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Integral Equations and Operator Theory

Characteristic Function of a Pure Commuting Contractive Tuple

T. Bhattacharyya, J. Eschmeier and J. Sarkar

Abstract. A theorem of Sz.-Nagy and Foias [9] shows that the characteristic function $\theta_T(z) = -T + zD_{T^*}(1_{\mathcal{H}} - zT^*)^{-1}D_T$ of a completely non-unitary contraction T is a complete unitary invariant for T. In this note we extend this theorem to the case of a pure commuting contractive tuple using a natural generalization of the characteristic function to an operator-valued analytic function defined on the open unit ball of $\mathbb{C}^n$. This function is related to the curvature invariant introduced by Arveson [3].

1. Introduction

A contraction T acting on a Hilbert space $\mathcal{H}$ is said to be completely non-unitary (c.n.u.) if there is no non-zero reducing subspace $\mathcal{M}$ of $\mathcal{H}$ such that $T|\mathcal{M}$ is a unitary operator. The class of completely non-unitary operators plays an important role in understanding general contractions because, given any contraction T on a Hilbert space $\mathcal{H}$, there is a decomposition $\mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_1$ of $\mathcal{H}$ into orthogonal subspaces each of which is a reducing subspace for T such that $T_0 = T|\mathcal{H}_0$ is unitary while $T_1 = T|\mathcal{H}_1$ is a c.n.u. contraction. A key ingredient for studying contraction operators on Hilbert spaces is the following analytic operator-valued function, called the characteristic function of T and introduced by Sz.-Nagy and Foias in [9]:

$$\theta_T(z) = -T + zD_{T^*}(1_{\mathcal{H}} - zT^*)^{-1}D_T, \quad z \in \mathbb{D}. \tag{1.1}$$

Here $\mathbb{D}$ is the open unit disk in the complex plane. The operators D_T and D_{T^*} are the so-called defect operators $(1_{\mathcal{H}} - T^*T)^{1/2}$ and $(1_{\mathcal{H}} - TT^*)^{1/2}$ of T and T^*, respectively. By virtue of the relation $TD_T = D_{T^*}T$ (see Section I.3 in [9]), the values $\theta_T(z)$ of the characteristic function can be regarded as bounded operators from $\mathcal{D}_T = \overline{\mathrm{Ran}} D_T$ into $\mathcal{D}_{T^*} = \overline{\mathrm{Ran}} D_{T^*}$.

It is shown in [9] that $\theta_T(z)$ is contraction valued and that $\|\theta_T(0)\xi\| < \|\xi\|$ for all $\xi \in \mathcal{D}_T$. The characteristic functions θ_T and θ_R of two contractions T and R are said to coincide if there are unitary operators $\sigma_1 : \mathcal{D}_T \to \mathcal{D}_R$ and $\sigma_2 : \mathcal{D}_{T^*} \to \mathcal{D}_{R^*}$ such that

$$\theta_T(z) = \sigma_2^{-1}\theta_R(z)\sigma_1 \quad \text{for all} \quad z \in \mathbb{D}. \tag{1.2}$$

It is easy to see that if T and R are two unitarily equivalent contractions, i.e., if there is a unitary operator U such that $T = URU^*$, then the characteristic functions θ_T and θ_R coincide. One can easily construct examples to show that the converse of this is not true in this generality (see page 240 in [9]). However, the converse is true if both T and R are c.n.u. contractions.

Theorem 1.1. *(***Sz.-Nagy and Foias***) Two completely non-unitary contractions are unitarily equivalent if and only if their characteristic functions coincide.*

This theorem shows that the characteristic function is a complete unitary invariant for c.n.u. contractions. The route to prove the theorem is via constructing a functional model for c.n.u. contractions which is also of independent interest. We briefly recall some essential features of this model theory relevant to us here. Let $\mathbb{B}^n$ be the open unit ball in $\mathbb{C}^n$. If $\mathcal{E}$ is a complex Hilbert space, we follow the notation of [4] and define $\mathcal{O}(\mathbb{B}^n, \mathcal{E})$ to be the class of all $\mathcal{E}$-valued analytic functions on $\mathbb{B}^n$. For any multi-index $k = (k_1, \dots, k_n) \in \mathbb{N}^n$, we write $|k| = k_1 + \cdots + k_n$. Then consider the Hilbert space

$$H(\mathcal{E}) = \{f \in \mathcal{O}(\mathbb{B}, \mathcal{E}) :\ f = \sum_{k \in \mathbb{N}^n} a_k z^k \text{ with } a_k \in \mathcal{E} \text{ and } \|f\|^2 = \sum_{k \in \mathbb{N}^n} \frac{\|a_k\|^2}{\gamma_k} < \infty\}, \tag{1.3}$$

where $\gamma_k = |k|!/k!$. One can show that $H(\mathcal{E})$ is the $\mathcal{E}$-valued functional Hilbert space given by the reproducing kernel $(1 - \langle z, w\rangle)^{-1}1_{\mathcal{E}}$. Of course, when $n = 1$ and $\mathcal{E} = \mathbb{C}$, this is the usual Hardy space on the disk. Given complex Hilbert spaces $\mathcal{E}$ and $\mathcal{E}_*$, the multiplier space $M(\mathcal{E}, \mathcal{E}_*)$ consists of all $\varphi \in \mathcal{O}(\mathbb{B}^n, \mathcal{B}(\mathcal{E}, \mathcal{E}_*))$ such that $\varphi H(\mathcal{E}) \subset H(\mathcal{E}_*)$. By the closed graph theorem, for each function $\varphi \in M(\mathcal{E}, \mathcal{E}_*)$,the induced multiplication operator $M_\varphi : H(\mathcal{E}) \to H(\mathcal{E}_*),\ f \mapsto \varphi f$ is continuous.

The Sz.-Nagy and Foias model theory works for c.n.u. contractions T. Here we shall confine ourselves to a more restricted class. The characteristic function of a single contraction T is a multiplier from the Hardy space $H(\mathcal{D}_T)$ to the Hardy space $H(\mathcal{D}_{T^*})$. A contraction T is said to be of class $C_{\cdot 0}$ if T^{*m} converges strongly to 0 as $m \to \infty$. It is easy to see that each $C_{\cdot 0}$ contraction is completely non-unitary. If T is a $C_{\cdot 0}$ contraction acting on a Hilbert space $\mathcal{H}$, then there is a unitary operator U from $\mathcal{H}$ onto $\mathbb{H} = H(\mathcal{D}_{T^*}) \ominus M_{\theta_T} H(\mathcal{D}_T)$ such that $UTU^* = P_{\mathbb{H}} M_z|\mathbb{H}$ where M_z is the multiplication operator with the independent variable z on $H(\mathcal{D}_{T^*})$. Thus any $C_{\cdot 0}$ contraction can be realized as $P_{\mathbb{H}} M_z|\mathbb{H}$ where the model space $\mathbb{H}$ is the orthocomplement of the range of M_{θ_T}.

In this note, we generalize Theorem 1.1 to the case of pure commuting contractive tuples. So we construct an operator-valued holomorphic function on the open unit ball in $\mathbb{C}^n$ and show that it is a complete unitary invariant for a pure commuting contractive tuple. En route we also construct a functional model for such a tuple.

Previously, Frazho [5] and Popescu [8] have considered characteristic functions for tuples of non-commuting operators. Since they are dealing with non-commuting families of operators, the characteristic function is actually an operator. The characteristic function in that case is a complete unitary invariant for a completely non-coisometric contractive family [8]. It is not clear how the characteristic function of a not necessarily commuting tuple is related to the one defined below in case the tuple consists of commuting operators.

2. Definition of the Characteristic Function

A commuting tuple of bounded operators $T = (T_1, \ldots, T_n)$ acting on a Hilbert space $\mathcal{H}$ is called contractive if $\|T_1h_1 + \cdots + T_nh_n\|^2 \leq \|h_1\|^2 \cdots + \|h_n\|^2$ for all $h_1, \ldots, h_n$ in $\mathcal{H}$. This is equivalent to demanding that $\sum_{i=1}^n T_iT_i^* \leq 1_{\mathcal{H}}$. The positive operator $(1_{\mathcal{H}} - \sum_{i=1}^n T_iT_i^*)^{1/2}$ and the closure of its range will be called the *defect operator* D_{T^*} and the *defect space* $\mathcal{D}_{T^*}$ of T^*.

We shall also denote by T the bounded operator from $\mathcal{H}^n$ to $\mathcal{H}$ which maps $(h_1, h_2, \ldots, h_n)$ to $T_1h_1 + T_2h_2 + \cdots + T_nh_n$. The adjoint $T^* : \mathcal{H} \to \mathcal{H}^n$ maps h to the column vector $(T_1^*h, T_2^*h, \ldots, T_n^*h)$ and, in fact, T is a contractive tuple if and only if the operator T is a contraction. Thus for a contractive tuple T one can also consider the defect operator $D_T = (1_{\mathcal{H}^n} - T^*T)^{1/2} = ((\delta_{ij}1_{\mathcal{H}} - T_i^*T_j))^{1/2}$ in $\mathcal{B}(\mathcal{H}^n)$ and the associated defect space $\mathcal{D}_T = \overline{\mathrm{Ran}} D_T \subset \mathcal{H}^n$.

Lemma 2.1. *For any commuting contractive tuple T, we obtain the identity*

$$TD_T = D_{T^*}T.$$

Proof. This follows from equation (I.3.4) of [9] where it is proved that $TD_T = D_{T^*}T$ for any contraction from a Hilbert space $\mathcal{H}'$ into a Hilbert space $\mathcal{H}$. Here we have the special case of the operator T defined above from $\mathcal{H}^n$ into $\mathcal{H}$. □

Note that, for $z = (z_1, \ldots, z_n) \in \mathbb{B}^n$, the operator Z from $\mathcal{H}^n$ to $\mathcal{H}$ which maps $(h_1, \ldots, h_n)$ to $z_1h_1 + \cdots + z_nh_n$ is a contraction because $ZZ^* = \sum |z_i|^2 1_{\mathcal{H}}$. Thus $Z = (z_1 1_{\mathcal{H}}, \ldots, z_n 1_{\mathcal{H}})$ is a commuting contractive tuple on $\mathcal{H}$ with $\|Z\| = (\sum |z_i|^2)^{1/2}$. Hence, given a commuting contractive tuple T, the operator ZT^* is a strict contraction for $z \in \mathbb{B}^n$ and hence $1_{\mathcal{H}} - ZT^*$ is invertible. We define the characteristic function of T to be the analytic operator-valued function $\theta_T : \mathbb{B}^n \to \mathcal{B}(\mathcal{D}_T, \mathcal{D}_{T^*})$ with

$$\theta_T(z) = -T + D_{T^*}(1_{\mathcal{H}} - ZT^*)^{-1}ZD_T, \quad z \in \mathbb{B}^n. \tag{2.1}$$

Lemma 2.2. *Given a commuting contractive tuple T, its characteristic function θ_T is a multiplier, that is $\theta_T \in M(\mathcal{D}_T, \mathcal{D}_{T*})$, with $\|M_{\theta_T}\| \leq 1$. For $z, w \in \mathbb{B}^n$, the* *identity*

$$1-\theta_T(w)\theta_T(z)^* = (1-WZ^*)D_{T^*}(1-WT^*)^{-1}(1-TZ^*)^{-1}D_{T^*} \tag{2.2}$$

holds.

Proof. It is an elementary exercise to check that

$$U = \begin{pmatrix} T^* & D_T \\ D_{T^*} & -T \end{pmatrix} \in \mathcal{B}(\mathcal{H}\oplus\mathcal{D}_T, \mathcal{H}^n \oplus \mathcal{D}_{T^*})$$

defines a unitary matrix operator. By Proposition 1.2 in [4] the transfer function of U, that is, the analytic operator-valued function $\theta_T : \mathbb{B}^n \to \mathcal{B}(\mathcal{H}\otimes\mathcal{D}_T, \mathcal{H}^n\otimes\mathcal{D}_{T^*})$,

$$\theta_T(z) = -T + D_{T^*}(1_{\mathcal{H}} - ZT^*)^{-1}ZD_T$$

defines a multiplier $\theta_T \in M(\mathcal{D}_T, \mathcal{D}_{T*})$ with $\|M_{\theta_T}\| \le 1$ such that formula (2.2) holds. $\square$

For $z = w$, the right-hand side of formula (2.2) defines a positive operator. Thus we obtain the following corollary.

Corollary 2.3. *Given a commuting contractive tuple T, its characteristic function θ_T is a bounded analytic function on $\mathbb{B}^n$ with* $\sup_{z\in\mathbb{B}^n}\|\theta_T(z)\| \le 1$.

3. Functional model of a pure commuting contractive tuple

The purpose of this section is to produce functional models for pure commuting contractive tuples. This functional model generalizes the corresponding model for $C_{\cdot 0}$ contractions (Theorem VI. 2.3 in [9]) to the multivariable case and reflects very clearly the important role that the characteristic function plays.

A prototype of a commuting contractive tuple is the so-called n-shift which we simply call the *shift* as long as the dimension n is fixed. By definition this is the commuting tuple $M_z = (M_{z_1}, \dots, M_{z_n})$ on the scalar-valued functional Hilbert space $H(\mathbb{C})$ consisting of the multiplication operators M_{z_i} with the coordinate functions z_i. It is not difficult to see that $\sum_{i=1}^n M_{z_i}M_{z_i}^* = 1 - E_0$ where 1 is the identity operator on $H(\mathbb{C})$ and E_0 is the projection onto the one-dimensional subspace consisting of all constant functions (see [2]). Hence the shift is a commuting contractive tuple. It is not hard to show that

$$\mathrm{SOT}-\lim_{k\to\infty} \sum_{1\le i_1,i_2,\dots i_k\le n} M_{z_{i_1}}M_{z_{i_2}}\dots M_{z_{i_k}}M_{z_{i_k}}^*\dots M_{z_{i_2}}^*M_{z_{i_1}}^* = 0.$$

Thus the shift is an example of a *pure* commuting contractive tuple in the sense of the following definition.

Definition 3.1. *For a commuting contractive tuple T on a Hilbert space $\mathcal{H}$, define a completely positive map $P_T : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$ by $P_T(X) = \sum_{i=1}^n T_iXT_i^*$. We denote by $A_\infty \in \mathcal{B}(\mathcal{H})$ the strong limit of the decreasing sequence of positive operators $I \ge P_T(I) \ge P_T^2(I) \ge \dots \ge 0$. The commuting contractive tuple T is called pure if $A_\infty = 0$.*

It is interesting to observe that the norm of A_∞ is either 0 or 1. For the proof, first define for any integer $m \geq 1$, the operator $T^m \in \mathcal{B}(\mathcal{H}^{n^m}, \mathcal{H})$ which sends an element $\underline{h}$ of $\mathcal{H}^{n^m}$ to the sum $\sum_{1\leq i_1,\dots,i_m \leq n} T_{i_1} ... T_{i_m} h_{i_1 \dots i_m}$. Its adjoint $T^{m*} \in \mathcal{B}(\mathcal{H}, \mathcal{H}^{n^m})$ maps a vector h to the n^m column vector $(T^*_{i_1} ... T^*_{i_m} h)_{1\leq i_1,\dots,i_m\leq n}$ in $\mathcal{H}^{n^m}$. By the above definition, $T^m T^{m*} = P_T^m(1)$. Thus we find that

$$\|A_\infty^{1/2} h\|^2 = \langle A_\infty h, h\rangle = \lim_{m\to\infty} \langle P_T^m(1)h, h\rangle = \lim_{m\to\infty} \langle T^m T^{m*} h, h\rangle = \lim_{m\to\infty} \|T^{m*}h\|^2.$$

Let $\underline{A}$ denote the operator $A_\infty \oplus A_\infty \oplus \cdots \oplus A_\infty : \mathcal{H}^{n^m} \to \mathcal{H}^{n^m}$. Then $T^m \underline{A} T^{m*} = P_T^m(A_\infty) = A_\infty$. It follows that

$$\begin{aligned} \|A_\infty{}^{\frac{1}{2}} h\|^2 &= \langle A_\infty h, h\rangle = \langle T^m \underline{A} T^{m*} h, h\rangle = \|\underline{A}^{\frac{1}{2}} T^{m*} h\|^2 \\ &\leq \|\underline{A}^{\frac{1}{2}}\|^2 \, \|T^{m*}h\|^2 = \|A_\infty\| \, \|T^{m*}h\|^2 \xrightarrow{m} \|A_\infty\| \, \|A_\infty{}^{\frac{1}{2}} h\|^2. \end{aligned}$$

Hence either $A_\infty{}^{1/2} = 0$ or $\|A_\infty\| \geq 1$. But A_∞ being a contraction, this means that $\|A_\infty\| = 1$.

Remark 3.2. *In the case $n = 1$ a contraction $T \in \mathcal{B}(\mathcal{H})$ is pure in the above sense if and only if it is of class $C_{\cdot 0}$.*

Arveson proved the following theorem for commuting contractive tuples in [2] (Theorem 4.5). In a way, the operator L below is a precursor of the functional model that we are going to construct.

Theorem 3.3. *Let T be a commuting contractive tuple of operators on some Hilbert space $\mathcal{H}$. Then there exists a unique bounded linear operator $L : H(\mathbb{C}) \otimes \mathcal{D}_{T^*} \to \mathcal{H}$ satisfying*

$$L(f \otimes \xi) = f(T) D_{T^*} \xi$$

for all f in $\mathbb{C}[z_1, ..., z_n]$, and ξ in $\mathcal{D}_{T^*}$. Furthermore, we have $LL^* = 1_{\mathcal{H}} - A_\infty$ and the identity $L(f(M_z) \otimes 1_{\mathcal{D}_{T^*}}) = f(T)L$ holds for all f in $\mathbb{C}[z_1, \dots, z_n]$ where $\mathbb{C}[z_1, \dots, z_n]$ is the algebra of all polynomials in n complex variables.

Remark 3.4. *The tuple T is pure if and only if L is a co-isometry.*

Given a Hilbert space $\mathcal{E}$, we denote by $M_z^{\mathcal{E}} = (M_{z_1}^{\mathcal{E}}, \dots, M_{z_n}^{\mathcal{E}}) \in \mathcal{B}(H(\mathcal{E}))^n$ the tuple of multiplication operators induced by the coordinate functions z_i. There is a canonical unitary operator $U_{\mathcal{E}} : H(\mathbb{C}) \otimes \mathcal{E} \to H(\mathcal{E})$ with $U_{\mathcal{E}}(f \otimes x) = fx$ for $f \in H(\mathbb{C})$ and $x \in \mathcal{E}$. In the following we shall identify the spaces $H(\mathbb{C}) \otimes \mathcal{E}$ and $H(\mathcal{E})$ via this unitary operator $U_{\mathcal{E}}$. In this way each multiplier $\varphi \in M(\mathcal{E}, \mathcal{E}_*)$ induces a bounded operator $M_\varphi : H(\mathbb{C}) \otimes \mathcal{E} \to H(\mathbb{C}) \otimes \mathcal{E}_*$.

As observed by Arveson in [2] (Proposition 1.12), the space $H(\mathbb{C})$ is a functional Hilbert space with reproducing kernel

$$K : \mathbb{B}^n \times \mathbb{B}^n \to \mathbb{C}, \quad K(z, w) = (1 - \langle z, w\rangle)^{-1}.$$

In particular, the space $H(\mathbb{C})$ is the closed linear span of the functions $k_w = K(\cdot, w)$ $(w \in \mathbb{B}^n)$.

Lemma 3.5. *Let $\varphi \in M(\mathcal{E}, \mathcal{E}_*)$ be a multiplier. Then the identity*

$$M_{\varphi^*}(k_z \otimes x) = k_z \otimes \varphi(z)^* x$$

holds for all $z \in \mathbb{B}^n$ and $x \in \mathcal{E}_*$.

Proof. Fix $z \in \mathbb{B}^n$ and $x \in \mathcal{E}_*$. Note first that

$$\langle f \otimes y, k_z \otimes x \rangle = f(z)\langle y, x \rangle = \langle (fy)(z), x \rangle$$

holds for all $f \in H(\mathbb{C})$ and $y \in \mathcal{E}_*$. Hence it follows that $\langle f, k_z \otimes x \rangle = \langle f(z), x \rangle$ for each function $f \in H(\mathcal{E}_*)$. Using this identity twice (for $\mathcal{E}$- and $\mathcal{E}_*$-valued functions), we obtain that

$$\langle f, M_\varphi^*(k_z \otimes x) \rangle = \langle \varphi(z) f(z), x \rangle = \langle f, k_z \otimes \varphi(z)^* x \rangle$$

for each function $f \in h(\mathcal{E})$. □

Next we relate the operator L described in Theorem 3.3 with the characteristic function.

Lemma 3.6. *Given a commuting contractive tuple T, we obtain the identity*

$$L^* L + M_{\theta_T} M_{\theta_T}^* = 1_{H(\mathbb{C}) \otimes \mathcal{D}_{T^*}}.$$

Proof. As observed by Arveson in the proof of Theorem 1.2 in [3], the operator L satisfies the identity

$$L(k_z \otimes \xi) = (1 - TZ^*)^{-1} D_{T^*} \xi \quad (z \in \mathbb{B}^n, \xi \in \mathcal{D}_{T^*}).$$

Therefore, for z, w in $\mathbb{B}^n$ and ξ, η in $\mathcal{D}_{T^*}$, we obtain that

$$\begin{aligned}
& \langle (L^* L + M_{\theta_T} M_{\theta_T}^*) k_z \otimes \xi, k_w \otimes \eta \rangle \\
= \; & \langle L(k_z \otimes \xi), L(k_w \otimes \eta) \rangle + \langle M_{\theta_T}^*(k_z \otimes \xi), M_{\theta_T}^*(k_w \otimes \eta) \rangle \\
= \; & \langle (1 - TZ^*)^{-1} D_{T^*} \xi, (1 - TW^*)^{-1} D_{T^*} \eta \rangle + \langle k_z \otimes \theta_T(z)^* \xi, k_w \otimes \theta_T(w)^* \eta \rangle \\
= \; & \langle D_{T^*}(1 - WT^*)^{-1}(1 - TZ^*)^{-1} D_T^* \xi, \eta \rangle + \langle k_z, k_w \rangle \langle \theta_T(w) \theta_T(z)^* \xi, \eta \rangle \\
= \; & \langle k_z \otimes \xi, k_w \otimes \eta \rangle.
\end{aligned}$$

To verify the last equality, the reader should use the formula obtained in Lemma 2.2. Using the fact that the vectors k_z form a total set in $H(\mathbb{C})$, the assertion follows. □

In [3] Arveson used abstract factorization results to prove the existence of a multiplier $\varphi \in M(\mathcal{D}, \mathcal{D}_{T^*})$ such that

$$1_{H(\mathbb{C}) \otimes \mathcal{D}_{T^*}} - L^* L = M_\varphi M_\varphi^*.$$

The above Lemma 3.6 shows that φ can be chosen as the characteristic function of T.

As usual we call two commuting tuples $T = (T_1, \ldots, T_n)$ and $R = (R_1, \ldots, R_n)$ of bounded operators on Hilbert spaces $\mathcal{H}$ and $\mathcal{K}$ unitarily equivalent if there exists a unitary operator U from $\mathcal{H}$ to $\mathcal{K}$ such that $R_i = UT_iU^*$ holds for all $i = 1, ..., n$. Now we are ready to prove the main theorem of this section.

Theorem 3.7. *Every pure commuting contractive tuple T on a Hilbert space $\mathcal{H}$ is unitarily equivalent to the commuting tuple $\mathbb{T} = (\mathbb{T}_1, \ldots, \mathbb{T}_n)$ on the functional space $\mathbb{H}_T = (H(\mathbb{C})\otimes\mathcal{D}_{T^*})\ominus M_{\theta_T}(H(\mathbb{C})\otimes\mathcal{D}_T)$ defined by $\mathbb{T}_i = P_{\mathbb{H}_T}(M_{z_i}\otimes 1_{\mathcal{D}_{T^*}})|\mathbb{H}_T$ for $1 \le i \le n$.*

Proof. Since T is pure, the map

$$L^* : \mathcal{H} \to H(\mathbb{C}) \otimes \mathcal{D}_{T^*}$$

is an isometry. Thus $\mathcal{H}$ is isometrically embedded into $H(\mathbb{C}) \otimes \mathcal{D}_{T^*}$ via the identification of $\mathcal{H}$ with the closed subspace $L^*\mathcal{H}$. Now L^*L is the projection of $H(\mathbb{C}) \otimes \mathcal{D}_{T^*}$ onto the closed subspace $L^*\mathcal{H}$. But then by Lemma 3.6, the operators L^*L and $M_{\theta_T}M_{\theta_T}^*$ are mutually orthogonal projections which add up to identity. Therefore the subspace $L^*\mathcal{H}$ is the orthocomplement of the range of M_{θ_T}:

$$L^*\mathcal{H} = (H(\mathbb{C}) \otimes \mathcal{D}_{T^*}) \ominus M_{\theta_T}(H(\mathbb{C}) \otimes \mathcal{D}_T).$$

Now by Theorem 3.3, $L^*T_i^* = (M_{z_i} \otimes 1_{\mathcal{D}_{T^*}})^*L^*$. Thus the subspace $L^*\mathcal{H}$ is co-invariant for the shift and, via the identification of $\mathcal{H}$ with $L^*\mathcal{H}$, the operators T_i in $\mathcal{B}(\mathcal{H})$ coincide with the compressions of the operators $M_{z_i} \otimes 1_{\mathcal{D}_{T^*}}$ to the space $\mathbb{H}_T$. □

So every pure commuting contractive tuple T on a Hilbert space $\mathcal{H}$ is unitarily equivalent to the commuting tuple $P_{\mathbb{H}_T}(M_z \otimes 1_{\mathcal{D}_{T^*}})|\mathbb{H}_T$, where $\mathbb{H}_T$ is the M_z^*-invariant subspace $(H(\mathbb{C}) \otimes \mathcal{D}_{T^*}) \ominus M_{\theta_T}(H(\mathbb{C}) \otimes \mathcal{D}_T)$ of $H(\mathbb{C}) \otimes \mathcal{D}_{T^*}$.

4. The characteristic function as a complete unitary invariant

Definition 4.1. *Given two commuting contractive tuples T and R on Hilbert spaces $\mathcal{H}$ and $\mathcal{K}$, the characteristic functions of T and R are said to coincide if there exist unitary operators $\tau : \mathcal{D}_T \to \mathcal{D}_R$ and $\tau_* : \mathcal{D}_{T^*} \to \mathcal{D}_{R^*}$ such that the following diagram commutes for all z in $\mathbb{B}^n$:*

$$\begin{tikzcd}
\mathcal{D}_T \arrow[r, "\theta_T(z)"] \arrow[d, "\tau"'] & \mathcal{D}_{T^*} \arrow[d, "\tau_*"] \\
\mathcal{D}_R \arrow[r, "\theta_R(z)"'] & \mathcal{D}_{R^*}
\end{tikzcd}$$

In this section, we prove that the characteristic function of a pure commuting contractive tuple is a complete unitary invariant.

Proposition 4.2. *The characteristic functions of two unitarily equivalent commuting contractive tuples coincide.*

Proof. Let T and R be two commuting contractive tuples on $\mathcal{H}$ and $\mathcal{K}$, respectively, such that there is a unitary operator $\sigma : \mathcal{H} \to \mathcal{K}$ satisfying $\sigma T_i \sigma^* = R_i$ for all i. Denote by $\underline{\sigma}$ and $\underline{\sigma}^*$ the operators

$$\oplus_{i=1}^{n} \sigma : \mathcal{H}^n \to \mathcal{K}^n \text{ and } i \oplus_{i=1}^{n} \sigma^* : \mathcal{K}^n \to \mathcal{H}^n.$$

Then it is easy to see that $\underline{\sigma} D_T^2 \underline{\sigma}^* = D_R^2$ and $\sigma D_{T^*}^2 \sigma^* = D_{R^*}^2$. Thus $\underline{\sigma} D_T \underline{\sigma}^* = D_R$ and $\sigma D_{T^*} \sigma^* = D_{R^*}$. Hence $\underline{\tau} : \mathcal{D}_T \to \mathcal{D}_R$ defined by $\underline{\tau} = \underline{\sigma} \,|_{\mathcal{D}_T}$ is a unitary operator between $\mathcal{D}_T$ and $\mathcal{D}_R$. Similarly, the restriction $\tau_* = \sigma \,|_{\mathcal{D}_{T^*}}$ defines a unitary operator from $\mathcal{D}_{T^*}$ to $\mathcal{D}_{R^*}$. Finally, note that

$$\begin{aligned}
\theta_R(z)\underline{\tau} &= (-R + D_{R^*}(1 - ZR^*)^{-1} Z D_R)\underline{\sigma} \,|_{\mathcal{D}_T} \,. \\
&= -\sigma T + D_{R^*}(1 - ZR^*)^{-1} \underline{Z}\,\underline{\sigma} D_T. \\
&= -\sigma T + D_{R^*}(1 - ZR^*)^{-1} \sigma Z D_T \\
&= -\sigma T + \sigma D_{T^*}(1 - ZT^*) Z D_T \\
&= \tau_* \theta_T(z),
\end{aligned}$$

for all $z \in \mathbb{B}^n$. Hence the two characteristic functions θ_T and θ_R coincide. □

Next we prove the converse of the above proposition for the case of pure tuples.

Proposition 4.3. *Let T and R be two pure commuting contractive tuples on $\mathcal{H}$ and $\mathcal{K}$, respectively. If their characteristic functions θ_T and θ_R coincide, then the tuples T and R are unitarily equivalent.*

Proof. Let $\tau' : \mathcal{D}_T \to \mathcal{D}_R$ and $\tau_*^{'} : \mathcal{D}_{T^*} \to \mathcal{D}_{R^*}$ be two unitary operators such that the diagram

$$\begin{tikzcd}
\mathcal{D}_T \arrow[r, "\theta_T(z)"] \arrow[d, "\tau^{'}"'] & \mathcal{D}_{T^*} \arrow[d, "\tau_*^{'}"] \\
\mathcal{D}_R \arrow[r, "\theta_R(z)"'] & \mathcal{D}_{R^*}
\end{tikzcd}$$

commutes for all z in $\mathbb{B}^n$. The operators τ' and τ_*' give rise to unitary operators $\tau = 1 \otimes \tau' : H(\mathbb{C}) \otimes \mathcal{D}_T \to H(\mathbb{C}) \otimes \mathcal{D}_R$ and $\tau_* = 1 \otimes \tau_*' : H(\mathbb{C}) \otimes \mathcal{D}_{T^*} \to H(\mathbb{C}) \otimes \mathcal{D}_{R^*}$ which satisfy the intertwining relation

$$M_{\theta_R} \tau = \tau_* M_{\theta_T}.$$

We conclude that

$$\tau_*(\mathbb{H}_T) = \tau_*((\mathrm{Ran} M_{\theta_T})^\perp) = \tau_*(\mathrm{Ran} M_{\theta_T})^\perp = (\mathrm{Ran} M_{\theta_R})^\perp = \mathbb{H}_R,$$

where $\mathbb{H}_T$ and $\mathbb{H}_R$ are the model spaces for T and R as in Theorem 3.7. Since the operator τ_* interwines the tuples $(M_z \otimes 1_{\mathcal{D}_{T^*}})^*$ and $(M_z \otimes 1_{\mathcal{D}_{R^*}})^*$ componentwise, the induced unitary operators $\tau_* : \mathbb{H}_T \to \mathbb{H}_R$ intertwines the adjoints of the restrictions of these tuples, which are precisely the model tuples $P_{\mathbb{H}_T}(M_z \otimes 1_{\mathcal{D}_{T^*}})|\mathbb{H}_T$ and $P_{\mathbb{H}_R}(M_z \otimes 1_{\mathcal{D}_{R^*}})|\mathbb{H}_R$. But then Theorem 3.7 shows that T and R are unitarily equivalent. □

Summarizing the last two propositions we obtain the main result of this paper.

Theorem 4.4. *Two pure commuting contractive tuples T and R on Hilbert spaces $\mathcal{H}$ and $\mathcal{K}$ are unitarily equivalent if and only if their characteristic functions coincide.*

Let $T \in \mathcal{B}(\mathcal{H})^n$ be a pure commuting contractive tuple on a separable Hilbert space $\mathcal{H}$. Arveson used in [3] the abstract solution of the factorization problem

$$1_{H(\mathbb{C}) \otimes \mathcal{D}_{T^*}} - L^*L = M_\varphi M_\varphi^*$$

to construct an invariant for pure commuting contractive tuples $T \in \mathcal{B}(\mathcal{H})^n$ with finite defect, that is, with $\dim(\mathcal{D}_{T^*}) < \infty$, called the *curvature invariant*. Since we know that the characteristic function θ_T of T can be used for φ, we see that the curvature invariant is completely determined by the characteristic function of T. We end this paper by briefly indicating this connection between the characteristic function and the curvature invariant.

By Corollary 2.3 the characteristic function θ_T is a bounded analytic function with values in $\mathcal{B}(\mathcal{D}_T, \mathcal{D}_{T^*})$ and supremum norm bounded by one. Suppose that the number $d = \dim(\mathcal{D}_{T^*})$ is finite. Then $\mathcal{B}(\mathcal{D}_T, \mathcal{D}_{T^*})$ is topologically isomorphic to a separable Hilbert space, and therefore θ_T has a pointwise radial limit almost everywhere defining a function $\tilde{\theta}_T : \partial\mathbb{B}^n \to \mathcal{B}(\mathcal{D}_T, \mathcal{D}_{T^*})$ belonging to the unit ball of $\mathrm{L}^\infty(\partial\mathbb{B}^n, \mathcal{B}(\mathcal{D}_T, \mathcal{D}_{T^*}))$. Define $k_T : \mathbb{B}^n \to \mathcal{B}(\mathcal{D}_{T^*}, \mathcal{H})$ by

$$k_T(z) = (1 - TZ^*)^{-1} D_{T^*}.$$

It follows from Lemma 2.2 that

$$1 - \theta_T(z)\theta_T(z)^* = (1 - \|z\|^2) k_T(z)^* k_T(z) \quad (z \in \mathbb{B}^n).$$

Using the definition given by Arveson in [3] we obtain the following representation of the curvature invariant of T in terms of the characteristic function

$$\begin{aligned} K(T) &= \lim_{r \uparrow 1} (1 - r^2) \int_S \mathrm{trace}\, k_T(rz)^* k_T(rz) \mathrm{d}\sigma(z) \\ &= \int_S \mathrm{trace}\, (1_{\mathcal{D}_{T^*}} - \tilde{\theta}_T(z) \tilde{\theta}_T(z)^*) \mathrm{d}\sigma(z). \end{aligned}$$

Here $S = \partial\mathbb{B}^n$ is the unit sphere and σ denotes the normalized surface measure on S.

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T. Bhattacharyya
Department of Mathematics, Indian Institute of Science, Bangalore 560012, India
e-mail: tirtha@math.iisc.ernet.in

J. Eschmeier
Fachbereich Mathematik, Universität des Saarlandes, 66123 Saarbrücken, Germany
e-mail: eschmei@math.uni-sb.de

J. Sarkar
Department of Mathematics, Indian Institute of Science, Bangalore 560012, India
e-mail: jaydeb@math.iisc.ernet.in

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