Journal of Mathematical Analysis and Applications **264,** 560–576 (2001)
doi:10.1006/jmaa.2001.7695, available online at http://www.idealibrary.com on IDEAL®

Multiparameter Sturm–Liouville Problems with Eigenparameter Dependent Boundary Conditions

T. Bhattacharyya[1]

Department of Mathematics, Indian Institute of Science, Bangalore 560012, India

P. A. Binding[2]

Department of Mathematics, University of Calgary, Alberta T2N 1N4, Canada

and

K. Seddighi[3]

Department of Mathematics, Shiraz University, Shiraz, Iran

Submitted by Joyce R. McLaughlin

Received December 8, 1998

A system of ordinary differential equations,

$$-y_j'' + q_j y_j = \left(\sum_{k=1}^{n} \lambda_k r_{jk} \right) y_j, \qquad j = 1, \dots, n, \tag{0.1}$$

with real valued and continuous coefficient functions q_j, r_{jk} is studied on $[0,1]$ subject to boundary conditions

$$\frac{y_j'(0)}{y_j(0)} = \cot \beta_j \quad \text{and} \quad b_j y_j(1) - d_j y_j'(1) = \mathbf{e}_j^T \boldsymbol{\lambda} \big(c_j y_j'(1) - a_j y_j(1) \big) \tag{0.2}$$

for $j = 1, \dots, n$. Here $E^T = [\mathbf{e}_1 \ \mathbf{e}_2 \ \cdots \ \mathbf{e}_n]$ is an arbitrary $n \times n$ matrix of real numbers and $\omega_j = a_j d_j - b_j c_j \neq 0$. A point $\boldsymbol{\lambda} = [\lambda_1 \ \cdots \ \lambda_n]^T \in \mathbb{C}^n$ satisfying (0.1) and (0.2) is called an *eigenvalue* of the system. Results are given on the existence and location of the eigenvalues and completeness and oscillation of the eigenfunctions. © 2001 Elsevier Science

[1] This work was done at the University of Calgary while this author was a Killam Post doctoral Fellow.

[2] Research supported by the Natural Sciences and Engineering Research Council of Canada.

[3] Prof. Seddighi passed away shortly after the work was completed. His research was carried out on sabbatical leave at the University of Calgary.

0022-247X/01 $35.00

1. INTRODUCTION

This paper is aimed at two of the many generalizations of the theory of Sturm–Liouville equations,

$$-y'' + qy = \lambda ry \qquad \text{on } [0,1]. \tag{1.1}$$

One involves λ-dependent boundary conditions of the form

$$(a_i\lambda + b_i)y(i) = (c_i\lambda + d_i)y'(i),$$
$$\mathbf{0} \neq [a_i \quad b_i \quad c_i \quad d_i]^T \in \boldsymbol{R}^4, i = 0,1, \tag{1.2}$$

in place of the usual condition for which $a_i = c_i = 0$. Roughly, the theory is in two parts. One concerns "Sturm" (e.g., oscillation and comparison) theory, for which we refer to [10] and the references therein. The other concerns "Liouville" (e.g., completeness and expansion) theory via the spectral decomposition of a self-adjoint operator on a Hilbert or a Pontryagin space. We refer to [20] for an approach in $L_2(\mu)$ (where μ is a partly atomic measure) and for many references. Further approaches in suitably weighted spaces of the form $L_2 \oplus \mathbb{C}^k$, perhaps for more general differential equations and boundary conditions, can be found in [15, 16] and the references therein. We remark that if the boundary condition at $x = 0$ is independent of λ and $r > 0$, then $k = 1$ and $L_2 \oplus \mathbb{C}$ is a Hilbert space (resp. Pontryagin space of index 1) if $\omega_1 = a_1d_1 - b_1c_1 > 0$ (resp. < 0). Both signs are of physical relevance. A Pontryagin space situation also arises if the weight function is indefinite (cf. [7, 13] for applications of indefinite weights).

The second generalization of Sturm–Liouville theory that we shall embrace deals with a "multiparameter" system of differential equations (0.1) involving an eigenvalue $\boldsymbol{\lambda} \in \mathbb{C}^n$. Standard self-adjoint (usually separated) boundary conditions are imposed for each equation, and "Sturm" and "Liouville" theories (as above) are known for a variety of so-called definiteness conditions: for example, if $\det[r_{jk}(x_j)]$ is of one sign for all $x_j \in [0,1]$, then (0.1) is called "right definite." We refer to the books [1, 18, 19] for various aspects of the theory, developed in either Cartesian or tensor products of L_2 spaces. We remark that these spaces are endowed with various inner products, leading to either Hilbert (as cited) or Pontryagin spaces (cf. [12]).

Our goal here is to consider multiparameter systems (0.1) subject to boundary conditions (0.2) generalizing (1.2) for $i = 1$. (Using similar methods, we could also consider $\boldsymbol{\lambda}$-dependent boundary conditions for $i = 0$.) We are aware of only two papers on this topic, [3, 14]. Browne and Sleeman considered a natural generalization of right-definiteness, using an abstract approach based on a tensor product H of $L_2(\mu)$ spaces of the type used by Walter in the case $n = 1$. A spectral decomposition was obtained via the joint spectral measure of certain operators Γ_k in H, but these operators are rather indirectly related to the original data in (0.1) and the boundary conditions. Here we shall apply abstract methods in tensor products of $L_2 \oplus \mathbb{C}$ spaces, to deal with various different definiteness conditions. We remark that even if the analogue of $\omega = ad - bc$ in just one boundary condition is of opposite sign from the situation in [14], then the Hilbert space tensor product setting is replaced by a Krein (not Pontryagin) space K. In principle the analysis then involves commuting self-adjoint operators in K, and their theory is not well developed. We have therefore taken a route involving fundamental symmetries, enabling us to consider operators in a Hilbert or Pontryagin space instead.

In Section 2 we set up our assumptions and the various definiteness conditions in terms of quadratic forms in a Cartesian product of spaces $L_2 \oplus \mathbb{C}$ with Hilbert or Pontryagin space inner products. The passage to tensor products is carried out in Section 3, along with the main completeness results. Section 4 details the special case $n = 2$; our definiteness conditions are then expressible in a simpler way in terms of the original data. As a consequence of our abstract setting, we are able to generalize our work on a special right-definite case in [3], and we give results on the location and asymptotics of eigenvalues as well as the oscillation of eigenfunctions.

2. PRELIMINARIES

In the notation of the Abstract, the system can be suitably scaled to make $\omega_j = \pm 1$, and we shall assume that this scaling has been done. As a referee pointed out, one could in fact choose each $\omega_j = 1$, but that would change the e_{jk} and hence the V_{jk}—see (2.2). We have chosen to make use of established work involving "definiteness conditions" (cf. Definitions 2.2 and 2.3) on the V_{jk}, accepting the possibility that some $\omega_j = -1$, rather than proceeding ab initio. See also Remark 4.2.

After a possible reordering of the equations we may then assume that there is a positive number $n_0 \le n$ such that $\omega_j = 1$ for $j \le n_0$ and $\omega_j = -1$ for $j > n_0$. Consider n copies of $L^2[0,1] \oplus \mathbb{C}$, each with two different inner products. For $j = 1, \dots, n$ let K_j be the vector space direct sum $L^2[0,1] \oplus \mathbb{C}$ with two inner products defined by

$$(\mathbf{Y}, \mathbf{Z})_{j\pm} = \int_0^1 y\bar{z} \pm \alpha\bar{\beta},$$

where

$$\mathbf{Y} = \begin{pmatrix} y \\ \alpha \end{pmatrix}, \qquad \mathbf{Z} = \begin{pmatrix} z \\ \beta \end{pmatrix}$$

belong to K_j. For $j \leq n_0$, we shall need only the $(.,.)_+$ inner product. We shall omit the subscript j when there is no chance of confusion. Note that K_j is a Hilbert space K_{j+} under $(.,.)_+$ and a Pontryagin space K_{j-} of index 1 under $(.,.)_-$. All topological concepts on K_j will be understood with respect to the $(.,.)_+$ inner product.

Let AC be the subspace of $L_2[0,1]$ consisting of absolutely continuous functions. We introduce two linear functionals on AC by

$$P_j(y) = b_j y(1) - d_j y'(1) \quad \text{and} \quad Q_j(y) = a_j y(1) - c_j y'(1), \qquad \text{for } y \in AC.$$

For each $j, k = 1, \dots n$ one can then consider the unbounded operators T_j and the bounded operators V_{jk} on K_j,

$$D(T_j) = \left\{ \mathbf{Y} \in K_j : y, y' \in AC, -y'' + q_j y \in L^2[0,1], \right. \\ \left. y'(0) = \cot \beta_j y(0), \alpha = -Q_j(y) \right\}$$

and

$$T_j \mathbf{Y} = \begin{pmatrix} -y'' + q_j y \\ b_j y(1) - d_j y'(1) \end{pmatrix} = \begin{pmatrix} -y'' + q_j y \\ P_j(y) \end{pmatrix} \tag{2.1}$$

for $\mathbf{Y} \in D(T_j)$, while

$$V_{jk} \mathbf{Y} = \begin{pmatrix} r_{jk} y \\ e_{jk} \alpha \end{pmatrix} \qquad \text{for } \mathbf{Y} \in K_j, \tag{2.2}$$

where the coefficients come from the Abstract; in particular $e_{jk} = (\mathbf{e_j})_k$. Then the system (0.1), (0.2) is equivalent to

$$\left(T_j - \sum_{k=1}^{n} V_{jk} \lambda_k \right) \mathbf{Y}_j = 0, \qquad j = 1, \dots, n, \tag{2.3}$$

where $\mathbf{Y}_j \in D(T_j)$. The operators T_j are densely defined for all $j = 1, \dots, n$.

LEMMA 2.1. *The operators T_j are selfadjoint for $j \leq n_0$ in K_{j+} and for $j > n_0$ in K_{j-}.*

Proof. For $j \leq n_0$, the T_j are Hilbert space operators and the self-adjointness was proved by Fulton (see [16]). For $j > n_0$, let J_j be the fundamental symmetry of the Krein space K_{j-} given by $J_j(y, \beta) = (y, -\beta)$. For $\mathbf{X}, \mathbf{Y} \in K_{j-}$, the operator J_j satisfies

$$J_j^2 = I, \qquad (\mathbf{X}, \mathbf{Y})_- = (J_j\mathbf{X}, \mathbf{Y})_+ = (\mathbf{X}, J_j\mathbf{Y})_+, \qquad (\mathbf{X}, \mathbf{Y})_+ = (J_j\mathbf{X}, \mathbf{Y})_-. \tag{2.4}$$

Define a Hilbert space operator $\tilde{T}_j$ on K_{j+} by (2.1) for $\mathbf{Y} \in D(\tilde{T}_j)$ where

$$D(\tilde{T}_j) = \{\mathbf{Y} \in K_j : y, y' \in AC, -y'' + q_j y \in L^2[0,1],$$
$$y'(0) = \cot \beta_j y(0), \alpha = Q_j(y)\}.$$

Then $D(\tilde{T}_j) = J_j D(T_j)$ and $T_j\mathbf{Y} = \tilde{T}_j J_j \mathbf{Y}$ for $\mathbf{Y} \in D(T_j)$. Now the result of Fulton in [16] applies to the operator $\tilde{T}_j$. Thus $\tilde{T}_j = T_j J_j$ is self-adjoint in K_{j+}, so T_j is self-adjoint in K_{j-}. ∎

The bounded operators V_{jk} are hermitian because r_{jk} are real-valued and e_{jk} are real. The corresponding quadratic forms (taken with respect to the Hilbert space inner product for $j \le n_0$ and the Pontryagin space inner product for $j > n_0$) are denoted by v_{jk}. For $\mathbf{Y} = (\mathbf{Y}_1, \dots, \mathbf{Y}_n)$, we set $v_{jk}(\mathbf{Y}) = v_{jk}(\mathbf{Y}_j)$, $\delta_0(\mathbf{Y}) = \det[v_{jk}(\mathbf{Y})]$, and $\delta_{0jk}(\mathbf{Y}) =$ the cofactor of $v_{jk}(\mathbf{Y})$ in $\delta_0(\mathbf{Y})$. The definiteness assumptions we shall consider with respect to the Hilbert and Pontryagin inner products are as follows. In the next section we shall relate them to Krein space definiteness conditions. Let $U_j = \{\mathbf{Y}_j \in K_j : \|\mathbf{Y}_j\| = 1\}$ and $U = U_1 \times \cdots \times U_n$.

DEFINITION 2.2. Uniform Right Definiteness (URD).

$$\text{For some } \gamma > 0 \text{ and for each } \mathbf{Y} = (\mathbf{Y}_1, \dots, \mathbf{Y}_n) \in U, \qquad \delta_0(\mathbf{Y}) > \gamma. \tag{2.5}$$

DEFINITION 2.3. Uniform Ellipticity (UE).

$$\text{For some } \gamma > 0 \text{ and for each } j, k \text{ and } \mathbf{Y} \in U, \qquad \delta_{0jk}(\mathbf{Y}) \ge \gamma. \tag{2.6}$$

LEMMA 2.4. *The quadratic forms t_j and v_{jk} can be expressed as*

$$t_j(\mathbf{Y}) \stackrel{\text{def}}{=} \langle T_j\mathbf{Y}_j, \mathbf{Y}_j\rangle_\sigma = \int_0^1 |y_j'|^2 + \int_0^1 q_j|y_j|^2 + [y_j(0)]^2 \cot \beta_j + \sigma D_j(y_j),$$
$$\text{for } \mathbf{Y}_j \in D(T_j),$$

where $\sigma = 1$ for $j \le n_0$, $\sigma = -1$ for $j > n_0$ and

$$D_j(y_j) = \begin{bmatrix} y_j(1) & y_j'(1) \end{bmatrix} \begin{pmatrix} -a_j b_j & b_j c_j \\ b_j c_j & -c_j d_j \end{pmatrix} \begin{bmatrix} y_j(1) \\ y_j'(1) \end{bmatrix}.$$

Moreover,

$$v_{jk}(\mathbf{Y}) = \int_0^1 r_{jk}|y_j|^2 + \sigma e_{jk}|\alpha_j|^2 \qquad \textit{for all } \mathbf{Y} \in K_j.$$

Proof. The expression for v_{jk} follows from the definition. For t_j, we write

$$t_j(\mathbf{Y}) = -\int_0^1 y_j'' y_j + \int_0^1 q_j|y_j|^2 \pm \big(b_j y_j(1) - d_j y_j'(1)\big)\big(c_j y_j'(1) - a_j y_j(1)\big),$$

with $+$ for $j \le n_0$ and $-$ for $j > n_0$. On simplification, using $a_j d_j - b_j c_j = +1$ or -1 depending on j, this is equal to the expression displayed above for t_j. ∎

Remark 2.5. The above expression for t_j is valid throughout the form domain of T_j (see [9] for details).

A strengthening of (2.6) is *uniform left definiteness*. Both uniform ellipticity and left definiteness arise in a variety of natural problems (see [18] and the references therein) and in several forms [4].

DEFINITION 2.9. Uniform Left Definiteness (ULD).

UE holds and for some $\gamma > 0$ and for each j and $\mathbf{Y} \in U$ with $\mathbf{Y}_j \in D(T_j)$,

$$t_j(\mathbf{Y}) \ge \gamma. \tag{2.7}$$

3. A KREIN SPACE FORMULATION

Let K_+ be the Hilbert space tensor product of the spaces $K_{1+}, \dots, K_{n+}$ and let K_- be the tensor product of the K_{j+} for $j \le n_0$ and K_{j-} for $j > n_0$. The inner products on K_+ and K_- will be denoted by $(.,.)_+$ and $(.,.)_-$, respectively. If $n_0 < n$, then K_- is a Krein (and not a Pontryagin) space. Its fundamental symmetry is $J = \otimes_{j=1}^n J_j$, where, for notational simplicity, we let J_j be the identity on K_j for $j \le n_0$. The operator J now satisfies an analogue of (2.4). For any $j = 1, \dots, n$ and any operator, say A, on K_j (or on the Krein space K_-), $\tilde{A}$ will denote the Hilbert space operator AJ_j (or AJ) on the domain $J_j D(A)$ (respectively $JD(A)$). Hence if $1 \le j \le n_0$, $\tilde{A} = A$.

The operators T_j and V_{jk} induce in a natural way operators $T_j^\dagger$ and $V_{jk}^\dagger$ on K. Note that $V_{jk}^\dagger$ and $V_{rs}^\dagger$ commute whenever $j \ne r$, and hence we can define the $n \times n$ determinant

$$\Delta_0 = \det\big[V_{jk}^\dagger\big]. \tag{3.1}$$

Henceforth, we shall assume that Δ_0 is one to one. This assumption could be weakened (cf. [12]). The cofactor of $V_{jk}^{\dagger}$ in Δ_0 is denoted by Δ_{0jk}. For each $k = 1, \ldots, n$, let the unbounded operator Δ_k be the closure of $\Delta_k|_D = \sum_{j=1}^n \Delta_{0jk} T_j^{\dagger}$ on domain $D = \bigcap_{j=1}^n D(T_j^{\dagger})$. The following lemma gives equivalent formulations of URD, UE, and ULD in terms of Δ_k. We use $\gg 0$ to denote uniform positive definiteness.

LEMMA 3.1. *The operators Δ_k are densely defined and self-adjoint in K_-, and* (2.5), (2.6), *and* (2.7) *are equivalent to*

$$\Delta_0 \gg 0, \tag{3.2}$$

$$\Delta_{0jk} \gg 0, \qquad \text{for } j, k = 1, \ldots, n, \tag{3.3}$$

and

$$(3.3) \text{ together with } T_j \gg 0 \text{ on } D(T_j), \tag{3.4}$$

respectively.

Proof. $\tilde{\Delta}_0$ is bounded and self-adjoint on K_+, and hence so is Δ_0 on K_-. Consider the operator $\tilde{\Delta}_k = \Delta_k J = \sum_{j=1}^n \tilde{\Delta}_{0jk} T_j^{\dagger} J = \sum_{j=1}^n \tilde{\Delta}_{0jk} T_j^{\dagger} J_j = \sum_{j=1}^n \tilde{\Delta}_{0jk} \tilde{T}_j^{\dagger}$ with domain $JD(\Delta_k)$. Note that all $\tilde{\Delta}_{0jk}$ and $\tilde{T}_j^{\dagger}$ are densely defined and self-adjoint on K_+ under UE [12]. Hence Δ_k is densely defined and self-adjoint on K_- by an argument similar to the proof of Lemma 2.1.

Next note that for all decomposable tensors $\mathbf{Y}$, $\det[(\tilde{V}_{jk}\mathbf{Y}, \mathbf{Y})_+] = \det[(V_{jk}\mathbf{Y}, \mathbf{Y})_-] = \delta_0(\mathbf{Y})$. By [4], $\delta_0(\mathbf{Y}) \geq \gamma \geq 0$ is thus equivalent to $\tilde{\Delta}_0 \gg 0$. This in turn is the same as $\Delta_0 \gg 0$, which shows that (2.5) and (3.2) are equivalent. The other conclusions follow similarly. ∎

Let PCI(K_-) (resp. K_+) be the set of all operators on K_- (resp. K_+), which have self-adjoint, positive, compact inverses. As a consequence of Lemma 3.1, Δ_0 has a positive bounded inverse under URD. In the next lemma, we use linear transformations of the eigenvalues given by

$$\lambda'_m = \sum_{k=1}^n \alpha_{mk} \lambda_k, \qquad \text{for } m = 1, \ldots, n,$$

where the $n \times n$ matrix $[\alpha_{mk}]$ is non-singular.

LEMMA 3.2. *Under ULD and UE, there are non-singular linear transformations of the eigenvalues, such that under ULD, each $\Delta_k \in PCI(K_-)$ and under UE, after an additional translation of each λ_j by ϵ, the Δ_k are bounded below and $\Delta_k + \epsilon\Delta_0 \in PCI(K_-)$ for some $\epsilon > 0$.*

Proof. First note that an operator $A \in \mathrm{PCI}(K_-)$ if and only if $\tilde{A} \in \mathrm{PCI}(K_+)$. The first assertion then follows by appealing to Theorem 3.3 of [5], with Δ_n there replaced by $\tilde{\Delta}_k$ in our case.

The ellipticity condition (3.3) assumes the form $\tilde{\Delta}_{0jk} \gg 0$ in the Hilbert space K_+, and then Theorem 2.5 in [12] implies the second conclusion. ∎

For $k = 1, \ldots, n$, having applied the transformation of Lemma 3.2 (and omitting primes), we define $B_k = \Delta_k^{-1}\Delta_0$ when ULD or UB holds. Theorem 6.1 of [5] and Theorem 3.2 of [12] show that the eigenvalues of the system (0.1), (0.2) are equivalent to

$$\mathbf{Y} = \lambda_k B_k \mathbf{Y}, \quad 0 \neq \mathbf{Y} \in K_-, \qquad k = 1, \ldots, n.$$

Hence the non-zero multiparameter eigenvalues under ULD or UE are the componentwise reciprocals of the joint eigenvalues of the operators $\Delta_k^{-1}\Delta_0$.

When URD holds, we let $\Gamma_k = \Delta_0^{-1}\Delta_k$. Then the multiparameter eigenvalues are the joint eigenvalues of $\Gamma = (\Gamma_1, \ldots, \Gamma_n)$ because the system (0.1), (0.2) is equivalent to

$$\Gamma_k \mathbf{Y} = \lambda_k \mathbf{Y}, \quad 0 \neq \mathbf{Y} \in K_-, \qquad k = 1, \ldots, n.$$

THEOREM 3.3. (i) *When ULD holds, $B = (B_1, \ldots, B_n)$ is a commuting tuple of compact operators in K_-. Moreover, the eigenvalues of the system* (0.1), (0.2) *are the non-zero eigenvalues of B_k and are real with finite multiplicity and with no accumulation points. The eigenvectors of* (0.1), (0.2) *generate decomposable tensors forming a set S of joint eigenvectors of the B_k, complete in K_-.*

(ii) *In the UE case, all but finitely many of the joint eigenvalues of the operator tuple $(B_1, \ldots, B_n)$ are in $\boldsymbol{R}^n$. Moreover, there is a vector space direct sum decomposition $K_- = F \oplus G$ where F is finite-dimensional and invariant under each B_k and the set S of joint eigenvectors of B_k is complete in G.*

(iii) *When URD holds, $\Gamma = (\Gamma_1, \ldots, \Gamma_n)$ is a commuting tuple, and its joint eigenvectors are complete in K_-.*

Proof. (i) Consider the operators $\tilde{B}_k = \tilde{\Delta}_k^{-1}\tilde{\Delta}_0$ which were introduced for Hilbert spaces in [5]. It is known that under the ULD assumption, the $\tilde{B}_k$ are commuting compact operators (see [5, Theorem 4.2]). In the Krein space formulation, using the fact that the fundamental symmetry J is an idempotent, we have $\tilde{B}_k$ equal to $\Delta_k^{-1}\Delta_0 = B_k$. Thus B is a commuting compact operator tuple. Suppose the non-singular eigenvalue transformations of Lemma 3.2 have been performed. Let $D_k = D(\tilde{\Delta}_k^{1/2})$ under the inner product $[\cdot\,,\cdot\,]_k$ given by

$$[x, y]_k = \left(\tilde{\Delta}_k^{1/2} x, \tilde{\Delta}_k^{1/2} y\right)_+, \qquad k = 1, \ldots, n,\ x, y \in K_+.$$

Then complete orthonormal bases for the D_k, $k = 1, \dots, n$, may be chosen from the set S (see again [5] and recall that Δ_0 is one-one). Since D_k is dense in K_+, the set S is complete in K_-.

(ii) The proof is similar to that of (i), replacing [5, Theorem 4.2] by [12, Theorem 3.1] and translating each λ_j by an $\epsilon > 0$ if necessary (see Lemma 3.2). Then D_k is defined via $\tilde{\Delta}_k + \epsilon\tilde{\Delta}_0$, and the existence of F and G as stated follow from [12].

(iii) Under URD, we consider $\tilde{\Delta}_0$, which is a bounded self-adjoint operator on K_+. Equip K_+ with the inner product $[\cdot\,,\cdot\,]_0$, which is defined by

$$[\mathbf{X}, \mathbf{Y}]_0 = \left(\tilde{\Delta}_0 \mathbf{X}, \mathbf{Y}\right)_+, \qquad \mathbf{X}, \mathbf{Y} \in K_+.$$

Then the set of joint eigenvectors of $\Gamma_k = \tilde{\Delta}_0^{-1}\tilde{\Delta}_k = \Delta_0^{-1}\Delta_k$ forms a complete orthonormal basis in K_+ with respect to the inner product above (see [19, Theorem 6.5.4]).

Hence in all of the above cases, the multiparameter eigenvectors are complete in the specified spaces. ∎

Remark 3.4. From the above proof, we note that the joint eigenvectors are also orthogonal in the inner product $[\cdot\,,\cdot\,]_k$ under ULD or UE and in the product $[\cdot\,,\cdot\,]_0$ under URD.

4. THE TWO-PARAMETER CASE

In this section we consider the problem (0.1), (0.2) with $n = 2$, assuming $\delta_0(\mathbf{Y}) \neq 0$ for $\mathbf{Y} \neq 0$. This problem has been studied under the conditions $E = I_2$, $\omega_j > 0$ and $\det[r_{ij}] > 0$ for all $y_1, y_2 \in L_2[0,1]$ in [3], providing existence, location, asymptotics, and perturbation of the eigenvalues λ_j and oscillation of the eigenfunctions y_i. Here we generalize and extend the results of [3]. By continuity of the r_{jk}, definiteness of δ_0 implies uniform definiteness, so after an affine transformation of the eigenvalues, UE holds [4]. In the sequel, we shall assume that this transformation has been performed. Sign-definiteness of r_{jk} and e_{jk} is a consequence of the UE condition, as follows.

LEMMA 4.1. *For each j and k, $(-1)^{j+k} r_{jk}(x) > 0$ for $x \in [0,1]$ and $(-1)^{j+k}\omega_j e_{jk} > 0$. In particular, $s_j = -e_{j1}/e_{j2}$ are distinct positive numbers for $j = 1, 2$.*

Proof. If $\mathbf{Y} = \mathbf{Y}_1 \otimes \mathbf{Y}_2$ is a decomposable tensor, then the cofactor of $v_{jk}(\mathbf{Y})$ in $\delta_0(\mathbf{Y})$ is $(-1)^{j+k}(r_{jk}(y_j) + \omega_j e_{jk}|\alpha_j|^2)$. Since UE holds, there is a

$\gamma > 0$, such that

$$(-1)^{j+k}\Big(r_{jk}(y_j) + \omega_j e_{jk}|\alpha_j|^2\Big) > \gamma \qquad \text{for all } y_j \text{ and } \alpha_j.$$

Thus,

$$(-1)^{j+k} r_{jk}(y_j) > \gamma \qquad \text{for all } y_j \in L_2[0,1]$$

and

$$(-1)^{j+k}\,\omega_j e_{jk}|\alpha_j|^2 > \gamma \qquad \text{for all } \alpha_j \in \mathbb{C}.$$

This is equivalent to $(-1)^{j+k}r_{jk}(x) > 0$ for $x \in [0,1]$ and $(-1)^{j+k}\omega_j e_{jk} > 0$. Positivity in the final contention is then immediate. Now let $\mathbf{U} = \binom{0}{1} \otimes \binom{0}{1}$. Evidently, $\Delta_0 \mathbf{U} = (\det E)\mathbf{U}$. So since Δ_0 is one-one, we have $\det E \neq 0$, and distinctness of s_j follows. ∎

Remark 4.2. It follows that, after transformation as above, UE dictates the sign of each $\omega_j e_{jk}$. By allowing ω_j to take both signs we increase the possibilities for the e_{jk}.

Our method of analysis will depend on the following lemma, which will, in particular, enable us to define the eigencurves.

LEMMA 4.3. *There are two sequences* $\lambda_{20}^1(\lambda_1) > \lambda_{21}^1(\lambda_1) > \cdots$ *and* $\lambda_{20}^2(\lambda_1) < \lambda_{21}^2(\lambda_1) < \cdots$ *of differentiable, monotone increasing functions of* λ_1 *and two sequences of eigenfunctions* y_{1k} *and* y_{2k} *such that for each integer* $k \geq 0$ *the pair* $(\lambda_1, \lambda_{2k}^i(\lambda_1))$ *and the function* y_{jk} *satisfy* (0.1), (0.2) *with* $n = 2$.

Moreover, the derivatives of λ_{2k}^i *are given by*

$$\frac{d\lambda_{2k}^1}{d\lambda_1} = -\frac{v_{11}(\mathbf{Y}_{1k})}{v_{12}(\mathbf{Y}_{1k})} \quad \textit{and} \quad \frac{d\lambda_{2k}^2}{d\lambda_1} = -\frac{v_{21}(\mathbf{Y}_{2k})}{v_{22}(\mathbf{Y}_{2k})}, \tag{4.1}$$

where

$$\mathbf{Y}_{jk} = \begin{pmatrix} y_{jk} \\ \alpha \end{pmatrix} \in D(T_j).$$

Proof. First rewrite (2.3) for $j = 2$ in the form

$$\big(V_{22}^{-1}T_2 - \lambda_1 V_{22}^{-1}V_{21} - \lambda_2\big)Y_2 = 0 \tag{4.2}$$

involving self-adjoint operators $V_{22}^{-1}T_2$ and $V_{22}^{-1}V_{21}$ on K_2 with a new inner product defined by $(.\,,.)_{22} = (.\,, V_{22}\,\cdot\,)$. Applying the results of [6], we obtain $(\lambda_1, \lambda_{2j}^2)$ eigencurves with eigenvectors $Y_{2j}^2(\lambda_1)$ and with

$$\frac{d\lambda_{2j}^i}{d\lambda_1} = -\big(V_{22}^{-1}V_{21}Y_{2j}^2, Y_{2j}^2\big)_{22}\big/\big(Y_{2j}^2, Y_{2j}^2\big)_{22} = -v_{21}\big(Y_{2j}^2\big)/v_{22}\big(Y_{2j}^2\big). \tag{4.3}$$

Strictness of the inequalities follows from the standard theory (cf. [10]). For the other derivative, one has to carry out a similar analysis for $j = 1$ with the roles of λ_1 and λ_2 interchanged and then take the reciprocal. ∎

The graphs of the functions λ_{2k}^j obtained in the above lemma for $j = 1, 2$ and $k = 1, 2, \ldots$ are called the *eigencurves*.

Remark 4.4. In the special case when the matrix E is the identity we obtain, for $\mathbf{Y}_j \in D(T_j)$,

$$v_{jk}(\mathbf{Y}_j) = r_{jk}(y_j) + \frac{1}{\omega_j} e_{jk}\big(a_j y_j(1) - c_j y_j'(1)\big)^2$$

$$= \begin{cases} r_{jk}(y_j) & \text{if } j \neq k \\ r_{jj}(y_j) + \dfrac{\omega_j^2 y_j^2(1)}{(c_j\lambda_j + d_j)^2} & \text{if } j = k, \end{cases}$$

by virtue of the end condition at 1. When inserted into (4.1), these expressions coincide with the formulae obtained in [3] by ordinary differential equation methods.

The following two quantities and the asymptotic results following them are useful in analyzing the eigencurves

$$M_j = \sup\{-r_{j1}(x)/r_{j2}(x) : 0 \le x \le 1\} \quad \text{and}$$

$$m_j = \inf\{-r_{j1}(x)/r_{j2}(x) : 0 \le x \le 1\} \qquad \text{for } j = 1, 2.$$

LEMMA 4.5. *All M_j and m_j are finite and*

$$\lim_{\lambda_1 \to \infty} \frac{\lambda_{2m}^1(\lambda_1)}{\lambda_1} = M_1 \quad \textit{for } m > 0 \quad \textit{and} \quad \lim_{\lambda_1 \to \infty} \frac{\lambda_{2n}^2(\lambda_1)}{\lambda_1} = m_2 \quad \textit{for } n > 0$$

$$\lim_{\lambda_1 \to -\infty} \frac{\lambda_{2m}^1(\lambda_1)}{\lambda_1} = m_1 \quad \textit{for } m > 0 \quad \textit{and}$$

$$\lim_{\lambda_1 \to -\infty} \frac{\lambda_{2n}^2(\lambda_1)}{\lambda_1} = M_2 \quad \textit{for } n > 0$$

$$\lim_{\lambda_1 \to \infty} \frac{\lambda_{20}^1(\lambda_1)}{\lambda_1} = \max\{s_1, M_1\} \quad \textit{and} \quad \lim_{\lambda_1 \to \infty} \frac{\lambda_{20}^2(\lambda_1)}{\lambda_1} = \min\{s_2, m_2\}$$

$$\lim_{\lambda_1 \to -\infty} \frac{\lambda_{20}^1(\lambda_1)}{\lambda_1} = \min\{s_1, m_1\} \quad \textit{and} \quad \lim_{\lambda_1 \to -\infty} \frac{\lambda_{20}^2(\lambda_1)}{\lambda_1} = \max\{s_2, M_2\},$$

where s_j are defined in Lemma 4.1.

Proof. The M_j and m_j are finite because the r_{jk} are continuous on $[0,1]$ and hence are bounded. Consider the first equation, i.e., $j = 1$ in (2.3). It is equivalent to the Hilbert space equation

$$\left(\tilde{T}_1 - \lambda_1 \tilde{V}_{11} - \tilde{V}_{12}\right)\mathbf{Y}_1 = 0 \qquad \text{for } \mathbf{Y}_1 \in K_{1+}.$$

Since $\tilde{V}_{11} \gg 0$ and $-\tilde{V}_{12} \gg 0$, it follows that the eigencurve corresponding to λ_{2l}^1 has two asymptotic directions forming ∂C_l^1, where C_l^1 is the cone of λ satisfying

$$0 < \sigma_l^1(\boldsymbol{\lambda}) \overset{\text{def}}{=} \sup_{\dim E = l} \inf_{\mathbf{Y} \in E^{\perp} \cap U} \left(\left(\lambda_1 \tilde{V}_{11} + \lambda_2 \tilde{V}_{12}\right)\mathbf{Y}, \mathbf{Y}\right)_+ \qquad \text{for } l = 1, 2, \dots .$$

(See the discussion following Lemma 6.3 in [6].) The spectrum of $\lambda_1 \tilde{V}_{11} + \lambda_2 \tilde{V}_{12}$ is

$$\sigma\left(\lambda_1 \tilde{V}_{11} + \lambda_2 \tilde{V}_{12}\right) = \{\omega_1(\lambda_1 e_{11} + \lambda_2 e_{12})\} \cup \left\{\lambda_1 r_{11}(x) + \lambda_2 r_{12}(x) : 0 \le x \le 1\right\},$$

so

$$\sigma_l^1(\boldsymbol{\lambda}) = \inf\left\{\lambda_1 r_{11}(x) + \lambda_2 r_{12}(x) : 0 \le x \le 1\right\} \qquad \text{for all } l = 1, 2, \dots$$

and

$$\sigma_0^1(\boldsymbol{\lambda}) = \min\left\{\omega_1(\lambda_1 e_{11} + \lambda_2 e_{12}), \inf\left\{\lambda_1 r_{11}(x) + \lambda_2 r_{12}(x) : 0 < x < 1\right\}\right\}.$$

Hence the results follow. The case $j = 2$ in (2.3) is similar. ▮

LEMMA 4.6. (i) *If* δ_0 *is positive*, *then* $\max\{s_2, M_2\} < \min\{s_1, m_1\}$.

(ii) *If* δ_0 *is negative*, *then* $\max\{s_1, M_1\} < \min\{s_2, m_2\}$.

Proof. Suppose δ_0 is positive. By virtue of Lemma 2.4, we have

$$\delta_0(\mathbf{Y}) = \left(r_{11}(y_1) + \omega_1 e_{11}|\alpha_1|^2\right)\left(r_{22}(y_2) + \omega_2 e_{22}|\alpha_2|^2\right) - \left(r_{12}(y_1) + \omega_1 e_{12}|\alpha_1|^2\right)\left(r_{21}(y_2) + \omega_2 e_{21}|\alpha_2|^2\right)$$

for all $y_1, y_2 \in L_2[0,1]$ and all $\alpha_1, \alpha_2 \in \mathbb{C}$. Choosing $\alpha_1 = \alpha_2 = 0$ we get $r_{11}(y_1)/r_{12}(y_1) > r_{21}(y_2)/r_{22}(y_2)$, which gives $m_1 > M_2$. Similarly choosing $y_2 = 0 = \alpha_1$ gives $m_1 > s_2$. Thus $m_1 > \max\{s_2, M_2\}$. Choosing $y_1 = y_2 = 0$ and $y_1 = 0 = \alpha_2$ gives, respectively, $s_1 > s_2$ and $s_1 > M_2$. The case of δ_0 negative is analogous. ▮

We are now ready for our basic existence and uniqueness result.

THEOREM 4.7. *The system* (0.1), (0.2) *has countably many two-parameter eigenvalues. For each non-negative integer pair* $\mathbf{n} = (n_1, n_2)$, *there is a unique eigenvalue* $\boldsymbol{\lambda}^{\mathbf{n}}$ *on the* n_i*th eigencurve of equation* i $(i = 1, 2)$.

Proof. We give the proof for δ_0 positive. The proof for the other case is similar. For any k, we have

$$\begin{aligned}\frac{d\lambda_{2k}^1}{d\lambda_1} &= -\frac{v_{11}(\mathbf{Y}_{1k})}{v_{12}(\mathbf{Y}_{1k})} \qquad \text{from Lemma 4.3}\\ &= \frac{r_{11}(y_{1k}) + \omega_1 e_{11}(c_1 y_{1k}'(1) - a_1 y_{1k}(1))^2}{-r_{12}(y_{1k}) - \omega_1 e_{12}(c_1 y_{1k}'(1) - a_1 y_{1k}(1))^2}\\ &\geq \frac{m_1(-r_{12}(y_{1k})) + s_1(-\omega_1 e_{12})(c_1 y_{1k}'(1) - a_1 y_{1k}(1))^2}{-r_{12}(y_{1k}) - \omega_1 e_{12}(c_1 y_{1k}'(1) - a_1 y_{1k}(1))^2}\\ &\geq \min\{s_1, m_1\}.\end{aligned}$$

Similarly $\max\{s_2, M_2\} \geq d\lambda_{2l}^2/d\lambda_1$ for any l. Thus by Lemma 4.6, if we plot $\lambda_{2n_1}^1$ and $\lambda_{2n_2}^2$ against λ_1, then these two curves meet exactly once, say at $\lambda_1^{\mathbf{n}}$. We denote the point $\lambda_{2n_1}^1(\lambda_1^{\mathbf{n}}) = \lambda_{2n_2}^2(\lambda_1^{\mathbf{n}})$ by $\lambda_2^{\mathbf{n}}$. Then it follows that $(\lambda_1^{\mathbf{n}}, \lambda_2^{\mathbf{n}})$ satisfy (0.1) and (0.2) with eigenfunctions given by $y_1(x) = y_{1n_1}(x, \boldsymbol{\lambda}^{\mathbf{n}})$ and $y_2(x) = y_{2n_2}(x, \boldsymbol{\lambda}^{\mathbf{n}})$. To complete the proof, we note that two eigencurves from the same equation (say $i = 2$) cannot intersect. For if they did, then the strict inequalities in Lemma 4.3 would be violated. So to each pair of eigencurves $(\lambda_{2n_1}^1, \lambda_{2n_2}^2)$ there corresponds the unique eigenvalue $\boldsymbol{\lambda}^{\mathbf{n}}$. ∎

By changing the $\boldsymbol{\lambda}$ origin to $-E^{-1}[d_1/c_1 \;\; d_2/c_2]^T$, we may (and shall) assume in what follows that $d_1 = d_2 = 0$. We define the *continuous cone* C_{c} for $\delta_0 > 0$ to be the cone of all points in the first quadrant of the $\boldsymbol{\lambda}$-plane such that $m_2 \leq \lambda_2/\lambda_1 \leq M_1$ and for $\delta_0 < 0$ to be the cone of all points in the third quadrant of the $\boldsymbol{\lambda}$-plane such that $m_1 \leq \lambda_2/\lambda_1 \leq M_2$. The *discrete cone* C_{d} consists of the two rays

$$\{\lambda_1 > 0,\ \lambda_2 = \max\{s_1, M_1\}\lambda_1\} \quad \text{and} \quad \{\lambda_1 > 0,\ \lambda_2 = \min\{s_2, m_2\}\lambda_1\} \qquad \text{for } \delta_0 > 0$$

and the two rays

$$\{\lambda_1 < 0,\ \lambda_2 = \min\{s_1, m_1\}\lambda_1\} \quad \text{and} \quad \{\lambda_1 < 0,\ \lambda_2 = \max\{s_2, M_2\}\lambda_1\} \qquad \text{for } \delta_0 < 0.$$

The union of C_c and C_d will be denoted by C. The *asymptotic spectrum*, denoted by AS, is the closure in S^1 of the set $\{\boldsymbol{\lambda}/\|\boldsymbol{\lambda}\|: \boldsymbol{\lambda}$ is an eigenvalue of the system (0.1), (0.2)$\}$, where S^1 denotes the unit circle.

The following result is a generalization of [3, Theorem 5.1] to the definiteness conditions used here.

THEOREM 4.8. *If the r_{jk} are absolutely continuous then, in the above notation, $AS = C \cap S^1$.*

Proof. Let AS_d denote the set of accumulation points of the sequences $\boldsymbol{\lambda}^{(0,n_2)}/\|\boldsymbol{\lambda}^{(0,n_2)}\|$ and $\boldsymbol{\lambda}^{(n_1,0)}/\|\boldsymbol{\lambda}^{(n_1,0)}\|$. From Lemma 4.5 and the definition of C_d, we see that

$$AS_d = C_d \cap S^1. \tag{4.4}$$

Now let AS_c be the set of accumulation points of $\boldsymbol{\lambda}^{\boldsymbol{n}}/\|\boldsymbol{\lambda}^{\boldsymbol{n}}\|$ for $n_1, n_2 > 0$. By [10, Theorems 3.6, 5.3] the eigenvalues of the one parameter problem in λ_1 given by (0.1), (0.2) with $j = 1$ and $\lambda_2 = \alpha\lambda_1$ (for fixed α) have asymptotics, to order $o(n^2)$, the same as for λ-independent boundary conditions, i.e., of the form $\lambda_1 = (c(\alpha) + o(1))n^2$ (see [2]). A similar statement holds for λ_2 if $\lambda_1 = 0$. The argument of [11, Corollary 6.3] can now be used directly to show that $AS_c = C_c \cap S^1$, and together with (4.4) this completes the proof. ∎

Although Theorem 4.7 resembles Klein's oscillation theorem, it says nothing directly about eigenfunction oscillation. To obtain a genuine oscillation theorem, we proceed as follows. By the oscillation count of an eigenvalue $\boldsymbol{\lambda}$ of (0.1), (0.2) we mean the pair $\boldsymbol{n} = (n_1, n_2)$, where n_i is the number of zeros of y_i in $]0,1[$. Thus each eigenvalue has a unique oscillation count, and the following result addresses the extent to which the converse is true. Define

Q_1 as the closed cone

$$\left\{\lambda_1, \lambda_2 \geq 0 : s_1 \geq \frac{\lambda_2}{\lambda_1} \geq s_2\right\} = \left\{\lambda_1, \lambda_2 \geq 0 : \mathbf{e}_1^T\boldsymbol{\lambda} \geq 0 \leq \mathbf{e}_2^T\boldsymbol{\lambda}\right\}$$

and

Q_3 as the open cone

$$\left\{\lambda_1, \lambda_2 < 0 : s_1 > \frac{\lambda_2}{\lambda_1} > s_2\right\} = \left\{\lambda_1, \lambda_2 < 0 : \mathbf{e}_1^T\boldsymbol{\lambda} < 0 > \mathbf{e}_2^T\boldsymbol{\lambda}\right\}.$$

Since $\boldsymbol{R}^2 \setminus \{Q_1 \cup Q_3\}$ is a disjoint union of two cones, we define Q_2 (respectively Q_4) to be the one which intersects the second quadrant (respectively fourth quadrant). Let $N_1 = \min\{n_1 : \lambda_2^{(n_1,0)} \in Q_4$ and $\lambda_2^{(n_1,1)} \in Q_1\}$ and $N_2 = \min\{n_2 : \lambda_1^{(0,n_2)} \in Q_3$ and $\lambda_1^{(1,n_2)} \in Q_1\}$.

THEOREM 4.9. *Suppose $\delta_0 > 0$. With the exceptions below, each oscillation count corresponds to one eigenvalue.*

(i) *There are always infinitely many double oscillation counts.*

(ii) *If $s_1 > M_1$, then for $n_2 \geq N_2$, $(0, n_2)$ is a sequence of double oscillation counts corresponding to exactly two eigenvalues $\boldsymbol{\lambda}^{(0,n_2)}$ and $\boldsymbol{\lambda}^{(1,n_2)}$. There are infinitely many other double oscillation counts, but only finitely many triple oscillation counts.*

(iii) *Similarly, if $s_2 < m_2$, then for $n_1 \geq N_1$, $(n_1, 0)$ is a sequence of double oscillation counts corresponding to exactly two eigenvalues $\boldsymbol{\lambda}^{(n_1,0)}$ and $\boldsymbol{\lambda}^{(n_1,1)}$. In this case, too, there are infinitely many other double and finitely many triple oscillation counts.*

(iv) *If both of the above situations occur, so $s_1 > M_1 > m_2 > s_2$, then only finitely many other double and triple oscillation counts exist, all corresponding to eigenvalues $\boldsymbol{\lambda}^{\boldsymbol{n}}$ with $n_j < N_j$.*

(v) *At most one oscillation count corresponds to four eigenvalues, and four is the maximum possible number.*

Remark 4.10. A similar result holds when $\delta_0 < 0$, involving redefined indices N_1 and N_2.

Remark 4.11. When $M_1 > s_1 > s_2 > m_2$, examples can be constructed with an infinite number of triple oscillation counts.

Proof of Theorem 4.9. We first note that y_{in} has n (respectively $n-1$) zeros in $]0,1[$ if $\mathbf{e}_i^T \boldsymbol{\lambda} < 0$ (respectively ≥ 0). This follows from [3, Lemma 4.1] (with λ_i replaced by $\mathbf{e}_i^T \boldsymbol{\lambda}$) and our assumption that $(d_1, d_2) \neq (0,0)$. Note that by Lemma 4.1 the signs of the r_{jk} are precisely as in [3].

It also follows that the oscillation count of the eigenvalue $\boldsymbol{\lambda}^{\boldsymbol{n}}$ is $(n_1 - 1, n_2 - 1)$ (respectively $(n_1, n_2 - 1), (n_1, n_2), (n_1 - 1, n_2)$) if $\boldsymbol{\lambda} \in Q_1$ (respectively Q_2, Q_3, Q_4). Thus an oscillation count can correspond to two or more eigenvalues only if they are in separate Q_j. This means there can be at most four occurrences of a particular oscillation count.

Let L_i, $i = 1,2$, denote the line $\lambda_2 = s_i \lambda_1$. To prove (i), consider the line L_1. All eigencurves from the second equation cross the line L_1 because their slopes are uniformly bounded above by s_1. There are at most a finite number of eigencurves $\lambda_{2n_1}^1$ from the first equation which do not intersect this line. This can be seen easily from the fact that for a fixed λ_1, the first equation in (0.1) is a negative right definite problem, and so its eigenvalues must accumulate only at $-\infty$. Since two consecutive first equation curves lying on different sides of the line L_1 have different oscillation counts corresponding to them, whenever a second equation eigencurve crosses the line, a double oscillation count occurs. Similarly, the line L_2 gives rise to another sequence of double oscillation counts. This proves (i).

An o.d.e. argument similar to the one in Lemma 3.3 of [3] shows that the graph of λ^1_{20} lies in $Q_2 \cup Q_3$. If $s_1 > M_1$, then each curve $\lambda^1_{2n_1}$ cuts L_1 exactly once for $n_1 > 0$. Thus the oscillation count on the curve λ^1_{21} is 1 (respectively 0) when the curve is not in Q_1 (respectively is in Q_1). Now for $n_2 \geq N_2$, the double occurrence of the oscillation count $(0, n_2)$ corresponding to $\boldsymbol{\lambda}^{(0,n_2)}$ and $\boldsymbol{\lambda}^{(1,n_2)}$ follows from the definition of the integer N_2. Another sequence of double oscillation counts arises from the line L_2 as in the proof of (i). A triple oscillation count has to have one eigenvalue from Q_3. Since there are only finitely many eigenvalues in Q_3, (ii) is proved, and the proof of (iii) is similar.

It is easy to see that if both $s_1 > M_1$ and $s_2 < m_2$ hold, then any other repeated oscillation count involves at least one eigenvalue in Q_3. Result (iv) follows because there can be only finitely many such eigenvalues.

Let $\Gamma^{\boldsymbol{n}}$ denote the curvilinear cell defined by the vertices $\boldsymbol{\lambda}^{\boldsymbol{n}}$, $\boldsymbol{\lambda}^{(n_1+1,n_2)}$, $\boldsymbol{\lambda}^{(n_1+1,n_2+1)}$, and $\boldsymbol{\lambda}^{(n_1,n_2+1)}$ and the corresponding eigencurve sections as edges. The oscillation count (n_1, n_2) corresponds to four eigenvalues if and only if all four vertices of $\Gamma^{\boldsymbol{n}}$ are in separate Q_j. This forces $(0,0) \in \Gamma^{\boldsymbol{n}}$, and so at most one oscillation count corresponds to four eigenvalues. Thus we prove (v). ∎

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