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J. Math. Anal. Appl. 321 (2006) 242–259

Journal of
MATHEMATICAL
ANALYSIS AND
APPLICATIONS

www.elsevier.com/locate/jmaa

Characteristic function for polynomially contractive commuting tuples

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Received 29 April 2005

Available online 12 September 2005

Submitted by E.J. Straube

Abstract

In this note, we develop the theory of characteristic function as an invariant for n-tuples of operators. The operator tuple has a certain contractivity condition put on it. This condition and the class of domains in $\mathbb{C}^n$ that we consider are intimately related. A typical example of such a domain is the open Euclidean unit ball. Given a polynomial P in $\mathbb{C}[z_1, z_2, \dots, z_n]$ whose constant term is zero, all the coefficients are nonnegative and the coefficients of the linear terms are nonzero, one can naturally associate a Reinhardt domain with it, which we call the P-ball (Definition 1.1). Using the reproducing kernel Hilbert space $H_P(\mathbb{C})$ associated with this Reinhardt domain in $\mathbb{C}^n$, S. Pott constructed the dilation for a polynomially contractive commuting tuple (Definition 1.2) [S. Pott, Standard models under polynomial positivity conditions, J. Operator Theory 41 (1999) 365–389. MR 2000j:47019]. Given any polynomially contractive commuting tuple T we define its characteristic function θ_T which is a multiplier. We construct a functional model using the characteristic function. Exploiting the model, we show that the characteristic function is a complete unitary invariant when the tuple is pure. The characteristic function gives newer and simpler proofs of a couple of known results: one of them is the invariance of the curvature invariant and the other is a Beurling theorem for the canonical operator tuple on $H_P(\mathbb{C})$. It is natural to study the boundary behaviour of θ_T in the case when the domain is the Euclidean unit ball. We do that and here essential differences with the single operator situation are brought out.

Keywords: Multivariable operator theory; Dilation theory; Characteristic function; Model theory

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1 Supported by DST grant no. SR/FTP/MS-16/2001.
2 Supported by a UGC research fellowship.

0022-247X/$ – see front matter
doi:10.1016/j.jmaa.2005.07.075

1. Introduction

Characteristic function for a single contraction on a Hilbert space has a long history and several significant results about it are well known (see Sz.-Nagy and Foias [30]).

For tuples of operators, the concept of dilation and model theory of contractive tuples (not necessarily commuting) has been systematically developed in several important papers. Early ideas of dilation of such tuples can be seen in Davis' paper [12]. Bunce [11] and Frazho [13] gave it a more concrete shape and finally Popescu (see [22–27]) has neat generalizations of many of the results of Sz.-Nagy and Foias including the characteristic function. In the noncommuting case, the characteristic function is a multi-analytic operator as defined by Popescu (see [17,25]). In the commuting case, Muller and Vasilescu [21] obtained the model theory for a large class of commuting tuples including the contractive ones. Tuples of operators satisfying the positivity condition of the type considered in this paper have also been considered before jointly by Arias and Popescu in [4] and then by Arias in [3].

In this note, all the n-tuples of bounded operators that we consider consist of commuting operators i.e., $T_iT_j = T_jT_i$ is satisfied for all $i, j = 1, 2, \ldots, n$. All the T_i act on a complex, separable Hilbert space $\mathcal{H}$. The algebra of all polynomials of n commuting variables $z_1, z_2, \ldots, z_n$ over $\mathbb{C}$ will be denoted by $\mathbb{C}[z_1, z_2, \ldots, z_n]$. The following definition is due to Pott [29].

Definition 1.1. A polynomial P in $\mathbb{C}[z_1, z_2, \ldots, z_n]$ is said to be a positive regular polynomial if the constant term of P is zero, all the coefficients are nonnegative and the coefficients of the linear terms are nonzero.

Let $\mathbb{N}^n$ denote the set of all multi-indices $k = (k_1, k_2, \ldots, k_n)$ and let $|k| = k_1 + k_2 + \cdots + k_n$. The multi-index which has 0 in all position except the ith. one, where it has 1, is denoted by e_i. Given $z = (z_1, z_2, \ldots, z_n)$ in $\mathbb{C}^n$ and a multi-index k, the monomial $z_1^{k_1} z_2^{k_2} \ldots z_n^{k_n}$ will be denoted by z^k. Let $P = \sum_{|k| \leqslant N} a_k z^k$ be a positive regular polynomial, where N is in $\mathbb{N}$. Denote by I_P the set of all k such that $a_k \neq 0$, which is ordered lexicographically and by A_P the set $\{a_k\colon k \in I_P\}$. Any such polynomial P and any commuting tuple $T = (T_1, T_2, \ldots, T_n)$, where T_i is in $\mathcal{B}(\mathcal{H})$ for each i, determines a completely positive map $C_{P,T}$ on $\mathcal{B}(\mathcal{H})$ by

$$C_{P,T}(X) = \sum_{k \in I_P} a_k T^k X T^{*k}, \quad \text{where } X \in \mathcal{B}(\mathcal{H}).$$

We shall be concerned with the following class of tuples.

Definition 1.2. If P is a positive regular polynomial, then a commuting tuple T of operators on $\mathcal{H}$ is called a P-contractive tuple if

$$C_{P,T}(I_{\mathcal{H}}) \leqslant I_{\mathcal{H}} \quad \text{or equivalently} \quad \sum_{k \in I_P} a_k T^k T^{*k} \leqslant I_{\mathcal{H}}, \quad \text{where } P = \sum_{k \in I_P} a_k z^k.$$

Associated with a positive regular polynomial P is the open set

$$\mathcal{P} = \left\{ z \in \mathbb{C}^n\colon \sum_{k \in I_P} a_k |z_1|^{2k_1} |z_2|^{2k_2} \ldots |z_n|^{2k_n} < 1 \right\}, \tag{1.1}$$

which we shall call the P-ball. It is easy to check that the P-ball is a Reinhardt domain. The function $z \to (1 - P(z))^{-1}$ in $\mathbb{C}^n$ has a power series expansion about 0 which converges uniformly on every compact subset of the P-ball. Let γ_k be the Taylor coefficient of z^k in this expansion and consider the Hilbert space $H_P(\mathbb{C})$ obtained by taking closure of the polynomials with respect to the inner product

$$\left\langle \sum_{|k|\leqslant M} c_k z^k, \sum_{|k|\leqslant M} b_k z^k \right\rangle = \sum_{|k|\leqslant M} \frac{c_k \bar{b}_k}{\gamma_k},$$

where M is a nonnegative integer. The elements of $H_P(\mathbb{C})$ are holomorphic functions on $\mathcal{P}$. If one defines

$$k_P(z,w) = \big(1 - P(z\bar{w})\big)^{-1}, \tag{1.2}$$

where $z\bar{w} = (z_1\bar{w}_1, z_2\bar{w}_2, \ldots, z_n\bar{w}_n)$ for $z = (z_1, z_2, \ldots, z_n)$, $w = (w_1, w_2, \ldots, w_n) \in \mathbb{C}^n$, then k_P is a positive definite kernel on the P-ball and $H_P(\mathbb{C})$ is the reproducing kernel Hilbert space corresponding to this kernel (see Pott [29, Lemma 3.2]). The set $\{\sqrt{\gamma_k} z^k\}_{k\in\mathbb{N}^n}$ forms an orthonormal basis for $H_P(\mathbb{C})$.

An important example is the polynomial $P(z) = z_1 + z_2 + \cdots + z_n$ which gives rise to the space H_n^2 studied by Arveson in [5–7]. We shall call this space the *Arveson space* and the polynomial the *Arveson polynomial*. Tuples which are P-contractive for this particular P are called *contractive*.

The multiplication operators M_{z_i} by co-ordinate functions are bounded operators on $H_P(\mathbb{C})$. In any Hilbert space with a reproducing kernel k defined on a domain $\Omega \subseteq \mathbb{C}^n$, the multiplication operators satisfy

$$\big(M_{z_i}^* k(\cdot, w)\big)z = \bar{w}_i k(z,w),$$

where $w \in \Omega$. Hence for $H_P(\mathbb{C})$,

$$\left(1 - \sum_{k\in I_P} a_k M_z^k M_z^{*k}\right) k_P(\cdot, w) = \left(1 - \sum_{k\in I_P} a_k z^k \bar{w}^k\right) k_P(\cdot, w) = 1 = E_0 k_P(\cdot, w), \tag{1.3}$$

by definition of the kernel, where E_0 is the projection onto the constant term. Since $k_P(\cdot, w)$ span the space, the operator $1 - \sum a_k M_z^k M_z^{*k}$ agrees with the projection E_0. Hence the multiplication operator tuple on $H_P(\mathbb{C})$ is a P-contractive tuple.

Let r be a positive integer or ∞ and $\mathcal{R}$ be a Hilbert space of dimension r. Then by $r.M_z$ we shall mean the operator tuple $(M_{z_1} \otimes I_{\mathcal{R}}, M_{z_2} \otimes I_{\mathcal{R}}, \ldots, M_{z_n} \otimes I_{\mathcal{R}})$ on the Hilbert space $H_P(\mathbb{C}) \otimes \mathcal{R}$. Also for another Hilbert space $\mathcal{N}$ and an n-tuple of operators $Z = (Z_1, Z_2, \ldots, Z_n)$ on $\mathcal{N}$, let $r.M_z \oplus Z$ denote the n-tuple of operators $((M_{z_1} \otimes I_{\mathcal{R}}) \oplus Z_1, (M_{z_2} \otimes I_{\mathcal{R}}) \oplus Z_2, \ldots, (M_{z_n} \otimes I_{\mathcal{R}}) \oplus Z_n)$ on the Hilbert space $(H_P(\mathbb{C}) \otimes \mathcal{R}) \oplus \mathcal{N}$. We are now in a position to state Pott's main result:

Theorem 1.3. *Let T be a commuting P-contractive tuple on a separable Hilbert space $\mathcal{H}$ such that rank of $\mathcal{D}_{T_P^*} := \overline{\mathrm{Range}}(I - \sum_{K\in I_P} a_k T^k T^{*k})^{1/2}$ is r. Then there is a Hilbert space $\mathcal{R}$ of dimension r and another Hilbert space $\mathcal{N}$ with a commuting tuple of normal operators $Z = (Z_1, Z_2, \ldots, Z_n)$ on it satisfying $C_{P,Z}(I) = I$ such that*

(1) *$\mathcal{H}$ is contained in $\widehat{\mathcal{H}} := (H_P(\mathbb{C}) \otimes \mathcal{D}_{T_P^*}) \oplus \mathcal{N}$ as a subspace and it is co-invariant under $A := r.M_z \oplus Z$.*
(2) *$T^k = P_{\mathcal{H}} A^k|_{\mathcal{H}}$, for all k in $\mathbb{N}^n$.*

Ambrozie, Englis and Muller in [1] worked with a setup more general than the above. They considered those open domains D in $\mathbb{C}^n$ which allow positive definite kernels C on D satisfying $C(z,w) \neq 0$ for all $z, w \in D$. Under the additional assumptions that $1/C$ is a polynomial, the corresponding reproducing kernel Hilbert space contains constant functions, polynomials are dense and the multiplication operators by the coordinate functions are bounded, they found that these are the suitable models for a *pure* (see Definition 3.5) commuting tuple of operators T which also satisfies $(1/C)(T, T^*) \geqslant 0$. Thus they had a generalization of Pott's model in the pure case. We shall stick to Pott's settings.

We start by studying the Hilbert space $H_P(\mathcal{E})$ and multipliers in Section 2. We shall construct the characteristic function for a commuting P-contractive tuple as an analytic operator valued function on the P-ball in Section 3. As in the case of $n = 1$ discussed in Sz.-Nagy and Foias [30], for each such T, we associate two defect spaces $\mathcal{D}_{T_P}$ and $\mathcal{D}_{T_P^*}$ respectively and the characteristic function is a $\mathcal{B}(\mathcal{D}_{T_P}, \mathcal{D}_{T_P^*})$ valued function defined on the $\mathcal{P}$ ball. We shall show in Section 4 that Pott's dilation space, in the case of a pure (see Definition 3.5) commuting P-contractive tuple can be written as a functional Hilbert space. From the explicit description of this model, we prove that the characteristic function of a pure P-contractive tuple of operators is a complete unitary invariant. As an application, we characterize the invariant subspaces for multiplication operators by the coordinate functions for certain functional Hilbert spaces which includes the Arveson space in Section 5. In Section 6, we study boundary properties when the domain is the Euclidean ball.

2. Properties of the space $H_P(\mathbb{C})$ and the multipliers

We begin with a minimality result on the space $H_P(\mathbb{C})$. Note that by definition of the inner product on $H_P(\mathbb{C})$, the monomials are orthogonal.

Theorem 2.1. *Let $M = (M_1, M_2, \dots, M_n)$ be a commuting P-contractive n-tuple of operators on a Hilbert space $\mathcal{H}$. Suppose there is a unit vector v in $\mathcal{H}$ such that for any nonzero $k \in \mathbb{N}^n$, $M^k v$ is orthogonal to v. Then there is a contraction $R : H_P(\mathbb{C}) \to \mathcal{H}$ such that $Rz^k = M^k v$ for all k in $\mathbb{N}^n$.*

Proof. Define R on the monomials by $Rz^k = M^k v$ for all k in $\mathbb{N}^n$ and extend linearly to polynomials. Since polynomials are dense in $H_P(\mathbb{C})$, to be able to extend R to $H_P(\mathbb{C})$, it is enough to show that R is a contraction on the space of polynomials. Take a natural number r and a polynomial $\sum_{|k| \leqslant r} b_k z^k$. We need to show that

$$\left\| \sum_{|k| \leqslant r} b_k M^k v \right\|^2 \leqslant \sum_{|k| \leqslant r} |b_k|^2 \left\| z^k \right\|^2.$$

This is equivalent to showing that

$$\left\| \sum_{|k| \leqslant r} \frac{b_k}{\|z^k\|} M^k v \right\|^2 \leqslant \sum_{|k| \leqslant r} |b_k|^2. \tag{2.1}$$

Let E_0 be the projection of $\mathcal{H}$ onto the one-dimensional subspace of $\mathcal{H}$ spanned by v. Then the above inequality is equivalent to showing that the tuple of operators $\{1/\|z^k\| M^k E_0\colon |k| \leqslant r\}$ is a contractive tuple. For this, recall that the completely positive map $C_{P,M}$ is defined on $B(\mathcal{H})$ by

$$C_{P,M}(X) = \sum_{k \in I_P} a_k M^k X M^{*k}.$$

Therefore $C_{P,M}$ is a contraction and so $C_{P,M}(E_0)$ is also a contraction. As $M^k v$ is orthogonal to v for all nonzero k in $\mathbb{N}^n$, so $\langle C_{P,M}(E_0)v, v\rangle = 0$. This gives $C_{P,M}(E_0) + E_0 \leqslant I_{\mathcal{H}}$. Since $C_{P,M}$ is a positive map,

$$C^2_{P,M}(E_0) + C_{P,M}(E_0) \leqslant I_{\mathcal{H}}.$$

Again as $\langle ((C^2_{P,M}(E_0) + C_{P,M})(E_0))v, v\rangle = 0$, so $C^2_{P,M}(E_0) + C_{P,M}(E_0) + E_0 \leqslant I_H$ and continuing this process, one gets

$$C^l_{P,M}(E_0) + C^{l-1}_{P,M}(E_0) + \cdots + C_{P,M}(E_0) + E_0 \leqslant I_{\mathcal{H}},$$

for any positive integer l. Define another completely positive map on $B(\mathcal{H})$ by

$$Q_M(X) = \sum_{|k| \leqslant r} \frac{1}{\|z^k\|^2} M^k E_0 X E_0 M^{*k}$$

and note that there is a positive integer l such that

$$Q_M(I) \leqslant C^l_{P,M}(E_0) + C^{l-1}_{P,M}(E_0) + \cdots + C_{P,M}(E_0) + E_0,$$

which in turn is a contractive completely positive map. Thus $\|Q_M\| = \|Q_M(I_{\mathcal{H}})\| \leqslant 1$. The theorem follows from the fact that $Q_M(I_{\mathcal{H}}) \leqslant I_{\mathcal{H}}$ implies

$$\sum_{|k| \leqslant r} \frac{1}{\|z^k\|^2} M^k E_0 M^{*k} \leqslant I_{\mathcal{H}}. \qquad \square$$

If $\mathcal{E}$ is a Hilbert space, we follow the notation of [16] and define $\mathcal{O}(\mathcal{P}, \mathcal{E})$ to be the class of all $\mathcal{E}$-valued holomorphic functions on $\mathcal{P}$. Then let $H_P(\mathcal{E})$ be the Hilbert space

$$H_P(\mathcal{E}) = \left\{ f \in \mathcal{O}(\mathcal{P}, \mathcal{E})\colon\ f = \sum_{k \in \mathbb{N}^n} a_k z^k \text{ and } \|f\|^2 = \sum_{k \in \mathbb{N}^n} \frac{\|a_k\|^2}{\gamma_k} < \infty \right\}. \tag{2.2}$$

It is well known that $H_P(\mathcal{E})$ is a reproducing kernel Hilbert space where the reproducing kernel on $\mathcal{P}$ is

$$K_{P,\mathcal{E}}(z, w) = \big(1 - P(z\bar{w})\big)^{-1} I_{\mathcal{E}}.$$

For two Hilbert spaces $\mathcal{E}$ and $\mathcal{E}_*$, the multiplier space $\mathcal{M}_P(\mathcal{E}, \mathcal{E}_*)$ consists of those $h \in \mathcal{O}(\mathcal{P}, \mathcal{B}(\mathcal{E}, \mathcal{E}_*))$ such that $h(H_P(\mathcal{E})) \subseteq H_P(\mathcal{E}_*)$. A simple application of closed graph theorem shows that each function $h \in \mathcal{M}_P(\mathcal{E}, \mathcal{E}_*)$ induces a continuous linear multiplication operator M_h from $H_P(\mathcal{E})$ to $H_P(\mathcal{E}_*)$ sending f to hf. With the operator norm, the space of multipliers becomes a Banach algebra.

Theorem 2.2. *Let h be a $\mathcal{B}(\mathcal{E}, \mathcal{E}_*)$ valued holomorphic function on $\mathcal{P}$, where $\mathcal{E}$ and $\mathcal{E}_*$ are two separable Hilbert space. Then the following conditions are equivalent*:

(1) *h is in $\mathcal{M}_P(\mathcal{E},\mathcal{E}_*)$ with $\|M_h\| \leqslant 1$.*
(2) *The kernel $k_h : \mathcal{P}\times\mathcal{P}\to\mathcal{B}(\mathcal{E}_*)$ defined by*

$$k_h(z,w) = \frac{I_{\mathcal{E}_*} - h(z)h(w)^*}{1-P(z\bar{w})}$$

is a positive $B(\mathcal{E}_*)$ valued kernel. Thus there exists an auxiliary Hilbert space $\mathcal{K}$ and a $\mathcal{B}(\mathcal{K},\mathcal{E}_*)$ valued function F such that

$$\frac{I_{\mathcal{E}_*} - h(z)h(w)^*}{1-P(z\bar{w})} = F(z)F(w)^*$$

for all z, w in $\mathcal{P}$.
(3) *There exists a Hilbert space $\mathcal{H}$ and a unitary operator*

$$U = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathcal{B}\big(\mathcal{H}\oplus\mathcal{E}, \mathcal{H}^{|I_P|}\oplus\mathcal{E}_*\big)$$

such that $h(z) = D + C(1 - Z_P A)^{-1} Z_P B$ for all z in $\mathcal{P}$.

Proof. This follows from a deep theorem of Ball and Bolotnikov [9, Theorem 1.5]. They proved this in more general settings. An earlier version of the theorem, where the function h above was scalar valued was proved by Ambrozie and Timotin [2]. □

We shall end this section with an example of a positive regular polynomial P such that the multiplication by the co-ordinate functions on $H_P(\mathbb{C})$ is essentially normal. It will have the following bearing on the theory of submodules of H^2_m.

For any integer $m \geqslant 1$, the Arveson space H^2_m is a Hilbert module over the polynomial ring. Moreover, it is a contractive Hilbert module (see, for example, [7] for definitions) because the multiplication operator tuple M_z is contractive. A submodule $\mathcal{M}$ of $\mathcal{H}$ is an invariant subspace of the M_{z_i} for $i = 1, 2, \ldots, n$. Let $R_i : \mathcal{M}\to\mathcal{M}$ be defined as $R_i = M_{z_i}|_{\mathcal{M}}$ for $i = 1, 2, \ldots, n$. It is an important question to decide when $R_1, R_2, \ldots, R_n$ are essentially normal. These submodules $\mathcal{M}$ are then called essentially reductive by Douglas and Paulsen [15]. Arveson showed that those submodules of H^2_n which are generated by a set of monomials are essentially reductive [8, Theorem 2.1]. He asked the general question of whether the same will be true for those $\mathcal{M}$ which are spanned by a set of all homogeneous polynomials [8, Conjecture A in Section 5]. Theorem 2.4 of Guo in [19] implies an affirmative answer to the question for $n = 2$. Any $H_P(\mathbb{C})$ with the operator tuple $\{\sqrt{a_k}M_z^k\colon k \in I_P\}$ is then a contractive Hilbert module where the operator tuple has length $m = |I_P|$. Note that this module is essentially reductive if and only if the operators $M_{z_1}, M_{z_2}, \ldots, M_{z_n}$ are essentially normal. The model theorem of Arveson [5, Theorem 8.5] tells us that $H_P(\mathbb{C})$ is isomorphic to a quotient module of H^2_m. Thus $M_{z_1}, M_{z_2}, \ldots, M_{z_n}$ are essentially normal if and only if this quotient module is essentially reductive. Douglas proved that a submodule is essentially reductive if and only if the corresponding quotient module is [14, Theorem 1]. Thus the example below gives an essentially reductive submodule of H^2_m. It is not clear to us whether the submodule of H^2_m that one obtains by this construction is generated by homogeneous polynomials. The $H_P(\mathbb{C})$ is a rich source of producing essentially reductive submodules of H^2_m, but to characterize all the positive regular polynomials for which the multiplication operators on $H_P(\mathbb{C})$ are essentially normal remains open.

For each $n \in \mathbb{N}$, the polynomial $P = (1 + \sum_{i=1}^n z_i)^2 - 1$ serves the purpose. For computational simplicity we give the proof for $n = 2$, i.e., the polynomial is $P = (1 + z_1 + z_2)^2 - 1$ in $\mathbb{C}[z_1, z_2]$.

To show that the multiplication operators $\{M_{z_1}, M_{z_2}\}$ on $H_P(\mathbb{C})$ are essentially normal for this P, we need to prove that $\lim_{|k|\to\infty}\big(\frac{\gamma_k}{\gamma_{k+e_i}} - \frac{\gamma_{k-e_i}}{\gamma_k}\big) = 0$, where $k = (k_1, k_2) \in \mathbb{N}^2$ and $i = 1, 2$.

Note that $(1-P)^{-1} = \frac{1}{2\sqrt{2}}\big[\frac{1}{a-(z_1+z_2)} + a\frac{1}{1+a(z_1+z_2)}\big]$, where $a = \sqrt{2}-1$. Denoting $\frac{\partial^{|k|}(1-P)^{-1}}{\partial z_1^{k_1}\partial z_2^{k_2}}$ by L_k, an easy induction gives the following identity:

$$L_k = \frac{|k|!}{2\sqrt{2}}\left[\frac{1}{(a-(z_1+z_2))^{|k|+1}} + (-1)^{|k|}a^{|k|+1}\frac{1}{(1+a(z_1+z_2))^{|k|+1}}\right].$$

So

$$\begin{aligned}\gamma_k = \frac{L_k(0,0)}{k!} &= \frac{1}{2\sqrt{2}}\frac{|k|!}{k!}\left[\frac{1}{a^{|k|+1}} + (-1)^{|k|}a^{|k|+1}\right]\\ &= \frac{1}{2\sqrt{2}}\frac{|k|!}{k!}\frac{(-1)^{|k|}}{a^{|k|+1}}\left[(-1)^{|k|} + a^{2(|k|+1)}\right].\end{aligned}$$

Due to the symmetry, we only need to show that $\lim_{|k|\to\infty}\big(\frac{\gamma_k}{\gamma_{k+e_1}} - \frac{\gamma_{k-e_1}}{\gamma_k}\big) = 0$. From the previous equality,

$$\frac{\gamma_k}{\gamma_{k+e_1}} = -a\frac{k_1+1}{|k|+1}\left[\frac{(-1)^{|k} + a^{2(|k|+1)}}{(-1)^{|k|+1} + a^{2(|k|+2)}}\right].$$

Thus

$$\frac{\gamma_k}{\gamma_{k+e_1}} - \frac{\gamma_{k-e_1}}{\gamma_k} = -a\left[\frac{k_1+1}{|k|+1}\frac{(-1)^{|k|} + a^{2(|k|+1)}}{(-1)^{|k|+1} + a^{2(|k|+2)}} - \frac{k_1}{|k|}\frac{(-1)^{|k|-1} + a^{2|k|}}{(-1)^{|k|} + a^{2(|k|+1)}}\right]. \tag{2.3}$$

Case 1. Let $|k|$ be even. Assume $d = |k|(|k|+1)(a^{2(|k|+1)}+1)(a^{2(|k|+2)}-1)$. Then the right side of the above equality becomes

$$\begin{aligned}&a\big\{k_1\big(|k|+1\big)\big(a^{2(|k|+2)}-1\big)\big(a^{2|k|}-1\big) - |k|(k_1+1)\big(a^{2(|k|+1)}+1\big)^2\big\}d^{-1}\\ &= a\big\{-k_2 + k_1\big(|k|+1\big)\big(a^{4(|k|+1)} - a^{2(|k|+2)} - a^{2|k|}\big)\\ &\quad - |k|(k_1+1)\big(a^{4(|k|+1)} + 2a^{2(|k|+1)}\big)\big\}d^{-1}\\ &= a\big\{-k_2\big(1+a^{4(|k|+1)}\big) - k_1\big(|k|+1\big)\big(a^{2(|k|+2)} + a^{2|k|}\big) - 2|k|\big(|k|+1\big)a^{2(|k|+1)}\big\}d^{-1}\\ &= -a\big\{k_2\big(1+a^{2(|k|+1)}\big)^2 + k_1\big(|k|+1\big)\big(a^{|k|+1} + a^{|k|}\big)^2\big\}d^{-1}\\ &= \frac{ak_2}{|k|(|k|+1)}\frac{1+a^{2(|k|+1)}}{1-a^{2(|k|+2)}} + \frac{ak_1}{|k|}\frac{a^{2|k|}(a^2+1)^2}{(a^{2(|k|+1)}+1)(1-a^{2(|k|+2)})}\\ &< \frac{1}{|k|+1}\frac{1+a^{2(|k|+1)}}{1-a^{2(|k|+2)}} + \frac{a^{2|k|}(a^2+1)^2}{(a^{2(|k|+1)}+1)(1-a^{2(|k|+2)})}.\end{aligned}$$

So taking limit as $|k| \to \infty$, one can say that the above limit exists and the limit is zero.

Case 2. Let $|k|$ be odd. If $l = |k|(|k|+1)(a^{2(|k|+1)}-1)(a^{2(|k|+2)}+1)$, then the right side of (2.3) becomes

$$\begin{aligned}&a\left\{\frac{k_1}{|k|}\frac{a^{2|k|}}{a^{2(|k|+1)}-1} - \frac{k_1+1}{|k|+1}\frac{a^{2(|k|+1)}-1}{a^{2(|k|+2)}+1}\right\}\\ &= a\big\{k_1\big(|k|+1\big)\big(a^{2(|k|+2)}+1\big)\big(a^{2|k|}+1\big) - |k|(k_1+1)\big(a^{2(|k|+1)}-1\big)^2\big\}l^{-1}\end{aligned}$$

$$
\begin{aligned}
&= a\bigl\{-k_2 + k_1\bigl(|k|+1\bigr)\bigl(a^{4(|k|+1)} + a^{2(|k|+2)} + a^{2|k|}\bigr) \\
&\quad - |k|(k_1+1)\bigl(a^{4(|k|+1)} - 2a^{2(|k|+1)}\bigr)\bigr\}l^{-1} \\
&= a\bigl\{-k_2\bigl(1 + a^{4(|k|+1)}\bigr) + k_1\bigl(|k|+1\bigr)\bigl(a^{2(|k|+2)} + a^{2|k|}\bigr) + 2|k|(k_1+1)a^{2(|k|+1)}\bigr\}l^{-1} \\
&= a\bigl\{-k_2\bigl(1 - a^{2(|k|+1)}\bigr)^2 + k_1\bigl(|k|+1\bigr)\bigl(a^{|k|+2} + a^{|k|}\bigr)^2\bigr\}l^{-1} \\
&= a\bigl\{-k_2\bigl(1 - a^{2(|k|+1)}\bigr)^2 + k_1\bigl(|k|+1\bigr)a^{2|k|}\bigl(1+a^2\bigr)^2\bigr\}l^{-1} \\
&= \frac{ak_2}{|k|(|k|+1)}\frac{1-a^{2(|k|+1)}}{1+a^{2(|k|+2)}} - \frac{ak_1}{|k|}\frac{a^{2|k|}(1+a^2)^2}{(1-a^{2(|k|+1)})(1+a^{2(|k|+2)})}.
\end{aligned}
$$

Note that either sequences arising from the last equality converges to zero. Hence for this case also the limit of (2.3) is zero as $|k| \to \infty$.

3. Characteristic function of a P-contraction

Given a P-contractive tuple T, we are interested in finding a complete unitary invariant for it. We begin the process of constructing this invariant, which is an operator valued analytic function.

Since T is a P-contractive tuple, the $m = |I_P|$-long operator tuple $\{\sqrt{a_k}T^k\colon k \in I_P\}$ is a contractive tuple on $\mathcal{H}$. It is convenient to denote both this operator tuple as well as the contraction it induces from $\mathcal{H}^m$ to $\mathcal{H}$ by T_P. Thus $I_{\mathcal{H}} \geqslant T_P T_P^*$ and $I_{\mathcal{H}^m} \geqslant T_P^* T_P$. Associate with T, the two defect operators $D_{T_P^*} = (I_{\mathcal{H}} - \sum_{k\in I_P} a_k T^k T^{*k})^{1/2}$ and $D_{T_P} = (I_{\mathcal{H}^m} - T_P^* T_P)^{1/2}$. The later is also conveniently represented by the square root of the $m \times m$ operator matrix $(\delta_{kl} I_{\mathcal{H}} - \sqrt{a_k a_l}T^{*k}T^l)$, where k and l belong to I_P. Thus D_{T_P} is a bounded operator on $\mathcal{H}^m$. The corresponding defect spaces are $\mathcal{D}_{T_P^*} = \overline{\mathrm{Range}} D_{T_P^*} \subseteq \mathcal{H}$ and $\mathcal{D}_{T_P} = \overline{\mathrm{Range}} D_{T_P} \subseteq \mathcal{H}^m$. The following lemma follows from (I.3.4) of Sz.-Nagy and Foias [30].

Lemma 3.1. *For any commuting P-contractive tuple of operator T,*

$$T_P D_{T_P} = D_{T_P^*} T_P.$$

Given z in $\mathcal{P}$, denote by z_P the m-tuple of complex numbers $\{\sqrt{a_k}z^k\colon k \in I_P\}$. Since z is in $\mathcal{P}$, the tuple z_P is in $\mathbb{B}_m$. The contractive operator tuple corresponding to z_P is denoted by Z_P. By virtue of the lemma above, we can thus define the characteristic function as follows.

Definition 3.2. Let T be a commuting P-contractive tuple of operators on some Hilbert space $\mathcal{H}$. Then the characteristic function of T is a bounded operator valued analytic function $\theta_T : \mathcal{P} \to \mathcal{B}(\mathcal{D}_{T_P}, \mathcal{D}_{T_P^*})$ defined by

$$\theta_T(z) = -T_P + D_{T_P^*}\bigl(I_H - Z_P T_P^*\bigr)^{-1} Z_P D_{T_P}, \quad z \in \mathcal{P}.$$

A remark is in order. The tuple T_P, being a commuting contracting tuple, has its own characteristic function on $\mathbb{B}_m$ as described in [10], viz.,

$$\theta_{T_P}(z) = -T_P + D_{T_P^*}\bigl(I_H - Z T_P^*\bigr)^{-1} Z D_{T_P}, \quad z \in \mathbb{B}_m.$$

We note that Popescu had studied the characteristic function for a (not necessarily commuting) contractive tuple in [24]. His characteristic function is a multi analytic operator. In a recent preprint [28], Popescu has proved that the characteristic function for a commuting contractive tuple can also be obtained by compressing Popescu's characteristic function to the symmetric Fock space. If $f_P : \mathcal{P} \to \mathbb{B}_m$ is the natural map sending $z \in \mathcal{P}$ to z_P, then $\theta_T(z) = \theta_{T_P}(f_P(z))$ for all $z \in \mathcal{P}$. Hence, θ_T is the restriction of θ_{T_P} to a lower-dimensional manifold in general, viz., $f_P(\mathcal{P})$.

The first result about θ_T shows that it is a multiplier.

Lemma 3.3. *Let T be a commuting P-contractive tuple of operators on some Hilbert space $\mathcal{H}$. Then its characteristic function θ_T is a multiplier with $\|M_{\theta_T}\| \leqslant 1$. Also for $z, w \in \mathcal{P}$, we have the following identity*:

$$I - \theta_T(w)\theta_T(z)^* = \left(I - W_P Z_P^*\right) D_{T_P^*} \left(I - W_P T_P^*\right)^{-1} \left(I - T_P Z_P^*\right)^{-1} D_{T_P^*}.$$

Proof. The fact that the characteristic function θ_T is a multiplier with $\|M_{\theta_T}\| \leqslant 1$ follows from Theorem 2.2 by considering the operator

$$U = \begin{pmatrix} T_P^* & D_{T_P} \\ D_{T_P^*} & -T_P \end{pmatrix} \in \mathcal{B}\left(\mathcal{H} \oplus \mathcal{D}_{T_P}, \mathcal{H}^{|I_P|} \oplus \mathcal{D}_{T_P^*}\right),$$

which is a unitary. Also using the condition (2) in the same theorem and some straightforward computation, one can show the identity of this lemma. □

Corollary 3.4. *Given a commuting P-contractive tuple of operators T, its characteristic function is a contraction.*

Proof. Put $w = z$ in the identity of the last lemma to get

$$I - \theta_T(z)\theta_T(z)^* = \left(1 - P\left(|z|^2\right)\right) D_{T_P^*} \left(I - Z_P T_P^*\right)^{-1} \left(I - T_P Z_P^*\right)^{-1} D_{T_P^*},$$

and note that the right-hand side is a positive operator. □

Also note that the characteristic function is *purely contractive*, this means that for all nonzero h in $\mathcal{D}_{T_P}$, $\|\theta_T(0)h\| < 1$.

By polynomial contractivity of the tuple T, we have

$$I_{\mathcal{H}} \geqslant C_{P,T}(I_{\mathcal{H}}) \geqslant \cdots \geqslant C_{P,T}^l(I_{\mathcal{H}}) \geqslant \cdots.$$

This is a decreasing sequence of positive operators, so converges strongly. Define $A_\infty \in \mathcal{B}(\mathcal{B}(\mathcal{H}))$ by

$$A_\infty := \text{s-}\lim_{l \to \infty} C_{P,T}^l(I_{\mathcal{H}}).$$

Clearly $0 \leqslant A_\infty \leqslant I_{\mathcal{H}}$.

Definition 3.5. The commuting P-contractive tuple T is called pure if the limit $A_\infty = 0$.

In the case of a single contraction T, the definition of pure is equivalent to T being in the $C_{\cdot 0}$ class considered by Sz.-Nagy and Foias. Further analysis of the characteristic function would require the following theorem which is a special case of Theorem 3.8 in S. Pott [29].

Theorem 3.6 *(Pott). Let P be a positive regular polynomial and let T be a commuting P-contractive n-tuple of operators on $\mathcal{H}$. Then there exists a unique bounded linear operator $L : H_P(\mathbb{C}) \otimes \mathcal{D}_{T_P^*} \to \mathcal{H}$ such that for all k in $\mathbb{N}^n$ and h in $\mathcal{D}_{T_P^*}$,*

$$L\big(z^k \otimes h\big) = T^k D_{T_P^*} h.$$

Also $L(f(M_z) \otimes I_{\mathcal{D}_{T_P^*}}) = f(T)L$ for all f in $\mathbb{C}[z_1, z_2, \ldots, z_n]$.

The following two lemmas relate the map L to the characteristic function.

Lemma 3.7. *The operator L obtained in Theorem* 3.6 *satisfies*

$$L\big(k_P(\cdot, z) \otimes \eta\big) = \big(I - T_P Z_P^*\big)^{-1} D_{T_P^*} \eta$$

for all $z \in \mathcal{P}$ and $\eta \in \mathcal{D}_{T_P^*}$.

Proof. For each $z \in \mathcal{P}$ and $k \in \mathbb{N}^n$ consider the monomial $g_{k,z}$ by $g_{k,z} = \gamma_k \bar{z}^k w^k$. Then for each $m \in \mathbb{N}$ define $f_{m,z} \in \mathbb{C}[z_1, z_2, \ldots, z_n]$ by

$$f_{m,z} = \sum_{|k| \leqslant m} g_{k,z}.$$

As $\{w^k \gamma_k^{-1}\}_{k \in \mathbb{N}^n}$ is an orthonormal basis for $H_P(\mathbb{C})$, we have

$$\big\|k_P(\cdot, z) - f_{m,z}\big\|^2 = \bigg\| \sum_{|k| > m} g_{k,z} \bigg\|^2 = \sum_{|k| > m} \gamma_k \big|z^k\big|^2 \to 0 \quad \text{as } m \to \infty$$

because $\|k_P(\cdot, z)\|^2 = \sum_{j=0}^{\infty} P(|z|^2)^j = \sum_{k \in \mathbb{N}^n} \gamma_k |z^k|^2$. Thus

$$\begin{aligned} L\big(k_P(\cdot, z) \otimes \eta\big) &= \lim_{m \to \infty} L(f_{m,z} \otimes \eta) = \lim_{m \to \infty} f_{m,z}(T) D_{T_P^*} \eta \\ &= \lim_{m \to \infty} \bigg(\sum_{|k| \leqslant m} \gamma_k T^k \bar{z}^k \bigg) D_{T_P^*} \eta. \end{aligned}$$

The last quantity is $(I - T_P Z_P^*)^{-1} D_{T_P^*} \eta$. $\square$

We have been informed by the referee of an earlier version of this note that the next lemma follows closely one of Popescu's results from [27].

Lemma 3.8.

$$L^* L + M_{\theta_T} M_{\theta_T}^* = I_{H_P(\mathbb{C}) \otimes \mathcal{D}_{T_P^*}}.$$

Proof. Observe that the set $\{k_P(\cdot, z) \otimes \eta \colon\ z \in \mathcal{P},\ \eta \in \mathcal{D}_{T_P^*}\}$ form a total set of $H_P(\mathbb{C})$. Take $z, w \in \mathcal{P}$ and $\xi, \eta \in \mathcal{D}_{T_P^*}$. Then, using Lemma 3.7,

$$\begin{aligned} &\big\langle \big(L^* L + M_{\theta_T} M_{\theta_T}^*\big) k_P(\cdot, z) \otimes \xi, k_P(\cdot, w) \otimes \eta \big\rangle \\ &\quad = \big\langle L\big(k_P(\cdot, z) \otimes \xi\big), L\big(k_P(\cdot, w) \otimes \eta\big)\big\rangle + \big\langle M_{\theta_T}^*\big(k_P(\cdot, z) \otimes \xi\big), M_{\theta_T}^*\big(k_P(\cdot, w) \otimes \eta\big)\big\rangle \\ &\quad = \big\langle \big(I - T_P Z_P^*\big)^{-1} D_{T_P^*} \xi, \big(I - T_P W_P^*\big)^{-1} D_{T_P^*} \eta \big\rangle \end{aligned}$$

$$
\begin{aligned}
&+\langle k_P(\cdot,z)\otimes\theta_T(z)^*\xi, k_P(\cdot,w)\otimes\theta_T(w)^*\eta\rangle\\
&=\langle D_{T_P^*}(I-W_PT_P^*)^{-1}(I-T_PZ_P^*)^{-1}D_{T_P^*}\xi,\eta\rangle+\langle k_P(\cdot,z),k_P(\cdot,w)\rangle\langle\theta_T(w)\theta_T(z)^*\xi,\eta\rangle\\
&=\langle D_{T_P^*}(I-W_PT_P^*)^{-1}(I-T_PZ_P^*)^{-1}D_{T_P^*}\xi,\eta\rangle+k_P(w,z)\langle\theta_T(w)\theta_T(z)^*\xi,\eta\rangle\\
&=k_P(w,z)\langle(I-\theta_T(w)\theta_T(z)^*)\xi,\eta\rangle+k_P(w,z)\langle\theta_T(w)\theta_T(z)^*\xi,\eta\rangle\\
&=k_P(w,z)\langle\xi,\eta\rangle\\
&=\langle k_P(\cdot,z)\otimes\xi, k_P(\cdot,w)\otimes\eta\rangle.
\end{aligned}
$$

Fifth equality follows from the fact that

$$k_P(w,z)(I-\theta_T(w)\theta_T(z)^*)=D_{T_P^*}(I-W_PT_P^*)^{-1}(I-T_PZ_P^*)^{-1}D_{T_P^*},$$

which is obtained from Lemma 3.3. Thus the result follows. □

If the tuple is pure, the operator L is a co-isometry. Hence L^*L is the projection onto the range of L^*. Thus $M_{\theta_T}M_{\theta_T}^*$ is the orthogonal projection. So Range L^* and Range M_{θ_T} are ortho-complements of each other. It is clear from the definition of L (Theorem 3.6) that Range L^* is an invariant subspace for $M_{z_i}^*$ for $i=1,2,\dots,n$. Thus Range M_{θ_T} is an invariant subspace for M_{z_i} for $i=1,2,\dots,n$. It is a remarkable fact (see Section 5) that the converse is true, i.e., given any invariant subspace $\mathcal{M}$ of $H_P(\mathbb{C})\otimes\mathcal{E}$, where $\mathcal{E}$ is a Hilbert space, one can produce a pure P-contractive tuple T such that $\mathcal{M}=$ Range M_{θ_T}. Moreover, M_{θ_T} is a partial isometry.

4. Functional model and a complete unitary invariant

Given a Hilbert space $\mathcal{E}$, we denote by $M_z^{\mathcal{E}}=(M_{z_1}^{\mathcal{E}},M_{z_2}^{\mathcal{E}},\dots,M_{z_n}^{\mathcal{E}})$, the tuple of multiplication operators on $H_P(\mathbb{C})$ induced by the coordinate functions z_i. There is a canonical unitary operator $U_{\mathcal{E}}:H_P(\mathbb{C})\otimes\mathcal{E}\to H_P(\mathcal{E})$ with $U_{\mathcal{E}}(f\otimes x)=fx$ for $f\in H_P(\mathbb{C})$ and $x\in\mathcal{E}$. This unitary $U_{\mathcal{E}}$ intertwines $M_{z_i}\otimes I_{\mathcal{E}}$ with $M_{z_i}^{\mathcal{E}}$ and hence is a module isomorphism. In the following we shall identify the spaces $H_P(\mathbb{C})\otimes\mathcal{E}$ and $H_P(\mathcal{E})$ via this unitary operator $U_{\mathcal{E}}$. In this way each multiplier $\varphi\in M_P(\mathcal{E},\mathcal{E}_*)$ induces a bounded operator $M_\varphi:H_P(\mathbb{C})\otimes\mathcal{E}\to H_P(\mathbb{C})\otimes\mathcal{E}_*$. It is easy to check that for a given multiplier $\varphi\in\mathcal{M}_P(\mathcal{E},\mathcal{E}_*)$, the following identity holds:

$$M_\varphi^*(k_P(\cdot,z)\otimes e_*)=k_P(\cdot,z)\otimes\varphi(z)^*e_*$$

for all $e_*\in\mathcal{E}_*$ and $z\in\mathcal{P}$ where k_P is as defined in (1.2).

Definition 4.1. Two commuting tuples $T=(T_1,T_2,\dots,T_n)$ and $R=(R_1,R_2,\dots,R_n)$ of bounded operators on Hilbert spaces $\mathcal{H}$ and $\mathcal{K}$ are said to be unitarily equivalent if there exists a unitary operator U from $\mathcal{H}$ to $\mathcal{K}$ such that $R_i=UT_iU^*$ holds for all $i=1,2,\dots,n$. This is the same as the two Hilbert modules $\mathcal{H}$ and $\mathcal{K}$ being module isomorphic.

The purpose of this section is to give the functional model of a given P-contractive tuple of operators. The multiplication operator tuple $M_z=(M_{z_1},M_{z_2},\dots,M_{z_n})$, where M_{z_i} is the multiplication by the co-ordinate function z_i, is known as the *shift* on the functional Hilbert space $H_P(\mathbb{C})$. We have noticed earlier that the shift tuple is a P-contractive, pure, commuting operator tuple.

Sz.-Nagy and Foias showed that for a given single $C_{.0}$ contraction T on some Hilbert space H, there is a unitary operator U from H onto $\mathbb{H}_T=H(\mathcal{D}_{T^*})\ominus M_{\theta_T}H(\mathcal{D}_T)$ such that $UTU^*=$

$P_{\mathbb{H}_T} M_z|_{\mathbb{H}_T}$. Our first theorem of this section is the generalization of the result of Sz.-Nagy and Foias to a P-contractive tuple of operators T.

Theorem 4.2. *Let T be a P-contractive, pure commuting n-tuple of operators on a Hilbert space $\mathcal{H}$. Then T is unitarily equivalent to the commuting tuple $\mathbb{T} = (\mathbb{T}_1, \mathbb{T}_2, \dots, \mathbb{T}_n)$ on the functional space $\mathbb{H}_T = (H_P(\mathbb{C}) \otimes \mathcal{D}_{T_P^*}) \ominus M_{\theta_T}(H_P(\mathbb{C}) \otimes \mathcal{D}_{T_P})$ defined by $\mathbb{T}_i = P_{\mathbb{H}_T}(M_{z_i} \otimes I_{\mathcal{D}_{T_P^*}})|_{\mathbb{H}_T}$ for $1 \leqslant i \leqslant n$.*

Proof. Since T is pure, the map L of Theorem 3.6 is a coisometry. Thus $\mathcal{H}$ is identified with its isometric image $L^*\mathcal{H}$ in $H_P(\mathbb{C}) \otimes \mathcal{D}_{T_P^*}$. The projection onto the closed subspace $L^*\mathcal{H}$ is L^*L. By Lemma 3.8, the subspace $L^*\mathcal{H}$ is thus the orthogonal complement of the range of the projection $M_{\theta_T} M_{\theta_T}^*$:

$$L^*\mathcal{H} = \left(H_P(\mathbb{C}) \otimes \mathcal{D}_{T_P^*}\right) \ominus M_{\theta_T}\left(H_P(\mathbb{C}) \otimes \mathcal{D}_{T_P}\right).$$

Theorem 3.6 also provides an intertwining property of L, i.e., $T_i L = L(M_{z_i} \otimes I_{\mathcal{D}_{T_P^*}})$ and this finishes the proof. □

Definition 4.3. Given two commuting P-contractive tuples T and R on Hilbert spaces $\mathcal{H}$ and $\mathcal{K}$, the characteristic functions of T and R are said to coincide if there exist unitary operators $\tau : \mathcal{D}_{T_P} \to \mathcal{D}_{R_P}$ and $\tau_* : \mathcal{D}_{T_P^*} \to \mathcal{D}_{R_P^*}$ such that the following diagram commutes for all z in $\mathcal{P}$:

$$\begin{tikzcd}
\mathcal{D}_{T_P} \arrow[r, "\theta_T(z)"] \arrow[d, "\tau"'] & \mathcal{D}_{T_P^*} \arrow[d, "\tau_*"] \\
\mathcal{D}_{R_P} \arrow[r, "\theta_R(z)"'] & \mathcal{D}_{R_P^*}.
\end{tikzcd}$$

The following proposition is immediate.

Proposition 4.4. *The characteristic functions of two unitary equivalent commuting P-contractive tuples coincide.*

Here is an interesting application of the proposition above. First specialize to the Arveson polynomial and then recall from [6,7] that Arveson defined the curvature invariant of a contractive finite rank Hilbert module $\mathcal{H}$ to be

$$K(\mathcal{H}) = \int_{\partial\mathbb{B}_n} \lim_{r \uparrow 1} \left(1 - r^2\right) \operatorname{trace} F(r\zeta)\, d\sigma(\zeta),$$

where

$$F(z)\xi = D_{T^*}(I - ZT^*)^{-1}(I - TZ^*)^{-1} D_{T^*}\xi, \quad z \in \mathbb{B}_n,\ \xi \in \mathcal{D}_{T^*}.$$

Corollary 4.5. *Let $\mathcal{H}$ and $\mathcal{K}$ be two isomorphic finite rank contractive Hilbert modules. Then their curvature invariants are the same.*

Proof. Note first that

$$I - \theta_T(z)\theta_T(z)^* = \left(1 - |z|^2\right) D_{T^*}(I - ZT^*)^{-1}(I - TZ^*)^{-1} D_{T^*}.$$

Let T and R be the associated operator tuples with $\mathcal{H}$ and $\mathcal{K}$, respectively. Thus there exist two unitary operators $\tau : \mathcal{D}_T \to \mathcal{D}_R$ and $\tau_* : \mathcal{D}_{T^*} \to \mathcal{D}_{R^*}$ such that the following diagram commutes for all z in $\mathbb{B}_n$:

$$\begin{array}{ccc} \mathcal{D}_T & \xrightarrow{\theta_T(z)} & \mathcal{D}_{T^*} \\ \downarrow{\scriptstyle \tau} & & \downarrow{\scriptstyle \tau_*} \\ \mathcal{D}_R & \xrightarrow[\theta_R(z)]{} & \mathcal{D}_{R^*}. \end{array}$$

Also it is clear that the dimension of the defect spaces $\mathcal{D}_{T^*}$ and $\mathcal{D}_{R^*}$ are the same. Since $\theta_T(z) = \tau_*^* \theta_R(z)\tau$, for all $z \in \mathbb{B}_n$, a calculation gives the following identity:

$$I_{\mathcal{D}_{T^*}} - \theta_T(z)\theta_T(z)^* = \tau_*^*\big(I_{\mathcal{D}_{R^*}} - \theta_T(z)\theta_R(z)^*\big)\tau_*.$$

Using the above equality with the fact that trace is preserved under unitary conjugation, one gets

$$\begin{aligned} K(\mathcal{H}) &= \lim_{r\uparrow 1} \int_{\partial\mathbb{B}_n} \operatorname{trace}\big(1_{\mathcal{D}_{T^*}} - \theta_T(rz)\theta_T(rz)^*\big)\, d\sigma(z) \\ &= \lim_{r\uparrow 1} \int_{\partial\mathbb{B}_n} \operatorname{trace}\big(\tau_*^*\big(I_{\mathcal{D}_R^*} - \theta_R(rz)\theta_R(rz)^*\big)\tau_*\big)\, d\sigma(z) \\ &= \lim_{r\uparrow 1} \int_{\partial\mathbb{B}_n} \operatorname{trace}\big(I_{\mathcal{D}_R^*} - \theta_R(rz)\theta_R(rz)^*\big)\, d\sigma(z) = K(\mathcal{K}). \qquad \square \end{aligned}$$

Now we prove the converse of Proposition 4.4 for pure tuple of operators.

Proposition 4.6. *Let T and R be two P-contractive, pure commuting tuples of operators acting on $\mathcal{H}$ and $\mathcal{K}$, respectively. If their characteristic functions θ_T and θ_R coincide, then the tuples T and R are unitarily equivalent.*

Proof. Let $\tau' : \mathcal{D}_{T_P} \to \mathcal{D}_{R_P}$ and $\tau'_* : \mathcal{D}_{T_P^*} \to \mathcal{D}_{R_P^*}$ be two unitary operators such that the following diagram:

$$\begin{array}{ccc} \mathcal{D}_{T_P} & \xrightarrow{\theta_T(z)} & \mathcal{D}_{T_P^*} \\ \downarrow{\scriptstyle \tau'} & & \downarrow{\scriptstyle \tau'_*} \\ \mathcal{D}_{R_P} & \xrightarrow[\theta_R(z)]{} & \mathcal{D}_{R_P^*} \end{array}$$

commutes for all z in $\mathcal{P}$. The operators τ' and τ'_* give rise to unitary operators $\tau = 1 \otimes \tau' : H_P(\mathbb{C}) \otimes \mathcal{D}_{T_P} \to H_P(\mathbb{C}) \otimes \mathcal{D}_{R_P}$ and $\tau_* = 1 \otimes \tau'_* : H_P(\mathbb{C}) \otimes \mathcal{D}_{T_P^*} \to H_P(\mathbb{C}) \otimes \mathcal{D}_{R_P^*}$ which satisfy the intertwining relation

$$M_{\theta_R}\tau = \tau_* M_{\theta_T}.$$

A little computation using this relation shows that $\tau_*|_{\mathbb{H}_T} : \mathbb{H}_T \to \mathbb{H}_R$ is an unitary operator, where $\mathbb{H}_T$ and $\mathbb{H}_R$ are the model spaces for T and R as in Theorem 4.2. Moreover, this unitary intertwines $(M_z^* \otimes I_{\mathcal{D}_{T^*}})|_{\mathbb{H}_T}$ with $(M_z^* \otimes I_{\mathcal{D}_{R^*}})|_{\mathbb{H}_R}$ componentwise. Thus $\tau_*|_{\mathbb{H}_T}$ intertwines the model tuples $P_{\mathbb{H}_T}(M_z \otimes I_{\mathcal{D}_{T^*}})|\mathbb{H}_T$ and $P_{\mathbb{H}_R}(M_z \otimes I_{\mathcal{D}_{R^*}})|\mathbb{H}_R$. But then Theorem 4.2 shows that T and R are unitarily equivalent. $\square$

Thus we can state that:

Theorem 4.7. *Two P-contractive, pure commuting tuple of operators T and R on the Hilbert spaces $\mathcal{H}$ and $\mathcal{K}$ respectively are unitarily equivalent if and only if their characteristic functions coincide.*

For two pure P-contractive n-tuple of operators T and R on $\mathcal{H}$, the tuples T_P and R_P are pure commuting contractive m-tuples of operators. Of course, T and R are unitarily equivalent if and only if T_P and R_P are so. It was shown in [10] that the characteristic function is a complete unitary invariant for a pure commuting contractive tuple. Popescu in a recent preprint [28] obtained a generalization of this fact using the characteristic function of a contractive tuple and then restricting it to a suitable subspace of the Fock space. Thus unitary equivalence of T_P and R_P happens if and only if θ_{T_P} and θ_{R_P} coincide. But these are bounded operator valued analytic functions on $\mathbb{B}_m$. The analysis above shows that to conclude unitary equivalence of T and R it is enough to show the coincidence of their characteristic functions as defined in Definition 3.2 which are bounded operator valued analytic functions on a lower dimensional manifold.

5. Invariant subspaces of M_z

In this section we give an elementary new proof of a known result, viz., the Beurling theorem for the tuple $(M_{z_1}, M_{z_2}, \dots, M_{z_n})$ on the functional Hilbert space $H_P(\mathbb{C})$ using the characteristic function. McCullough and Trent proved existentially that Beurling–Lax–Halmos (BLH) Theorem holds for complete Nevanlinna–Pick kernels, see [20, Theorem 0.7] for details. However, we construct the Beurling function as the characteristic function for the restriction of M_z to the orthocomplement of the invariant subspace.

Definition 5.1. A joint reducing subspace for a commuting tuple of operators T on a Hilbert space $\mathcal{H}$ is a closed subspace $\mathcal{M}$ of $\mathcal{H}$ such that $T_i\mathcal{M} \subseteq \mathcal{M}$ and $T_i^*\mathcal{M} \subseteq \mathcal{M}$ for all $i = 1, 2, \dots, n$.

It is not hard to see that there are many reducing subspaces for each M_{z_i} on $H_P(\mathbb{C})$. The next theorem gives an elementary proof of the fact that in $H_P(\mathbb{C})$, the tuple M_z has no joint reducing subspace.

Theorem 5.2. *A joint reducing subspace $\mathcal{M}$ for the tuple M_z is either $\{0\}$ or the full space $H_P(\mathbb{C})$.*

Proof. Let $\mathcal{M}$ be a nonzero reducing subspace of $H_P(\mathbb{C})$. Take $f \in \mathcal{M}$ such that $f \neq 0$. Let $f = \sum a_k z^k$ and without loss of generality we can assume that the constant term of f is nonzero. Otherwise if a_l is the first nonzero coefficient, then apply M_z^{*l} on f to get an element of $\mathcal{M}$ whose constant term is nonzero. Then using the identity in 1.3, we see that $P_{E_0} f \in \mathcal{M}$. Thus the constant term of f is in $\mathcal{M}$, so all polynomial are in $\mathcal{M}$ and as polynomials are dense in $H_P(\mathbb{C})$, $\mathcal{M} = H_P(\mathbb{C})$. So there is no nontrivial joint reducing subspace of M_z. □

Definition 5.3. An operator valued analytic function $\varphi : \mathcal{P} \to B(\mathcal{E}, \mathcal{E}_*)$, where $\mathcal{E}$ and $\mathcal{E}_*$ are Hilbert spaces is called *inner* if $\varphi \in \mathcal{M}_P(\mathcal{E}, \mathcal{E}_*)$ and if M_φ is a partial isometry.

Now we deduce the Beurling theorem for the functional Hilbert space $H_P(\mathbb{C})$.

Theorem 5.4. *Let $\mathcal{M}$ be an invariant subspace for M_z on $H_P(\mathbb{C})$. Then there exists a Hilbert space $\mathcal{E}$ and an inner function $\Phi : \mathcal{P} \to B(\mathcal{E}, \mathbb{C})$ such that*

$$\mathcal{M} = \Phi H_P(\mathcal{E}).$$

Proof. Let T be the n-tuple of operators on $\mathcal{M}^\perp$ defined by

$$T = (P_{\mathcal{M}^\perp} M_{z_1}|_{\mathcal{M}^\perp}, P_{\mathcal{M}^\perp} M_{z_2}|_{\mathcal{M}^\perp}, \dots, P_{\mathcal{M}^\perp} M_{z_n}|_{\mathcal{M}^\perp}).$$

As M_z is a pure commuting tuple, so T is also a pure and commuting operator tuple on $\mathcal{M}^\perp$.

Let us prove that for the tuple T, the defect space $\mathcal{D}_{T_P^*}$ is of rank one. By the dilation theorem (Theorem 1.3), $(M_{z_1} \otimes I_{\mathcal{D}_{T_P^*}}, M_{z_2} \otimes I_{\mathcal{D}_{T_P^*}}, \dots, M_{z_n} \otimes I_{\mathcal{D}_{T_P^*}})$ is the minimal dilation of T on the minimal dilation space $H_P(\mathbb{C}) \otimes \mathcal{D}_{T_P^*}$. Also by the definition of the tuple T, $(M_{z_1}, M_{z_2}, \dots, M_{z_n})$ is a dilation of T on the dilation space $H_P(\mathbb{C})$. Since the minimal dilation space of a tuple of operators is unique, it is enough to show that the later dilation is also minimal, i.e., to show that

$$\overline{sp}\{M_z^k h \colon k \in \mathbb{N}^n,\ h \in \mathcal{M}^\perp\} = H_P(\mathbb{C}).$$

But the subspace in the left-hand side of the equality is a proper joint reducing subspace which is a contradiction by Theorem 5.2. Thus $\mathcal{D}_{T_P^*}$ is of rank one.

Now let us consider the characteristic function for the tuple T. As the tuple is pure, so Theorem 4.2 says that

$$\mathcal{M}^\perp = \big(H_P(\mathbb{C}) \otimes \mathcal{D}_{T_P^*}\big) \ominus M_{\theta_T}\big(H_P(\mathbb{C}) \otimes \mathcal{D}_{T_P}\big).$$

By the identification of $\mathcal{D}_{T_P^*}$ with $\mathbb{C}$, we have $\mathcal{M} = M_{\theta_T}(H_P(\mathbb{C}) \otimes \mathcal{D}_{T_P})$. □

6. Boundary behaviour

We conclude the paper with the boundary behaviour of the characteristic function in the case of the Euclidean unit ball. Also, a result of Arveson follows easily from this boundary behaviour. First note that the radial limit of θ_T exists for σ-a.e. $\lambda \in \partial\mathbb{B}_n$ where σ is the unique rotation invariant measure on the boundary because θ_T is a bounded analytic function on $\mathbb{B}_n$. We shall call the limit $\theta_T(\lambda)$.

Theorem 6.1. *Let T be a pure commuting contractive tuple of operators on the Hilbert space $\mathcal{H}$ with $\mathcal{D}_{T^*}$ being finite-dimensional. Then $\theta_T(\lambda)$ is a partial isometry σ-a.e.*

Proof. To show that $\theta_T(\lambda)$ is a partial isometry σ-a.e. for $\lambda \in \partial\mathbb{B}_n$, we need to show that $\theta_T(\lambda)^*$ is isometric σ-a.e. on Range $\theta_T(\lambda)$. Now for $z \in \mathbb{B}_n$ and $h \in \mathcal{D}_T$,

$$\begin{aligned}
&\big\|\theta_T(z)^*\theta_T(z)h\big\|^2 \\
&\quad = \big\langle \theta_T(z)\theta_T(z)^*\theta_T(z)h, \theta_T(z)h\big\rangle \\
&\quad = \big\langle \big(I - \big(1-|z|^2\big)D_{T^*}(I - ZT^*)^{-1}(I - TZ^*)^{-1}D_{T^*}\big)\theta_T(z)h, \theta_T(z)h\big\rangle \\
&\quad = \big\|\theta_T(z)h\big\|^2 - \big(1-|z|^2\big)\big\langle D_{T^*}(I - ZT^*)^{-1}(I - TZ^*)^{-1}D_{T^*}\theta_T(z)h, \theta_T(z)h\big\rangle \\
&\quad = \big\|\theta_T(z)\big\|^2 - \big(1-|z|^2\big)\big\langle k_T(z)^*k_T(z)\theta_T(z)h, \theta_T(z)h\big\rangle,
\end{aligned}$$

where $k_T(z) = (I - TZ^*)^{-1} D_{T^*}$. So for $0 < r < 1$ and $\lambda \in \partial\mathbb{B}_n$, we have

$$\left\|\theta_T(r\lambda)^*\theta_T(r\lambda)h\right\|^2 = \left\|\theta_T(r\lambda)h\right\|^2 - \left(1 - r^2\right)\left\langle k_T(r\lambda)^* k_T(r\lambda)\theta_T(r\lambda)h, \theta_T(r\lambda)h\right\rangle.$$

So taking the limit as $r \to 1$, one gets

$$\left\|\theta_T(\lambda)^*\theta_T(\lambda)h\right\|^2 = \left\|\theta_T(\lambda)h\right\|^2,$$

σ-a.e. on $\partial\mathbb{B}_n$. $\square$

Let $\mathcal{M}$, T and θ_T be as in Theorem 5.4. Then $P_{\mathcal{M}} = M_{\theta_T} M_{\theta_T}^*$. Given any orthonormal basis $\{e_i\}_{i\in\mathbb{N}}$ of $\mathcal{D}_T$, define $\varphi_i \in H_n^2$ by $\varphi_i(z) = \theta_T(z)e_i$ for $i \in \mathbb{N}$. Then φ_i is a multiplier on H_n^2 for each i. For z in $\mathbb{B}_n$, we have

$$\begin{aligned}\left\langle M_{\theta_T} M_{\theta_T}^* k(\cdot, z), k(\cdot, z)\right\rangle &= \left\|M_{\theta_T}^* k(\cdot, z)\right\|^2 = \left\|k(\cdot, z) \otimes \theta_T(z)^* 1\right\|^2 \\ &= \left\|k(\cdot, z)\right\|^2 \sum_{i\geqslant 1} \left|\left\langle\theta_T(z)^* 1, e_i\right\rangle\right|^2.\end{aligned}$$

Now $|\langle\theta_T(z)^* 1, e_i\rangle| = |\langle 1, \theta_T(z)e_i\rangle|$ and since the last inner product is in $\mathcal{D}_{T^*}$, which is one-dimensional, we have $|\langle\theta_T(z)^* 1, e_i\rangle| = |\theta_T(z)e_i| = |\varphi_i(z)|$. So

$$\left\langle M_{\theta_T} M_{\theta_T}^* k(\cdot, z), k(\cdot, z)\right\rangle = \left\|k(\cdot, z)\right\|^2 \sum_{i\geqslant 1} \left|\varphi_i(z)\right|^2.$$

It is easy to see that the right-hand side of the equality above is the same as $\langle\sum_i M_{\varphi_i} M_{\varphi_i}^* k(\cdot, z), k(\cdot, z)\rangle$. Thus the following identity is immediate:

$$P_{\mathcal{M}} = M_{\theta_T} M_{\theta_T}^* = \sum_{i\geqslant 1} M_{\varphi_i} M_{\varphi_i}^*,$$

where the convergence is in the strong operator topology. Such a sequence $\{\varphi_i\}_{i\geqslant 1}$ is called inner if the boundary functions (also denoted by φ_i) satisfy $\sum |\varphi_i(\lambda)|^2 = 1$ almost everywhere with respect to σ. Arveson showed in [7] that if $\mathcal{M}$ contains a nonzero polynomial, then the corresponding sequence $\{\varphi_i\}_{i\geqslant 1}$ is inner and he conjectured that the same will be true for any submodule $\mathcal{M}$. Greene, Richter and Sundberg proved the conjecture in [18] in a more general setting. They considered those complete Nevanlinna–Pick kernels which satisfied a certain natural condition. Their class included the class of all $\mathcal{U}$-invariant kernels. We give a simple proof in the case of Arveson space using the characteristic function.

Theorem 6.2. *Let $\mathcal{M}$ be a multiplication invariant subspace of the Arveson space H_n^2. Then there exists an inner sequence $\{\varphi_i\}_{i\geqslant 1}$ corresponding to $\mathcal{M}$ such that $M_{\theta_T} M_{\theta_T}^* = \sum_{i\geqslant 1} M_{\varphi_i} M_{\varphi_i}^*$.*

Proof. Observe that

$$\sum_{i\geqslant 1} \left|\varphi_i(z)\right|^2 = \frac{1}{\|k_z\|^2} \sum_{i\geqslant 1} \left\langle M_{\varphi_i} M_{\varphi_i}^* k_z, k_z\right\rangle = \frac{1}{\|k_z\|^2}\left\langle M_{\theta_T} M_{\theta_T}^* k_z, k_z\right\rangle = \frac{1}{\|k_z\|^2}\left\|M_{\theta_T}^* k_z\right\|^2.$$

This gives

$$\sum_{i\geqslant 1} \left|\varphi_i(z)\right|^2 = \left\|\theta_T(z)^* 1\right\|^2.$$

But $\|\theta_T(\lambda)^* 1\|^2 = 1$ a.e. for $\lambda \in \partial\mathbb{B}_n$. Hence for a.e. $[\sigma]$, $\lambda \in \partial\mathbb{B}_n$, $\sum_{i\geqslant 1} |\varphi_i(z)|^2 = 1$. $\square$

Acknowledgments

Both authors are thankful to Gadadhar Misra for many lively discussions.

References

[1] C. Ambrozie, M. Englis, V. Muller, Operator tuples and analytic models over general domains in $\mathbb{C}^n$, J. Operator Theory 47 (2002) 287–302. MR 2004c:47013.
[2] C. Ambrozie, D. Timotin, On an intertwining lifting theorem for certain reproducing kernel Hilbert spaces, Integral Equations Operator Theory 42 (2002) 373–384. MR 2002m:47012.
[3] A. Arias, Projective modules on Fock spaces, J. Operator Theory 52 (2004) 139–172. MR 2005g:46094.
[4] A. Arias, G. Popescu, Noncommutative interpolation and Poisson transforms. II, Houston J. Math. 25 (1999) 79–98. MR 2000i:47148.
[5] W.B. Arveson, Subalgebras of C^*-algebras III, Multivariable operator theory, Acta Math. 181 (1998) 159–228. MR 2000e:47013.
[6] W.B. Arveson, The curvature of a Hilbert module over $\mathbb{C}[z_1, \dots, z_d]$, Proc. Natl. Acad. Sci. USA 96 (1999) 11096–11099. MR 2001k:47010.
[7] W. Arveson, The curvature invariant of a Hilbert module over $\mathbb{C}[z_1, \dots, z_d]$, J. Reine Angew. Math. 522 (2000) 173–236. MR 2003a:47013.
[8] W. Arveson, p-Summable commutators in dimension d, J. Operator Theory, in press.
[9] J.A. Ball, V. Bolotnikov, Realization and interpolation for Schur–Agler-class functions on domains with matrix polynomial defining function in $\mathbb{C}^n$, J. Funct. Anal. 213 (2004) 45–87. MR 2069781.
[10] T. Bhattacharyya, J. Eschmeier, J. Sarkar, Characteristic function of a pure commuting contractive tuple, Integral Equations Operator Theory (2005) (available online).
[11] J.W. Bunce, Models for n-tuples of noncommuting operators, J. Funct. Anal. 57 (1984) 21–30. MR 85k:47019.
[12] C. Davis, Some dilation and representation theorems, in: Proc. Second International Symposium in West Africa on Functional Analysis and Its Applications, Kumasi, 1979, Forum for Funct. Anal. Appl., Kumasi, Ghana, 1979, pp. 159–182. MR 84e:47012.
[13] A.E. Frazho, Models for noncommuting operators, J. Funct. Anal. 48 (1982) 1–11. MR 84h:47010.
[14] R.G. Douglas, Essentially reductive Hilbert modules, J. Operator Theory, in press.
[15] R.G. Douglas, V.I. Paulsen, Hilbert Modules over Function Algebras, Longman Res. Notes, vol. 217, 1989. MR 91g:46084.
[16] J. Eschmeier, M. Putinar, Spherical contractions and interpolation problems on the unit ball, J. Reine Angew. Math. 542 (2002) 219–236. MR 2002k:47019.
[17] A. Gheondea, G. Popescu, Joint similarity and dilations for noncontractive sequences of operators, J. Funct. Anal. 197 (2003) 68–109. MR 2003m:47011.
[18] D. Greene, S. Richter, C. Sundberg, The structure of inner multipliers on spaces with complete Nevanlinna–Pick kernels, J. Funct. Anal. 194 (2002) 311–331. MR 2003h:46038.
[19] K. Guo, Defect operators for submodules of H_d^2, J. Reine Angew. Math. 573 (2004) 181–209. MR 2084587.
[20] S. McCullough, T.T. Trent, Invariant subspaces and Nevanlinna–Pick kernels, J. Funct. Anal. 178 (2000) 226–249. MR 2002b:47006.
[21] V. Muller, F.-H. Vasilescu, Standard models for some commuting multioperators, Proc. Amer. Math. Soc. 117 (1993) 879–989. MR 93e:47016.
[22] G. Popescu, Isometric dilations for infinite sequences of noncommuting operators, Trans. Amer. Math. Soc. 316 (1989) 523–536. MR 90c:47006.
[23] G. Popescu, Models for infinite sequences of noncommuting operators, Acta Sci. Math. (Szeged) 53 (1989) 355–368. MR 91b:47025.
[24] G. Popescu, Characteristic functions for infinite sequences of noncommuting operators, J. Operator Theory 22 (1989) 51–71. MR 91m:47012.
[25] G. Popescu, Multi-analytic operators on Fock spaces, Math. Ann. 303 (1995) 31–46. MR 96k:47049.
[26] G. Popescu, Poisson transforms on some C^*-algebras generated by isometries, J. Funct. Anal. 161 (1999) 27–61. MR 2000m:46117.
[27] G. Popescu, Curvature invariant for Hilbert modules over free semigroup algebras, Adv. Math. 158 (2001) 264–309. MR 2002b:46097.
[28] G. Popescu, Constrained multivariable operator theory, Indiana Univ. Math. J. (2005) (available online).

[29] S. Pott, Standard models under polynomial positivity conditions, J. Operator Theory 41 (1999) 365–389. MR 2000j:47019.

[30] B. Sz.-Nagy, C. Foias, Harmonic Analysis of Operators on Hilbert Space, North-Holland, 1970. MR 43:947.

Further reading

J. Agler, J.E. McCarthy, Complete Nevanlinna–Pick kernels, J. Funct. Anal. 157 (2000) 111–124. MR 2001h:47019.

J. Agler, J.E. McCarthy, Pick Interpolation and Hilbert Function Spaces, Grad. Stud. Math., vol. 44, Amer. Math. Soc., 2002. MR 2003b:47001.

S.W. Drury, A generalization of Von Neumann's inequality to the complex ball, Proc. Amer. Math. Soc. 68 (1978) 300–304. MR 80c:47010.