On finite-dimensional commutative non-hermitian fusion algebras

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Abstract

We characterize the three and four dimensional commutative non-hermitian fusion algebras and construct some new examples of these objects. These algebras arise naturally in the study of graphs, specially those associated with von Neumann algebras. Characterisations of hermitian fusion algebras have been given earlier by Sunder and Wildberger. Commutative finite-dimensional non-hermitian fusion algebras are algebraically isomorphic to certain special Cartan subalgebras of matrices. Every Cartan subalgebra of $M_n$ is a conjugate of the standard Cartan algebra by an orthogonal matrix. We characterize the orthogonal matrices that can occur here and thus characterize the four dimensional non-hermitian fusion algebras. The three dimensional ones are parametrized by the hyperbola $\{(x,y) : y^2 - x^2 = 1$ and $x, y > 0\}$. By restricting to a special subclass of orthogonal matrices obtained by the above characterization, we construct a family of new commutative finite-dimensional non-hermitian fusion algebras. © 1999 Elsevier Science Inc. All rights reserved.

1. Preliminaries

Finite-dimensional commutative fusion algebras arise naturally in the context of graphs. These have been studied for general graphs and certain special graphs associated with von Neumann algebras in [4–6]. The hermitian ones

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have been characterized in [6]. In this note, we endeavour to study the non-hermitian ones which are defined below. The result about the three dimensional commutative non-hermitian fusion algebras can be deduced straight from the definition (see Lemma 1.2). In the rest of the first section we set up notations and state the known results about the isomorphic images of an \((n+1)\)-dimensional fusion algebra in \(M_{n+1}\). These are known to be Cartan subalgebras. In Section 2 we characterize those orthogonal matrices which can conjugate the standard Cartan algebra into a fusion algebra. Theorem 2.5 is the main result of this section. This is applied in Sections 3 and 4, respectively to get the examples in any finite dimension and to obtain the characterization of the four dimensional commutative non-hermitian fusion algebras.

**Definition 1.1.** A (finite-dimensional) fusion algebra is an associative unital \(*\)-algebra \(\mathcal{A}\) with a distinguished basis \(\{x_0, x_1, \ldots, x_n\}\) such that the 'structure constants' \(N_{ij}^k\) defined by the equation

\[
x_i x_j = \sum_{k=0}^{n} N_{ij}^k x_k
\]

satisfy the following conditions:

(i) \(N_{00}^k = N_{0k}^0 = \delta_k^0\); in other words, \(x_0\) is the identity of the algebra \(\mathcal{A}\);

(ii) \(N_{ij}^k \geq 0\);

(iii) there exists an involution \(i \mapsto i^*\) of the index set \(\{0, 1, \ldots, n\}\) such that:

(a) \(x_i^* = x_{i^*}\),

(b) \(N_{ij}^0 = \begin{cases} 1 & \text{if } i = j^*, \\ 0 & \text{if } i \neq j^*. \end{cases}\)

We shall call \(\mathcal{A}\) a signed fusion algebra if it satisfies conditions (i) and (iii) (a), (b) above and the requirement that \(N_{ij}^k \in \mathbb{R}\) (rather than (ii) above).

Finally, we shall call a (signed) fusion algebra hermitian if it is the case that \(x_i^* = x_i\ \forall i\).

We shall restrict our attention to commutative non-hermitian fusion algebras. We start with the three dimensional ones.

**Lemma 1.2.** The three dimensional commutative non-hermitian fusion algebras are in one-one correspondence with the points on the hyperbola \(\{(x,y) : y^2 - x^2 = 1, x,y > 0\}\).

**Proof.** The structure constants \(N_{ij}^k\) of a fusion algebra are such that for any \(i, j, k\) in the index set, \(N_{ij}^k = N_{kj}^i\). This can be seen as follows.

\[
x_i x_j x_k^* = \sum_{l=0}^{n} N_{ij}^l x_l x_k^* = \sum_{l,m=0}^{n} N_{ij}^l N_{lm}^m x_m,
\]
whence we find that $N_{ij}^k$ is the coefficient of $x_0$ in $(x_ix_jx_k)$. Thus $N_{ij}^{i*}$ is the coefficient of $x_0$ in $(x_ix_jx_{i*})$. Since these two are conjugates of each other, $x_0$ has the same coefficient in both of them.

Using the above, it is possible to characterize a three dimensional non-hermitian signed fusion algebra. The basis of such an algebra is of the form $\{1, \alpha, \alpha^*\}$. The relation of the structure constants shows that the coefficient of $\alpha$ in $\alpha^2$ and $\alpha \alpha^*$ are same. Let

$$\alpha^2 = \xi \alpha + \eta \alpha^* \quad \text{and} \quad \alpha \alpha^* = 1 + \xi(\alpha - \alpha^*).$$

So the matrix of multiplication by $\alpha$ is:

$$L_\alpha = \begin{bmatrix} 0 & 0 & 1 \\ 1 & \xi & \xi \\ 0 & \eta & \xi \end{bmatrix}.$$

Thus

$$L_{\alpha \alpha^*} = L_\alpha L_\alpha^* = \begin{bmatrix} 1 & \xi & \xi \\ \xi & 1 + 2\xi^2 & \xi \eta + \xi^2 \\ \xi & \xi \eta + \xi^2 & \eta^2 + \xi^2 \end{bmatrix}.$$

On the other hand, since $\alpha \alpha^* = 1 + \xi(\alpha + \alpha^*)$,

$$L_{\alpha \alpha^*} = I_3 + \xi(L_\alpha + L_\alpha^*) = \begin{bmatrix} 1 & \xi & \xi \\ \xi & 1 + 2\xi^2 & \xi \eta + \xi^2 \\ \xi & \xi \eta + \xi^2 & 1 + 2\xi^2 \end{bmatrix}.$$

Equating these two matrices, we get $\eta^2 - \xi^2 = 1$. Conversely, let $(\xi, \eta)$ be any point on the hyperbola $\eta^2 - \xi^2 = 1$. Then any three dimensional associative unital *-algebra with a basis of the form $\{1, \alpha, \alpha^*\}$ satisfying the multiplication rules

$$\alpha^2 = \xi \alpha + \eta \alpha^* \quad \text{and} \quad \alpha \alpha^* = 1 + \xi(\alpha + \alpha^*) = \alpha^* \alpha$$

is a signed fusion algebra. If we restrict to the positive $\xi$-axis, we get a genuine fusion algebra. A commutative three dimensional non-hermitian (signed) fusion algebra thus corresponds to a point on the hyperbola $\eta^2 - \xi^2 = 1$.

Given an $(n + 1)$-dimensional fusion algebra $\mathcal{A}$, there is a natural *-algebra homomorphism, which we denote by $\varphi$, of $\mathcal{A}$ into $M_{n+1}(\mathbb{R})$ if we map $x_i$ to the matrix $L_i$ of left multiplication by $x_i$ with respect to the ordered basis $\{x_0, x_1, \ldots, x_n\}$ and extend linearly. Note that the matrix $L_i$ thus obtained has the property that the first column has all zeros except the entry on the $i$th row, which is 1 (because $x_0$ is the identity of the algebra) and the first row too has all zeros except the entry on column $i^*$, which is 1 (because of the restrictions on $N_{ij}^0$). Let the algebra generated by $L_0, L_1, \ldots, L_n$ be $\mathcal{G}$. This is the matrix algebra
which is \( \varphi \)-isomorphic to \( \mathcal{A} \). It is an \((n + 1)\)-dimensional commutative \( C^* \)-subalgebra of \( M_{n+1}(\mathbb{R}) \). The following result says that it is also a maximal commutative \( C^* \)-subalgebra.

**Lemma 1.3.** The following conditions on a commutative \( C^* \)-subalgebra \( \mathcal{C} \) of \( M_{n+1}(\mathbb{R}) \) are equivalent:

(a) \( \dim \mathcal{C} = n + 1 \);

(b) \( \mathcal{C} \) is a maximal commutative subalgebra of \( M_{n+1}(\mathbb{R}) \).

For a proof of this result, see [1,2]. Such a subalgebra is called a Cartan subalgebra. So every commutative fusion algebra \( \mathcal{A} \) with basis \( \{x_0, x_1, \ldots, x_n\} \) is isomorphic to a Cartan subalgebra \( \mathcal{C} \) of \( M_{n+1}(\mathbb{R}) \) generated by \( L_0, L_1, \ldots, L_n \). We briefly sketch below those properties of a Cartan subalgebra which we shall need.

**Lemma 1.4.** Any Cartan subalgebra \( \mathcal{C} \) of \( M_{n+1}(\mathbb{R}) \) determines a unique pair \((p, q)\) of non-negative integers. The set of minimal projections of \( \mathcal{C} \) is given by \( \{P_1, \ldots, P_p, Q_1, \ldots, Q_q\} \), where \( P_i \) and \( Q_j \) are projections of rank one and two, respectively. One of the integers \( p \) and \( q \), however, can be zero.

The set of minimal projections \( \{P_1, \ldots, P_p, Q_1, \ldots, Q_q\} \) forms a basis for the real vector subspace of \( \mathcal{C} \) consisting of symmetric matrices. The subspace of skew-symmetric matrices in \( \mathcal{C} \) has dimension \( q \). There is a basis \( \{S_1, S_2, \ldots, S_q\} \) of this vector space satisfying \( S_j = Q_j S_j Q_j \) and \( \|S_j\| = 1 \), for all \( j = 1, \ldots, q \). Moreover, the matrices \( S_j \) are unique (upto sign) i.e., if \( T_j \) form another basis satisfying \( T_j = Q_j T_j Q_j \) and \( \|T_j\| = 1 \), then \( T_j = \pm S_j \). For more details on Cartan subalgebras, see [1–3]. The Cartan subalgebras which we are concerned about viz., the ones which are isomorphic images of fusion algebras have the following property.

**Remark 1.5.** If the Cartan algebra \( \mathcal{C} \) arises from a fusion algebra through the isomorphism \( \varphi \), then at least one projection of rank 1 exists in \( \mathcal{C} \). This can be seen from the fact that every fusion algebra admits a unique dimension function (see [4]). Thus we number the rank one projections as \( P_0, P_1, \ldots, P_p \). This, in particular, means that \( p + 2q = n \). From now on, we shall consider only such Cartan algebras.

If \( \mathcal{A} \) is an \((n + 1)\)-dimensional fusion algebra among whose basis elements there are exactly \( p \) selfadjoint elements other than identity and \( q \) non-selfadjoint elements then the Cartan subalgebra \( \mathcal{C} \) is as in Remark 1.5. The following proposition from [6] provides a complete characterization and one–one correspondence between fusion algebras and Cartan subalgebras of \( M_{n+1}(\mathbb{R}) \). Let \( \{v_0, v_1, \ldots, v_n\} \) be the standard basis for \( \mathbb{R}^{n+1} \).
Proposition 1.6. (a) Let $\mathcal{A}$ be a fusion algebra with basis $\{x_0, x_1, \ldots, x_n\}$. Let $I = \{i : i = i^*\}$ and $J \subset \{1, 2, \ldots, n\} \setminus I$ be such that $j \in J \Rightarrow j^* \notin J$ and $|I| + 2|J| = n$. Let $\mathcal{C}$ be that Cartan algebra which is the $\varphi$-isomorphic image of $\mathcal{A}$. Then,

(i) $v_0$ is a cyclic vector for $\mathcal{C}$; and

(ii) $\{S_j v_0 : 1 \leq j \leq q\}$ and $\{v_j - v_{j^*} : j \in J\}$ are both orthogonal bases for the same subspace of $\mathbb{R}^{n+1}$.

(b) Conversely, suppose $\mathcal{C}$ is a Cartan subalgebra of $M_{n+1}(\mathbb{R})$, with minimal projections $P_0, \ldots, P_p, Q_1, \ldots, Q_q$ as in Remark 1.4. Suppose there exists a $2q$ element set $J \cup \{j^* : j \in J\} \subset \{1, 2, \ldots, n\}$ such that conditions (i) and (ii) of (a) are satisfied. Then there exists a signed fusion algebra $\mathcal{A}$ with precisely $p$ self-adjoint basis elements such that $\mathcal{C}$ is the Cartan subalgebra associated with $\mathcal{A}$.

(c) Suppose $\mathcal{C}$ is as in (b) above; let $\{A_j : 0 \leq j \leq n\}$ be any basis for $\mathcal{C}$. (For instance, we may take the basis as $P_0, P_1, \ldots, P_p, Q_1, S_1, Q_2, S_2, \ldots, Q_q, S_q$.) Define the matrix $B \in M_{n+1}(\mathbb{R})$ by $b_{ij} = \langle A_i v_0, v_j \rangle$. (Thus, the $j$th column of the matrix $B$ is just $A_j v_0$.) Then $B$ is a non-singular matrix. Let $C = B^{-1}$. Define

$$L_k = \sum_{j=0}^{n} c_{jk} A_j.$$ 

Then these are precisely the matrices $L_k$ obtained from the signed fusion algebra corresponding to $\mathcal{C}$ as in (b) above.

2. Fusion algebras and orthogonal matrices

Henceforth, we shall concentrate on commutative non-hermitian fusion algebras which have two non-selfadjoint elements in their distinguished bases; consequently, we shall only consider Cartan algebras in $M_{n+1}(\mathbb{R})$ whose 'antisymmetric part' is one-dimensional, or equivalently, for which $q = 1$.

Lemma 2.1. Given any Cartan subalgebra $\mathcal{C}$ of $M_{n+1}(\mathbb{R})$, there is an orthogonal matrix $U$ such that $\mathcal{C} = U^* \mathcal{C}_0 U$ where $\mathcal{C}_0$ is the Cartan subalgebra generated by the following set of matrices.

$$\left\{ \begin{bmatrix} \lambda_0 & 0 & \cdots & 0 & 0 \\ 0 & \lambda_1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & \lambda_p & 0 \\ 0 & 0 & \cdots & 0 & aR_\theta \end{bmatrix} : \lambda_i, \theta \in \mathbb{R}, \quad \theta \in [0, 2\pi] \right\},$$

where
This is called the standard Cartan subalgebra.

The above lemma is a very well-known fact. See, for example [1], p. 35 for the proof. The standard Cartan subalgebra is cyclic. Given \( (n + 1) \) real numbers \( y_0, y_1, \ldots, y_n \) satisfying

\[
y_0, \ldots, y_{n-2} \neq 0 \quad \text{and} \quad y^2_{n-1} + y^2_n \neq 0,
\]

the vector \( y = (y_0, y_1, \ldots, y_n) \) is a cyclic vector for the Cartan algebra \( \mathcal{C}_0 \). The next lemma tells how to construct fusion algebras by conjugating the standard Cartan subalgebra with a suitable orthogonal matrix. It gives a sufficient condition which an orthogonal matrix \( U \) should satisfy to make \( U^* \mathcal{C}_0 U \) a fusion algebra. A full characterization of such orthogonal matrices is presented in Lemma 2.3.

**Lemma 2.2.** Let \( y = (y_0, y_1, \ldots, y_n) \) be a unit vector and \( U \) be an \((n + 1) \times (n + 1)\) orthogonal matrix where \( y \) satisfies Eq. (1) and \( U \) satisfies

\[
U v_0 = y \quad \text{and} \quad U(v_{n-1} - v_n) = \left( \frac{2}{y^2_{n-1} + y^2_n} \right)^{1/2} (0, \ldots, 0, -y_n, y_{n-1}). \quad (*)
\]

Then the matrix algebra \( U^* \mathcal{C}_0 U \) can be endowed with a (signed) fusion algebra structure with exactly two elements of the basis being non-selfadjoint.

**Proof.** Let \( \mathcal{C} = U^* \mathcal{C}_0 U \). First note that \( \mathcal{C} \) is a Cartan subalgebra by Lemma 1.3 because conjugation by an orthogonal matrix does not change the dimension and also retains commutativity. All that is needed to prove is that \( \mathcal{C} \) satisfies the two conditions of Proposition 1.6(a). Then the rest will follow from 1.6(b) and (c).

The vector \( y = (y_0, y_1, \ldots, y_n) \) is of the above form so that it is a cyclic vector of the standard Cartan algebra. This, in view of condition (i), means that \( v_0 \) becomes a cyclic vector for the algebra \( \mathcal{C} \).

The minimal projections for \( \mathcal{C}_0 \) are given by

\[
\begin{align*}
F_0 &= \text{diag}(1, 0, 0, \ldots, 0), \\
F_1 &= \text{diag}(0, 1, 0, \ldots, 0), \\
&\vdots \\
F_{n-2} &= \text{diag}(0, 0, 0, \ldots, 0, 1, 0, 0), \\
F_{n-1} &= \text{diag}(0, 0, 0, \ldots, 0, 0, 1, 1).
\end{align*}
\]
Conjugating these with the orthogonal matrix $U$ and setting $A_i = U^* F_i U$ for $i = 1, \ldots, n - 1$, we get the minimal projections $\{A_0, A_1, \ldots, A_{n-1}\}$ for the Cartan subalgebra $\mathcal{C}$. The only two dimensional projection is $A_{n-1}$. The anti-symmetric part of $\mathcal{C}_0$ is the span of $F_n = O_{n-1} \oplus \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. Let $A_n = U^* F_n U$.

The unique (upto sign) anti-symmetric matrix $S_1 \in \mathcal{C}$ which satisfies $A_{n-1} S_1 A_{n-1} = S_1$ and $\|S_1\| = 1$ is, in this case, just $A_n$. Since condition (*) is true for $U$, we see that

$$A_n v_0 = U^* F_n U v_0 = U^* F_n (v_0, y_1, \ldots, y_n)$$

$$= U^* (0, \ldots, 0, -y_n, y_{n-1}) = \left( \frac{y_{n-1}^2 + y_n^2}{2} \right)^{1/2} (v_{n-1} - v_n).$$

This means $A_n v_0 / \|A_n v_0\| = (1/\sqrt{2})(v_{n-1} - v_n)$. Thus the algebra $\mathcal{C}$ satisfies the two conditions of Proposition 1.6(a) with $q = 1$. Hence it can be endowed with a fusion algebra structure following 1.6(c).

**Lemma 2.3.** Let $y = (y_0, y_1, \ldots, y_n)$ be any unit vector in $\mathbb{R}^{n+1}$; define $c_k = (y_0^2 + \cdots + y_k^2)^{1/2}$, for $0 \leq k \leq n$, then there exists a unique $(n+1) \times (n+1)$-orthogonal matrix $U_0$ whose action on the standard basis is as follows:

$$U_0 v_0 = (y_0, \ldots, y_n),$$

$$U_0 v_k = \frac{y_{k+1}}{c_k c_{k+1}} \left( y_0, \ldots, y_k, \frac{-c_k^2}{y_{k+1}}, 0, \ldots, 0 \right) \quad \text{for } k = 1, \ldots, n - 3,$$

$$U_0 v_{n-2} = \frac{(y_{n-1}^2 + y_n^2)^{1/2}}{c_{n-2}} \left( y_0, \ldots, y_{n-2}, \frac{-y_n c_{n-2}^2}{y_{n-1}^2 + y_n^2}, \frac{-y_n^2 c_{n-2}^2}{y_{n-1}^2 + y_n^2} \right),$$

$$U_0 v_{n-1} = \frac{1}{\sqrt{2}} \left( y_1, -y_0, c_1, c_1, 0, \ldots, 0, \frac{-y_n}{(y_{n-1}^2 + y_n^2)^{1/2}}, \frac{y_{n-1}}{(y_{n-1}^2 + y_n^2)^{1/2}} \right),$$

$$U_0 v_n = \frac{1}{\sqrt{2}} \left( y_1, -y_0, c_1, c_1, 0, \ldots, 0, \frac{y_n}{(y_{n-1}^2 + y_n^2)^{1/2}}, \frac{-y_{n-1}}{(y_{n-1}^2 + y_n^2)^{1/2}} \right).$$

Given any $(n-1) \times (n-1)$ orthogonal matrix $W_0$ let $W$ be the $(n+1) \times (n+1)$ matrix:

$$W = \left( I_{n-1} \oplus (1/\sqrt{2}) \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \right) \left( I_1 \oplus W_0 \oplus I_1 \right) \times \left( I_{n-1} \oplus (1/\sqrt{2}) \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \right).$$

Then, an orthogonal matrix $U$ satisfies...
Proof. First note that by construction $U_0$ satisfies

$$U_0v_0 = y \quad \text{and} \quad U_0(v_{n-1} - v_n) = \left(\frac{2}{y_{n-1}^2 + y_n^2}\right)^{1/2}(0, \ldots, 0, -y_n, y_{n-1}).$$

The matrix $W$ is such that it leaves $v_0$ and $v_{n-1} - v_n$ unchanged. So if $U = U_0 W$, then $U$ clearly satisfies Eq. (3).

Conversely, let the orthogonal matrix $U$ be specified on just the two dimensional subspace spanned by $v_0$ and $v_{n-1} - v_n$ as in Eq. (3). Let $W = U_0^{-1}U$. Since both $U_0$ and $U$ satisfy Eq. (3), the orthogonal matrix $W$ leaves the vectors $v_0$ and $v_{n-1} - v_n$ unchanged. Consider the following orthogonal matrix:

$$\tilde{W} = \left(I_{n-1} \oplus \left(1/\sqrt{2}\right)\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\right) W \left(I_{n-1} \oplus \left(1/\sqrt{2}\right)\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\right).$$

Note that $\tilde{W}v_0 = v_0$ and $\tilde{W}v_n = v_n$. Thus $\tilde{W} = I_1 \oplus W_0 \oplus I_1$ for some $(n-1) \times (n-1)$ orthogonal matrix $W_0$. That gives the result. □

This lemma leads us to a converse of Lemma 2.2 in the following way.

Lemma 2.4. Let $\mathcal{C}$ be a Cartan subalgebra of $M_{n+1}(\mathbb{R})$ arising from a fusion algebra through the isomorphism $\varphi$. Then $\mathcal{C} = U^*\mathcal{C}_0 U$ for some orthogonal matrix $U$ of the form $U_0 W$ where $U_0$ and $W$ are as above.

Proof. Let $\mathcal{C}$ be a cyclic Cartan subalgebra of $M_{n+1}(\mathbb{R})$ with minimal projections $P_0, \ldots, P_{n-2}, Q$ and cyclic vector $v_0$. Also let $S$ be the unique (up to sign) skew-symmetric matrix in $\mathcal{C}$ such that $QS = S$ and $\|S\| = 1$. Then there is an orthogonal matrix $U$ such that $\mathcal{C} = U^*\mathcal{C}_0 U$ and $Uv_0 = y = (y_0, y_1, \ldots, y_n)$ is a cyclic vector for the standard Cartan algebra $\mathcal{C}_0$. If, moreover, we demand that $Sv_0/\|Sv_0\| = (v_{n-1} - v_n)/\sqrt{2}$, which we may in case if $\mathcal{C} = \varphi(\mathcal{A})$ for some fusion algebra $\mathcal{A}$, then $U(v_{n-1} - v_n) = \sqrt{2/((y_{n-1}^2 + y_n^2)}$. Lemma 2.3 then implies that $U = U_0 W$. Note that the Cartan algebra $\mathcal{C}$ exactly satisfies the requirements which would associate with it a fusion algebra according to the isomorphism $\varphi$. Thus if we fix a cyclic vector $y$ for the standard Cartan algebra, then to any fusion algebra there corresponds to an $(n-1) \times (n-1)$ orthogonal matrix $U = U_0 W$ where $U_0$ and $W$ are as in Lemma 2.3. □

The results of this section are summarised in the following theorem.
Theorem 2.5. Let \( \mathcal{A} \) be an \((n + 1)\)-dimensional non-hermitian fusion algebra with exactly two non-selfadjoint elements in the basis. Let \( \mathcal{C} \) be the Cartan subalgebra of \( \mathbb{M}_{n+1}(\mathbb{R}) \) which is isomorphic to \( \mathcal{A} \) via the map \( \varphi \). Then \( \mathcal{C} = U^* \mathcal{C}_0 U \) if and only if \( U \) is an orthogonal matrix satisfying Eq. (3).

3. A family of non-hermitian fusion algebras

Now we use the theorems of the last section to construct non-hermitian fusion algebras of all finite dimensions with exactly two non-hermitian elements in their bases.

Theorem 3.1. Let \( y_0, y_1, \ldots, y_n \) be real numbers satisfying Eq. (1) and \( y_0^2 + y_1^2 + \cdots + y_n^2 = 1 \). Let \( c_k = (y_0^2 + y_1^2 + \cdots + y_k^2)^{(1/2)} \) for \( k = 1, \ldots, n-2 \). The following set of \((n + 1)\) matrices forms the basis of a (signed) fusion algebra:

\[
L_0 = I_{n+1},
\]

\[
L_k = \begin{bmatrix}
0 & 0 & \cdots & 0 & 1 & 0 & \cdots & 0 & 0 \\
0 & \frac{y_{k+1}}{c_k c_{k+1}} & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & \frac{y_{k+1}}{c_k c_{k+1}} & 0 & 0 & \cdots & 0 & 0 \\
1 & 0 & \cdots & 0 & \frac{y_{k+1}^2 - c_k}{c_{k+1} c_{k+2}} & \frac{y_{k+2}}{c_{k+1} c_{k+2}} & \cdots & \frac{y_{n-2}}{c_{n-3} c_{n-2}} & \frac{(1-c_{n-2}^2)^{1/2}}{c_{n-2}} \\
0 & 0 & \cdots & 0 & \frac{y_{k+1}}{c_k c_{k+1}} & 0 & \cdots & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & 0 & \frac{(1-c_{n-2}^2)^{1/2}}{c_{n-2}} & 0 & \cdots & 0 & 0 \\
0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 \\
\end{bmatrix}
\]

\[\oplus \frac{y_k}{2c_k c_{k+1}} I_2\] for \( k = 1, \ldots, n-3 \),

\[
L_{n-2} = \begin{bmatrix}
0 & 0 & \cdots & 0 & 1 \\
0 & \frac{(1-c_{n-2}^2)^{1/2}}{c_{n-2}} & \cdots & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & \frac{(1-c_{n-2}^2)^{1/2}}{c_{n-2}} & 0 \\
1 & 0 & \cdots & 0 & \frac{(1-2c_{n-2}^2)}{c_{n-2}(1-c_{n-2}^2)^{1/2}} \\
\end{bmatrix}
\]

\[\oplus \frac{(1-2c_{n-2}^2)}{c_{n-2}(1-c_{n-2}^2)^{1/2}} I_2,\]
Proof. We take the orthogonal matrix \( U_0 \) defined earlier and conjugate the basis \( \{ F_0, \ldots, F_n \} \) of the standard Cartan algebra defined in Eq. (2) with it to get the basis \( \{ A_0, \ldots, A_n \} \). We form the matrix \( B \), whose \( j \)th column is the 1st column of \( A_j \). It is as follows.

\[
B = \begin{bmatrix}
U_0 F_1 U_0 v_0, U_0 F_2 U_0 v_0, \ldots, U_0 F_n U_0 v_0
\end{bmatrix}
= U_0 \begin{bmatrix}
F_1 y, F_2 y, \ldots, F_n y
\end{bmatrix}
= U_0 \left( \text{diag}(y_0, y_1, \ldots, y_{n-2}) \otimes \begin{pmatrix} y_{n-1} & -y_n \\ y_n & y_{n-1} \end{pmatrix} \right).
\]

Set \( C = B^{-1} \). Then

\[
C = \begin{pmatrix}
\text{diag}(1/y_0, 1/y_1, \ldots, 1/y_{n-2}) \otimes (1/(1 - c_{n-2}^2))
\end{pmatrix} \begin{pmatrix} y_{n-1} & -y_n \\ y_n & y_{n-1} \end{pmatrix} U_0.
\]

Then the columns of \( C \) are as follows:

- \( C_{v_0} = (1, \ldots, 1, 0, 0) \),
- \( C_{v_k} = \frac{y_{k+1}}{c_k c_{k+1}} \begin{pmatrix} 1, \ldots, 1, \frac{-c_{n-2}^2}{y_{k+1}^2}, 0, \ldots, 0 \end{pmatrix} \) for \( k = 1, \ldots, n-3 \),
- \( C_{v_{n-2}} = \frac{(1 - c_{n-2}^2)^{1/2}}{c_{n-2}} \begin{pmatrix} 1, \ldots, 1, \frac{-c_{n-2}^2}{y_{n-2}^2}, 0 \end{pmatrix} \),
- \( C_{v_{n-1}} = \frac{1}{\sqrt{2}} \begin{pmatrix} \frac{y_1}{y_0 c_1}, \frac{-y_0}{y_1 c_1}, 0, \ldots, 0, 0, (1 - c_{n-2}^2)^{-1/2} \end{pmatrix} \),
- \( C_{v_n} = \frac{1}{\sqrt{2}} \begin{pmatrix} \frac{y_1}{y_0 c_1}, \frac{-y_0}{y_1 c_1}, 0, \ldots, 0, 0, -(1 - c_{n-2}^2)^{-1/2} \end{pmatrix} \).

Now we form the matrices
\[ L_k = \sum_{j=0}^{n} c_{jk} A_j = U_0^* \left( \sum_{j=0}^{n} c_{jk} F_j \right) U_0. \]

The matrices \( L_0, L_1, \ldots, L_n \) are of the above form. They form a basis of the algebra \( U_0^* \mathcal{C}_0 U_0 \).

**Remark 3.2.** If we choose

\[ y_0 > 0, \quad y_{k+1} > c_k \quad \text{for} \quad k = 0, 1, \ldots, n - 3, \quad \text{and} \quad c_{n-2}^2 < 1/2 \]

we get a genuine fusion algebra.

Here are some numerical examples of the matricial fusion algebras of the above theorem for special values of the parameters for \( n = 4 \), i.e., in dimension five. Note that the \( (n+1) \)-dimensional ones which we obtained depend on \( n-1 \) parameters. Since \( L_0 = I_{n+1} \) and \( L_n = L_{n-1} \) always, we are concerned about \( L_1, \ldots, L_{n-1} \) only, in this case, \( L_1, L_2, L_3 \). The parameters are \( y_0, y_1 \) and \( y_2 \) subject to \( y_1 \geq y_0 > 0, y_2 \geq \sqrt{y_0^2 + y_1^2} \) and \( y_0^2 + y_1^2 + y_2^2 < 1/2 \). We further specialize by putting \( y_2 = \sqrt{y_0^2 + y_1^2} \). Then \( c_2 = c_1 \sqrt{2} \) and \( c = c_1 = c_2 / \sqrt{2} < 1/2 \). This can be made further simpler by the following substitution:

\[ c \sqrt{2} = \sin \theta, \quad y_0 = c \cos \phi, \quad y_1 = c \sin \phi. \]

Then

\[ L_1 = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & \cot \theta & 0 & 0 \\ 0 & \cot \theta & 0 & 0 & 0 \\ 0 & 0 & \sin \phi & \sin \theta & \sin \phi & \sin \theta \\ 0 & 0 & 0 & \sin \phi & \sin \theta & \sin \theta \end{pmatrix}, \]

\[ L_2 = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & \cot \theta & 0 & 0 & 0 \\ 1 & 0 & \frac{1}{2} \cot 2\theta & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2} \cot 2\theta & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{2} \cot 2\theta \end{pmatrix}, \]

\[ L_3 = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & \frac{1}{2} \sin \theta & \frac{1}{2} \sin \theta \\ 0 & 0 & 0 & \frac{1}{2} \sin \theta & \frac{1}{2} \sin \theta & \cot 2\theta \\ 1 & \frac{1}{2} \sin \theta & \cot 2\theta & -\frac{\cot 2\theta}{\sin \theta} & -\frac{\cot 2\theta}{\sin \theta} \\ 0 & \frac{1}{2} \sin \theta & \frac{1}{2} \sin \theta & \frac{1}{2} \sin \theta & \frac{1}{2} \sin \theta \end{pmatrix}. \]
By the conditions on $y_0, y_1, y_2$, one has $0 < \theta < \pi/4$ and $\pi/4 < \varphi < \pi/2$. Putting $\theta = \pi/6$ and $\varphi = \pi/3$, we get

$$L_1 = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & \sqrt{3} & 0 & 0 \\ 0 & \sqrt{3} & 0 & 0 & 0 \\ 0 & 0 & 0 & \sqrt{3} & \sqrt{3} \\ 0 & 0 & 0 & \sqrt{3} & \sqrt{3} \end{pmatrix},$$

$$L_2 = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & \sqrt{3} & 0 & 0 & 0 \\ 1 & 0 & \frac{1}{2} \sqrt{3} & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{\sqrt{3}} & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{\sqrt{3}} \end{pmatrix},$$

$$L_3 = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & \frac{2}{\sqrt{3}} & \sqrt{3} \end{pmatrix}.$$

4. Four dimensional non-hermitian fusion algebras

**Theorem 4.1.** Let $\mathcal{A}$ be a four dimensional signed fusion algebra with basis $\{x_0, x_1, x_2, x_3\}$ such that $x_1^* = x_1$ and $x_2^* = x_3$. Then the matrices $L_i$ of multiplication by $x_i$ with respect to the above basis are as follows: $L_0 = I_4, L_3 = L_2^* \text{ and } L_1$ and $L_2$ are either

$$L_1(a, r, \theta) = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & f_1(a, r, \theta) & f_2(a, r, \theta) \sin \theta & f_2(a, r, \theta) \sin \theta \\ 0 & f_2(a, r, \theta) \sin \theta & f_3(a, r, \theta) \cos \theta & \frac{1}{2} f_2(a, r, \theta) \cos \theta \\ 0 & f_2(a, r, \theta) \sin \theta & \frac{1}{2} f_2(a, r, \theta) \cos \theta & f_3(a, r, \theta) \cos \theta \\ 0 & f_2(a, r, \theta) \sin \theta & \frac{1}{2} f_2(a, r, \theta) \cos \theta & f_3(a, r, \theta) \cos \theta \end{bmatrix},$$

$$L_2(a, r, \theta) = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & f_2(a, r, \theta) \sin \theta & \frac{1}{2} f_2(a, r, \theta) \cos \theta & f_3(a, r, \theta) \cos \theta \\ 1 & f_3(a, r, \theta) \cos \theta & f_4(a, r, \theta) & f_4(a, r, \theta) \\ 0 & \frac{1}{2} f_2(a, r, \theta) \cos \theta & f_5(a, r, \theta) & f_4(a, r, \theta) \end{bmatrix}.$$

(4)
or

\[
L_1(a,r,\theta) = \begin{bmatrix}
0 & 1 & 0 & 0 \\
1 & -f_1(a,r,\theta) & f_2(a,r,\theta) \sin \theta & f_2(a,r,\theta) \sin \theta \\
0 & f_2(a,r,\theta) \sin \theta & -f_3(a,r,\theta) \cos \theta & -\frac{1}{2} f_2(a,r,\theta) \cos \theta \\
0 & f_2(a,r,\theta) \sin \theta & -\frac{1}{2} f_2(a,r,\theta) \cos \theta & -f_3(a,r,\theta) \cos \theta \\
\end{bmatrix}
\]

\[
L_2(a,r,\theta) = \begin{bmatrix}
0 & 0 & 0 & 1 \\
0 & f_2(a,r,\theta) \sin \theta & -\frac{1}{2} f_2(a,r,\theta) \cos \theta & -f_3(a,r,\theta) \cos \theta \\
1 & -f_3(a,r,\theta) \cos \theta & f_4(a,r,\theta) & f_4(a,r,\theta) \\
0 & -\frac{1}{2} f_2(a,r,\theta) \cos \theta & f_5(a,r,\theta) & e(a,r,\theta) \\
\end{bmatrix}
\]

where

\[
f_1(a,r,\theta) = \frac{-(2 - r^2) \cos^3 \theta + 3(1 - r^2) \cos \theta + a \sqrt{1 - r^2} \sin^3 \theta}{r \sqrt{1 - r^2}},
\]

\[
f_2(a,r,\theta) = \frac{(r^2 + (2 - r^2) \cos 2\theta + a \sqrt{1 - r^2} \sin 2\theta)}{2 \sqrt{2} r \sqrt{1 - r^2}},
\]

\[
f_3(a,r,\theta) = \frac{(- 3r^2 + (2 - r^2) \cos 2\theta + a \sqrt{1 - r^2} \sin 2\theta)}{4 \sqrt{2} r \sqrt{1 - r^2}},
\]

\[
f_4(a,r,\theta) = \frac{(a \sqrt{1 - r^2} \cos \theta + (2 - r^2) \sin^3 \theta - (3 - 4r^2) \sin \theta)}{2 \sqrt{2} r \sqrt{1 - r^2}},
\]

\[
f_5(a,r,\theta) = \frac{(a \sqrt{1 - r^2} \cos \theta + (2 - r^2) \sin^3 \theta - 3 \sin \theta)}{2 \sqrt{2} r \sqrt{1 - r^2}}.
\]

**Proof.** In dimension four, the matrix \( U_0 \) of Lemma 2.3 is given by,

\[
\begin{bmatrix}
y_0 & y_0 \sqrt{\frac{y_0^2 + y_2^2}{y_0^2 + y_1^2}} & y_1 & y_1 \sqrt{\frac{y_1^2}{y_0^2 + y_1^2}} \\
y_1 & y_1 \sqrt{\frac{y_1^2 + y_3^2}{y_0^2 + y_1^2}} & -y_0 & -y_0 \sqrt{\frac{y_0^2}{y_0^2 + y_1^2}} \\
y_2 & -y_2 \sqrt{\frac{y_2^2 + y_3^2}{y_2^2 + y_3^2}} & -y_3 & -y_3 \sqrt{\frac{y_3^2}{y_2^2 + y_3^2}} \\
y_3 & -y_3 \sqrt{\frac{y_3^2 + y_1^2}{y_2^2 + y_3^2}} & y_2 & y_2 \sqrt{\frac{y_2^2}{y_2^2 + y_3^2}} \\
\end{bmatrix}
\]

where \( y = (y_0, y_1, y_2, y_3) \) is a unit vector such that \( y_0 \neq 0, y_1 \neq 0 \) and \( y_0^2 + y_1^2 < 1 \).

From the form of it, we see that the matrix \( U_0 \) depends on two parameters \( y_0 \) and \( y_1 \) only because \( y_2^2 + y_3^2 \) can be replaced by \( 1 - (y_0^2 + y_1^2) \). The matrix \( W_0 \) is a \( 2 \times 2 \) orthogonal matrix. So it is either...
\[
\begin{pmatrix}
\cos \theta & \sin \theta \\
-\sin \theta & \cos \theta
\end{pmatrix}
\]
or
\[
\begin{pmatrix}
-\cos \theta & \sin \theta \\
\sin \theta & \cos \theta
\end{pmatrix}
\]
where \( \theta \in [0, 2\pi] \). Since \( W_0 \) depends on \( \theta \) (only), let us call it \( W_0(\theta) \). Accordingly, \( W \), the matrix defined in the statement of Lemma 2.3, will be called \( W(\theta) \). Thus the four dimensional non-hermitian fusion algebras depend on three real parameters. Two substitutions come in handy for simplifying the calculations. They are as follows.

\[ a = \frac{y_1^2 - y_0^2}{y_0 y_1} \text{(note that } y_0 y_1 \neq 0) \text{ and } r = \sqrt{y_0^2 + y_1^2}. \]

Thus \( L_1 \) and \( L_2 \) are functions of \( a, r \) and \( \theta \). Among the basis matrices, \( L_0 = I_4 \) and \( L_3 = L_2^* \).

When \( W_0(\theta) = \begin{pmatrix} 
\cos \theta & \sin \theta \\
-\sin \theta & \cos \theta
\end{pmatrix} \), we have

\[
W(\theta) = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & \cos \theta & -\sin \theta/\sqrt{2} & -\sin \theta/\sqrt{2} \\
0 & \sin \theta/\sqrt{2} & (1 + \cos \theta)/2 & (-1 + \cos \theta)/2 \\
0 & \sin \theta/\sqrt{2} & (-1 + \cos \theta)/2 & (1 + \cos \theta)/2
\end{pmatrix}.
\]

When \( \theta = 0 \) i.e., \( W(\theta) \) is \( I_4 \), then \( L_1 \) and \( L_2 \) are as follows.

\[
L_1(a, r, 0) = \begin{pmatrix}
0 & 1 & 0 & 0 \\
1 & \frac{(1-2r^2)}{\sqrt{1-r^2}} & 0 & 0 \\
0 & 0 & \frac{1}{2\sqrt{1-r^2}} & \frac{(1-2r^2)}{2\sqrt{1-r^2}} \\
0 & 0 & \frac{1}{2\sqrt{1-r^2}} & \frac{(1-2r^2)}{2\sqrt{1-r^2}}
\end{pmatrix},
\]

\[
L_2(a, r, 0) = \begin{pmatrix}
0 & 0 & 1 & 0 \\
0 & 0 & \frac{1}{2\sqrt{1-r^2}} & \frac{(1-2r^2)}{2\sqrt{1-r^2}} \\
1 & \frac{(1-2r^2)}{2\sqrt{1-r^2}} & \frac{a}{2\sqrt{2r}} & \frac{a}{2\sqrt{2r}} \\
0 & \frac{1}{2\sqrt{1-r^2}} & \frac{a}{2\sqrt{2r}} & \frac{a}{2\sqrt{2r}}
\end{pmatrix}.
\]

It is interesting to see the numerical form of these matrices for special values of \( r \).
For $r = 1/\sqrt{2}$, $L_1 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$, $L_2 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & a & a \\ 0 & 1 & a & a \end{bmatrix}$.

For $r = 1/2$, $L_1 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 2/\sqrt{3} & 0 & 0 \\ 0 & 0 & 1/\sqrt{3} & 2/\sqrt{3} \\ 0 & 0 & 2/\sqrt{3} & 1/\sqrt{3} \end{bmatrix}$,

$L_2 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 2/\sqrt{3} & 1/\sqrt{3} \\ 1 & 1/\sqrt{3} & a & a \\ 1 & 2/\sqrt{3} & a & a \end{bmatrix}$,

where $a$ is any non-negative real number.

A long computation which consists of first forming the matrix $U(\theta) = U_0 W(\theta)$ and then conjugating the standard Cartan algebra with it, yields

$$L_1(a, r, \theta) = W(\theta)^* \left( \cos \theta L_1(a, r, 0) + \frac{\sin \theta}{\sqrt{2}} (L_2(a, r, 0) + L_2(a, r, 0)^*) \right) W(\theta),$$

$$L_2(a, r, \theta) = W(\theta)^* \left( -\frac{\sin \theta}{\sqrt{2}} L_1(a, r, 0) + \frac{1 + \cos \theta}{2} (L_2(a, r, 0) - L_2(a, r, 0)^*) \right) W(\theta).$$

After completing the calculation using the forms of $L_1(a, r, 0)$ and $L_2(a, r, 0)$ above, one gets the above form (4).

If, on the other hand, we had started with $W_0(\theta) = \begin{pmatrix} -\cos \theta & \sin \theta \\ \sin \theta & \cos \theta \end{pmatrix}$, then following a similar procedure as above we arrive at the form (5).

Since these are the only two possibilities, thus the matrices $L_i$ associated with any four dimensional signed fusion algebra are bound to be of one of the above forms. 

The characterization of genuine four dimensional fusion algebras is as follows.
Theorem 4.4. For $0 < r < 1$ and $\theta \in [0, 2\pi]$, let

$$
\alpha_1(r, \theta) = \frac{3\sin \theta - (2 - r^2)\sin^3 \theta}{\sqrt{1 - r^2}\cos^3 \theta},
$$

$$
\alpha_2(r, \theta) = \frac{3r^2 - (2 - r^2)\cos 2\theta}{\sqrt{1 - r^2}\sin 2\theta},
$$

$$
\alpha_3(r, \theta) = \frac{(2 - r^2)\cos^3 \theta - 3(1 - r^2)\cos \theta}{\sqrt{1 - r^2}\sin^3 \theta},
$$

$$
\alpha_4(r, \theta) = \frac{-(2 - r^2)\cos 2\theta}{\sqrt{1 - r^2}\sin 2\theta},
$$

$$
\alpha_5(r, \theta) = \frac{(3 - 4r^2)\sin \theta - (2 - r^2)\sin^3 \theta}{\sqrt{1 - r^2}\cos^3 \theta}.
$$

Define

$$
M_1(r, \theta) = \max\{\alpha_1, \alpha_2, \alpha_3\},
$$

$$
M_2(r, \theta) = \max\{\alpha_3, \alpha_4, \alpha_5\},
$$

$$
m_1(r, \theta) = \min\{\alpha_1, \alpha_2, \alpha_3\},
$$

$$
m_2(r, \theta) = \min\{\alpha_3, \alpha_4, \alpha_5\}.
$$

Let $A$ be a four dimensional fusion algebra with basis $\{x_0, x_1, x_2, x_3\}$ such that $x_1^* = x_1$ and $x_2^* = x_3$. Let $L_i$ be the matrices of multiplication by $x_i$ with respect to the above basis. Then $L_0 = I_4$, $L_3 = L_2^*$ and $L_1$ and $L_2$ are necessarily among one of the following four types:

Type I: $L_1$ and $L_2$ as in Eq. (4), $0 \leq \theta < \pi/2$ and $a > M_1(r, \theta),

Type II: $L_1$ and $L_2$ as in Eq. (4), $\pi/2 < \theta < 3\pi/2$ and $a \leq m_2(r, \theta),

Type III: $L_1$ and $L_2$ as in Eq. (5), $\pi/2 < \theta < \pi$ and $a \leq m_1(r, \theta),

Type IV: $L_1$ and $L_2$ as in Eq. (5), $3\pi/2 < \theta < 2\pi$ and $a \geq M_2(r, \theta).

Proof. We have already seen that the matrices $L_1$ and $L_2$ for a four dimensional signed fusion algebra are of the form (4) or (5). For genuine fusion algebras, the entries have to be non-negative. This forces the above ranges where $\theta$ has to lie and the corresponding intervals for $a$. \qed

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References