F_3, H)) \subset K^3$ is a nonempty open dense subset (this follows from Lemma 3.1), we may choose these x_i, $i = 1, 2, 3$, to lie in $q_0^{-1}(R'(F_3, H))$.

Consider the orbit $\mathrm{Orb}_{K^3}(x_1, x_2, x_3)$ of $(x_1, x_2, x_3) \in K^3$ for the adjoint action of the group K^3 on itself. Clearly we have the following identification of this orbit:

$$\mathrm{Orb}_{K^3}\left(x_1, x_2, x_3\right) = \frac{K}{C(x_1)} \times \frac{K}{C(x_2)} \times \frac{K}{C(x_3)} \tag{3.29}$$

with $C(x_i) \subset K$ being the centralizer of x_i. The conjugation action of K on $\mathrm{Hom}(F_3, K)$ coincides with the restriction of the above action of K^3 to the image of the diagonal map $K \hookrightarrow K^3$. Therefore, the fibre K/Z through the point $(x_1, x_2, x_3) \in q_0^{-1}(R'(F_3, H))$ of the K/Z-bundle

$$q_0^{-1}\left(R'\left(F_3, H\right)\right) \longrightarrow R'\left(F_3, H\right) \tag{3.30}$$

is contained in the orbit

$$\frac{K}{C(x_1)} \times \frac{K}{C(x_2)} \times \frac{K}{C(x_3)}. \tag{3.31}$$

Now if we choose a point $(x_1, x_2, x_3) \in K^3$ which is general enough, then by the definition of the inverse image $q_0^{-1}(R'(F_3, H))$ and Lemma 3.1(2) it follows immediately that the complement of the open dense subset

$$q_0^{-1}\left(R'\left(F_3, H\right)\right) \bigcap \left(\frac{K}{C(x_1)} \times \frac{K}{C(x_2)} \times \frac{K}{C(x_3)}\right) \subset \frac{K}{C(x_1)} \times \frac{K}{C(x_2)} \times \frac{K}{C(x_3)} \tag{3.32}$$

is of codimension at least four.

Since the image of K/Z in $q_0^{-1}(R'(F_3, H))$ lies in $(q^{-1}(R'(F_3, H))) \bigcap \mathrm{Orb}_{K^3}(x_1, x_2, x_3)$, whose complement is of codimension at least four in

$$\mathrm{Orb}_{K^3}\left(x_1, x_2, x_3\right) = \frac{K}{C(x_1)} \times \frac{K}{C(x_2)} \times \frac{K}{C(x_3)}, \tag{3.33}$$

it follows that the homomorphism $\pi_1(K/Z) \to \pi_1(q_0^{-1}(R'(F_3, H)))$ in the proposition factors through

$$\begin{aligned}\pi_1 & ((q_0^{-1}(R'(F_3, H))) \cap \mathrm{Orb}_{K^3}\ (x_1, x_2, x_3)) \\ & = \pi_1\left(\frac{K}{C(x_1)} \times \frac{K}{C(x_2)} \times \frac{K}{C(x_3)}\right) = \pi_1(K/T)^3,\end{aligned} \tag{3.34}$$

where T is a maximal torus of K.

For any maximal torus $T \subset K$, the quotient K/T is diffeomorphic to H/B, where B is a Borel subgroup of H. Since H/B is simply connected, we conclude that $(K/C(x_1)) \times (K/C(x_2)) \times (K/C(x_3))$ is simply connected. This completes the proof of the proposition. ■

We now complete the proof of the nonexistence of the covering $\widetilde{M} \to q^{-1}(R'(F_3, H))$ *as in (3.16).*

Since the homomorphism of fundamental groups $\pi_1(K/Z) \to \pi_1(q_0^{-1}(R'(F_3, H)))$ is trivial (see Proposition 3.2), the induced covering $K \to K/Z$ is trivial (see the diagram (3.16)). But this is a contradiction to the fact that K is connected.

Therefore, we have proved the following theorem.

Theorem 3.3. If the centre $Z(H)$ is nontrivial, then there is no universal bundle over $X \times \mathcal{M}'_X(H)$. □

Acknowledgments

We thank A. King, A. Nair, and P. Sankaran for useful discussions. All authors are members of the international research group Vector Bundles on Algebraic Curves (VBAC), which is partially supported by EAGER (EC FP5 Contract no. HPRN-CT-2000-00099) and by EDGE (EC FP5 Contract no. HPRN-CT-2000-00101). The fourth author would also like to thank the Royal Society and CSIC, Madrid, for supporting a visit to CSIC during which his contribution to this work was completed.

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V. Balaji: Chennai Mathematical Institute, 92 G.N. Chetty Road, Chennai 600017, India
E-mail address: balaji@cmi.ac.in

I. Biswas: School of Mathematics, Tata Institute of Fundamental Research, Homi Bhabha Road, Mumbai 400005, India
E-mail address: indranil@math.tifr.res.in

D. S. Nagaraj: The Institute of Mathematical Sciences, CIT Campus, Taramani, Chennai 600113, India
E-mail address: dsn@imsc.res.in

P. E. Newstead: Department of Mathematical Sciences, The University of Liverpool, Peach Street, Liverpool, L69 7ZL, England
E-mail address: newstead@liverpool.ac.uk