ELSEVIER Linear Algebra and its Applications 287 (1999) 87–103

LINEAR ALGEBRA AND ITS APPLICATIONS

On finite-dimensional commutative non-hermitian fusion algebras

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Received 12 February 1998; accepted 6 July 1998

Submitted by R. Bhatia

Dedicated to Ludwig Elsner on the occasion of his 60th birthday

Abstract

We characterize the three and four dimensional commutative non-hermitian fusion algebras and construct some new examples of these objects. These algebras arise naturally in the study of graphs, specially those associated with von Neumann algebras. Characterisations of hermitian fusion algebras have been given earlier by Sunder and Wildberger. Commutative finite-dimensional non-herimitian fusion algebras are algebraically isomorphic to certain special Cartan subalgebras of matrices. Every Cartan subalgebra of M_n is a conjugate of the standard Cartan algebra by an orthogonal matrix. We characterize the orthogonal matrices that can occur here and thus characterize the four dimensional non-hermitian fusion algebras. The three dimensional ones are parametrized by the hyperbola $\{(x,y) : y^2 - x^2 = 1 \text{ and } x, y > 0\}$. By restricting to a special subclass of orthogonal matrices obtained by the above characterization, we construct a family of new commutative finite-dimensional non-hermitian fusion algebras.

1. Preliminaries

Finite-dimensional commutative fusion algebras arise naturally in the context of graphs. These have been studied for general graphs and certain special graphs associated with von Neumann algebras in [4–6]. The hermitian ones

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0024-3795/99/$ – see front matter
PII: S0024-3795(98)10135-0

have been characterized in [6]. In this note, we endeavour to study the non-hermitian ones which are defined below. The result about the three dimensional commutative non-hermitian fusion algebras can be deduced straight from the definition (see Lemma 1.2). In the rest of the first section we set up notations and state the known results about the isomorphic images of an $(n+1)$-dimensional fusion algebra in M_{n+1}. These are known to be Cartan subalgebras. In Section 2 we characterize those orthogonal matrices which can conjugate the standard Cartan algebra into a fusion algebra. Theorem 2.5 is the main result of this section. This is applied in Sections 3 and 4, respectively to get the examples in any finite dimension and to obtain the characterization of the four dimensional commutative non-hermitian fusion algebras.

Definition 1.1. A (finite-dimensional) *fusion algebra* is an associative unital $*$-algebra $\mathscr{A}$ with a distinguished basis $\{x_0, x_1, \dots, x_n\}$ such that the 'structure constants' N_{ij}^k defined by the equation

$$x_i x_j = \sum_{k=0}^{n} N_{ij}^k x_k$$

satisfy the following conditions:

(i) $N_{i0}^k = N_{0i}^k = \delta_i^k$; in other words, x_0 is the identity of the algebra $\mathscr{A}$;
(ii) $N_{ij}^k \geqslant 0$;
(iii) there exists an involution $i \mapsto i^*$ of the index set $\{0, 1, \dots, n\}$ such that:
(a) $x_i^* = x_{i^*}$,
(b)

$$N_{ij}^0 = \begin{cases} 1 & \text{if } i = j^*, \\ 0 & \text{if } i \neq j^*. \end{cases}$$

We shall call $\mathscr{A}$ a signed *fusion algebra* if it satisfies conditions (i) and (iii) (a), (b) above and the requirement that $N_{ij}^k \in \mathbb{R}$ (rather than (ii) above).

Finally, we shall call a (signed) fusion algebra *hermitian* if it is the case that $x_i^* = x_i \; \forall i$.

We shall restrict our attention to commutative non-hermitian fusion algebras. We start with the three dimensional ones.

Lemma 1.2. *The three dimensional commutative non-hermitian fusion algebras are in one–one correspondence with the points on the hyperbola* $\{(x, y) : y^2 - x^2 = 1, x, y > 0\}$.

Proof. The structure constants N_{ij}^k of a fusion algebra are such that for any i, j, k in the index set, $N_{ij}^k = N_{kj^*}^i$. This can be seen as follows.

$$x_i x_j x_k^* = \sum_{l=0}^{n} N_{ij}^l x_l x_k^* = \sum_{l,m=0}^{n} N_{ij}^l N_{lk^*}^m x_m,$$

whence we find that N_{ij}^k is the coefficient of x_0 in $(x_i x_j x_{k^*})$. Thus $N_{kj^*}^i$ is the coefficient of x_0 in $(x_k x_{j^*} x_{i^*})$. Since these two are conjugates of each other, x_0 has the same coefficient in both of them.

Using the above, it is possible to characterize a three dimensional non-hermitian signed fusion algebra. The basis of such an algebra is of the form $\{1, \alpha, \alpha^*\}$. The relation of the structure constants shows that the coefficient of α in α^2 and $\alpha\alpha^*$ are same. Let

$$\alpha^2 = \xi\alpha + \eta\alpha^* \quad \text{and} \quad \alpha\alpha^* = 1 + \xi(\alpha - \alpha^*).$$

So the matrix of multiplication by α is:

$$L_\alpha = \begin{bmatrix} 0 & 0 & 1 \\ 1 & \xi & \xi \\ 0 & \eta & \xi \end{bmatrix}.$$

Thus

$$L_{\alpha\alpha^*} = L_\alpha L_\alpha^* = \begin{bmatrix} 1 & \xi & \xi \\ \xi & 1 + 2\xi^2 & \xi\eta + \xi^2 \\ \xi & \xi\eta + \xi^2 & \eta^2 + \xi^2 \end{bmatrix}.$$

On the other hand, since $\alpha\alpha^* = 1 + \xi(\alpha + \alpha^*)$,

$$L_{\alpha\alpha^*} = I_3 + \xi(L_\alpha + L_\alpha^*) = \begin{bmatrix} 1 & \xi & \xi \\ \xi & 1 + 2\xi^2 & \xi\eta + \xi^2 \\ \xi & \xi\eta + \xi^2 & 1 + 2\xi^2 \end{bmatrix}.$$

Equating these two matrices, we get $\eta^2 - \xi^2 = 1$. Conversely, let (ξ, η) be any point on the hyperbola $\eta^2 - \xi^2 = 1$. Then any three dimensional associative unital *-algebra with a basis of the form $\{1, \alpha, \alpha^*\}$ satisfying the multiplication rules

$$\alpha^2 = \xi\alpha + \eta\alpha^* \quad \text{and} \quad \alpha\alpha^* = 1 + \xi(\alpha + \alpha^*) = \alpha^*\alpha$$

is a signed fusion algebra. If we restrict to the positive ξ-axis, we get a genuine fusion algebra. A commutative three dimensional non-hermitian (signed) fusion algebra thus corresponds to a point on the hyperbola $\eta^2 - \xi^2 = 1$. □

Given an $(n+1)$-dimensional fusion algebra $\mathscr{A}$, there is a natural *-algebra homomorphism, which we denote by φ, of $\mathscr{A}$ into $M_{n+1}(\mathbb{R})$ if we map x_i to the matrix L_i of left multiplication by x_i with respect to the ordered basis $\{x_0, x_1, \ldots, x_n\}$ and extend linearly. Note that the matrix L_i thus obtained has the property that the first column has all zeros except the entry on the ith row, which is 1 (because x_0 is the identity of the algebra) and the first row too has all zeros except the entry on column i^*, which is 1 (because of the restrictions on N_{ij}^0). Let the algebra generated by $L_0, L_1, \ldots, L_n$ be $\mathscr{C}$. This is the matrix algebra which is φ-isomorphic to $\mathscr{A}$. It is an $(n+1)$-dimensional commutative C^*-subalgebra of $M_{n+1}(\mathbb{R})$. The following result says that it is also a maximal commutative C^*-subalgebra.

Lemma 1.3. *The following conditions on a commutative C^*-subalgebra $\mathscr{C}$ of $M_{n+1}(\mathbb{R})$ are equivalent:*
(a) $\dim \mathscr{C} = n+1$;
(b) *$\mathscr{C}$ is a maximal commutative subalgebra of $M_{n+1}(\mathbb{R})$.*

For a proof of this result, see [1,2]. Such a subalgebra is called a *Cartan subalgebra*. So every commutative fusion algebra $\mathscr{A}$ with basis $\{x_0, x_1, \dots, x_n\}$ is isomorphic to a Cartan subalgebra $\mathscr{C}$ of $M_{n+1}(\mathbb{R})$ generated by $L_0, L_1, \dots, L_n$. We briefly sketch below those properties of a Cartan subalgebra which we shall need.

Lemma 1.4. *Any Cartan subalgebra $\mathscr{C}$ of $M_{n+1}(\mathbb{R})$ determines a unique pair (p,q) of non-negative integers. The set of minimal projections of $\mathscr{C}$ is given by $\{P_1, \dots, P_p, Q_1, \dots, Q_q\}$, where P_i and Q_j are projections of rank one and two, respectively. One of the integers p and q, however, can be zero.*

The set of minimal projections $\{P_1, \dots, P_p, Q_1, \dots, Q_q\}$ forms a basis for the real vector subspace of $\mathscr{C}$ consisting of symmetric matrices. The subspace of skew-symmetric matrices in $\mathscr{C}$ has dimension q. There is a basis $\{S_1, S_2, \dots, S_q\}$ of this vector space satisfying $S_j = Q_j S_j Q_j$ and $\|S_j\| = 1$, for all $j = 1, \dots, q$. Moreover, the matrices S_j are unique (upto sign) i.e., if T_j form another basis satisfying $T_j = Q_j T_j Q_j$ and $\|T_j\| = 1$, then $T_j = \pm S_j$. For more details on Cartan subalgebras, see [1–3]. The Cartan subalgebras which we are concerned about viz., the ones which are isomorphic images of fusion algebras have the following property.

Remark 1.5. If the Cartan algebra $\mathscr{C}$ arises from a fusion algebra through the isomorphism φ, then at least one projection of rank 1 exists in $\mathscr{C}$. This can be seen from the fact that every fusion algebra admits a unique dimension function (see [4]). Thus we number the rank one projections as $P_0, P_1, \dots, P_p$. This, in particular, means that $p + 2q = n$. From now on, we shall consider only such Cartan algebras.

If $\mathscr{A}$ is an $(n+1)$-dimensional fusion algebra among whose basis elements there are exactly p selfadjoint elements other than identity and q non-selfadjoint elements then the Cartan subalgebra $\mathscr{C}$ is as in Remark 1.5. The following proposition from [6] provides a complete characterization and one–one correspondence between fusion algebras and Cartan subalgebras of $M_{n+1}(\mathbb{R})$. Let $\{v_0, v_1, \dots, v_n\}$ be the standard basis for $\mathbb{R}^{n+1}$.

Proposition 1.6. (a) *Let $\mathscr{A}$ be a fusion algebra with basis $\{x_0, x_1, \ldots, x_n\}$. Let $I = \{i : i = i^*\}$ and $J \subset \{1, 2, \ldots, n\} \setminus I$ be such that $j \in J \Rightarrow j^* \notin J$ and $|I| + 2|J| = n$. Let $\mathscr{C}$ be that Cartan algebra which is the φ-isomorphic image of $\mathscr{A}$. Then,*

(i) *v_0 is a cyclic vector for $\mathscr{C}$; and*

(ii) *$\{S_j v_0 : 1 \leqslant j \leqslant q\}$ and $\{v_j - v_{j^*} : j \in J\}$ are both orthogonal bases for the same subspace of $\mathbb{R}^{n+1}$.*

(b) *Conversely, suppose $\mathscr{C}$ is a Cartan subalgebra of $M_{n+1}(\mathbb{R})$, with minimal projections $P_0, \ldots, P_p, Q_1, \ldots, Q_q$ as in Remark 1.4. Suppose there exists a 2q element set $J \cup \{j^* : j \in J\} \subset \{1, 2, \ldots, n\}$ such that conditions* (i) *and* (ii) *of* (a) *are satisfied. Then there exists a signed fusion algebra $\mathscr{A}$ with precisely p selfadjoint basis elements such that $\mathscr{C}$ is the Cartan subalgebra associated with $\mathscr{A}$.*

(c) *Suppose $\mathscr{C}$ is as in* (b) *above; let $\{A_j : 0 \leqslant j \leqslant n\}$ be any basis for $\mathscr{C}$. (For instance, we may take the basis as $P_0, P_1, \ldots, P_p, Q_1, S_1, Q_2, S_2, \ldots, Q_q, S_q$.) Define the matrix $B \in M_{n+1}(\mathbb{R})$ by $b_{ij} = \langle A_j v_0, v_i \rangle$. (Thus, the jth column of the matrix B is just $A_j v_0$.) Then B is a non-singular matrix. Let $C = B^{-1}$. Define*

$$L_k = \sum_{j=0}^{n} c_{jk} A_j.$$

Then these are precisely the matrices L_{v_k} obtained from the signed fusion algebra corresponding to $\mathscr{C}$ as in (b) *above.*

2. Fusion algebras and orthogonal matrices

Henceforth, we shall concentrate on commutative non-hermitian fusion algebras which have two non-selfadjoint elements in their distinguished bases; consequently, we shall only consider Cartan algebras in $M_{n+1}(R)$ whose 'anti-symmetric part' is one-dimensional, or equivalently, for which $q = 1$.

Lemma 2.1. *Given any Cartan subalgebra $\mathscr{C}$ of $M_{n+1}(\mathbb{R})$, there is an orthogonal matrix U such that $\mathscr{C} = U^* \mathscr{C}_0 U$ where $\mathscr{C}_0$ is the Cartan subalgebra generated by the following set of matrices.*

$$\left\{ \begin{bmatrix} \lambda_0 & 0 & \cdots & 0 & 0 \\ 0 & \lambda_1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & \lambda_p & 0 \\ 0 & 0 & \cdots & 0 & aR_\theta \end{bmatrix} : \lambda_i, a \in \mathbb{R}, \quad \theta \in [0, 2\pi] \right\},$$

where

$$R_\theta = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}.$$

This is called the standard Cartan subalgebra.

The above lemma is a very well-known fact. See, for example [1], p. 35 for the proof. The standard Cartan subalgebra is cyclic. Given $(n+1)$ real numbers $y_0, y_1, \dots, y_n$ satisfying

$$y_0, \dots, y_{n-2} \neq 0 \quad \text{and} \quad y_{n-1}^2 + y_n^2 \neq 0, \tag{1}$$

the vector $y = (y_0, y_1, \dots, y_n)$ is a cyclic vector for the Cartan algebra $\mathscr{C}_0$. The next lemma tells how to construct fusion algrebras by conjugating the standard Cartan subalgebra with a suitable orthogonal matrix. It gives a sufficient condition which an orthogonal matrix U should satisfy to make $U^*\mathscr{C}_0 U$ a fusion algebra. A full characterization of such orthogonal matrices is presented in Lemma 2.3.

Lemma 2.2. *Let $y = (y_0, y_1, \dots, y_n)$ be a unit vector and U be an $(n+1) \times (n+1)$ orthogonal matrix where y satisfies Eq.* (1) *and U satisfies*

$$Uv_0 = y \quad \textit{and} \quad U(v_{n-1} - v_n) = \left(\frac{2}{y_{n-1}^2 + y_n^2}\right)^{1/2} (0, \dots, 0, -y_n, y_{n-1}). \quad (*)$$

Then the matrix algebra $U^*\mathscr{C}_0 U$ can be endowed with a (signed) fusion algebra structure with exactly two elements of the basis being non-selfadjoint.

Proof. Let $\mathscr{C} = U^*\mathscr{C}_0 U$. First note that $\mathscr{C}$ is a Cartan subalgebra by Lemma 1.3 because conjugation by an orthogonal matrix does not change the dimension and also retains commutativity. All that is needed to prove is that $\mathscr{C}$ satisfies the two conditions of Proposition 1.6(a). Then the rest will follow from 1.6(b) and (c).

The vector $y = (y_0, y_1, \dots, y_n)$ is of the above form so that it is a cyclic vector of the standard Cartan algebra. This, in view of condition (i), means that v_0 becomes a cyclic vector for the algebra $\mathscr{C}$.

The minimal projections for $\mathscr{C}_0$ are given by

$$\begin{aligned} F_0 &= \operatorname{diag}(1, 0, 0, \dots, 0), \\ F_1 &= \operatorname{diag}(0, 1, 0, \dots, 0), \\ &\vdots \\ F_{n-2} &= \operatorname{diag}(0, 0, \dots, 0, 1, 0, 0), \\ F_{n-1} &= \operatorname{diag}(0, 0, \dots, 0, 0, 1, 1). \end{aligned} \tag{2}$$

Conjugating these with the orthogonal matrix U and setting $A_i = U^*F_iU$ for $i = 1, \ldots, n-1$, we get the minimal projections $\{A_0, A_1, \ldots, A_{n-1}\}$ for the Cartan subalgebra $\mathscr{C}$. The only two dimensional projection is A_{n-1}. The anti-symmetric part of $\mathscr{C}_0$ is the span of $F_n = O_{n-1} \oplus \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. Let $A_n = U^*F_nU$. The unique (upto sign) anti-symmetric matrix $S_1 \in \mathscr{C}$ which satisfies $A_{n-1}S_1A_{n-1} = S_1$ and $\|S_1\| = 1$ is, in this case, just A_n. Since condition (*) is true for U, we see that

$$\begin{aligned} A_nv_0 &= U^*F_nUv_0 = U^*F_n(y_0, y_1, \ldots, y_n) \\ &= U^*(0, \ldots, 0, -y_n, y_{n-1}) = \left(\frac{y_{n-1}^2 + y_n^2}{2}\right)^{1/2} (v_{n-1} - v_n). \end{aligned}$$

This means $A_nv_0/\|A_nv_0\| = (1/\sqrt{2})(v_{n-1} - v_n)$. Thus the algebra $\mathscr{C}$ satisfies the two conditions of Proposition 1.6(a) with $q = 1$. Hence it can be endowed with a fusion algebra structure following 1.6(c). □

Lemma 2.3. *Let* $y = (y_0, y_1, \ldots, y_n)$ *be any unit vector in* $\mathbb{R}^{n+1}$; *define* $c_k = (y_0^2 + \cdots + y_k^2)^{1/2}$, *for* $0 \leqslant k \leqslant n$; *then there exists a unique* $(n+1) \times (n+1)$*-orthogonal matrix* U_0 *whose action on the standard basis is as follows:*

$$\begin{aligned} U_0v_0 &= (y_0, \ldots, y_n), \\ U_0v_k &= \frac{y_{k+1}}{c_kc_{k+1}} \left(y_0, \ldots, y_k, \frac{-c_k^2}{y_{k+1}}, 0, \ldots, 0\right) \quad \textit{for } k = 1, \ldots, n-3, \\ U_0v_{n-2} &= \frac{(y_{n-1}^2 + y_n^2)^{1/2}}{c_{n-2}} \left(y_0, \ldots, y_{n-2}, \frac{-y_{n-1}c_{n-2}^2}{y_{n-1}^2 + y_n^2}, \frac{-y_nc_{n-2}^2}{y_{n-1}^2 + y_n^2}\right), \\ U_0v_{n-1} &= \frac{1}{\sqrt{2}} \left(\frac{y_1}{c_1}, \frac{-y_0}{c_1}, 0, \ldots, 0, \frac{-y_n}{(y_{n-1}^2 + y_n^2)^{1/2}}, \frac{y_{n-1}}{(y_{n-1}^2 + y_n^2)^{1/2}}\right), \\ U_0v_n &= \frac{1}{\sqrt{2}} \left(\frac{y_1}{c_1}, \frac{-y_0}{c_1}, 0, \ldots, 0, \frac{y_n}{(y_{n-1}^2 + y_n^2)^{1/2}}, \frac{-y_{n-1}}{(y_{n-1}^2 + y_n^2)^{1/2}}\right). \end{aligned}$$

Given any $(n-1) \times (n-1)$ *orthogonal matrix* W_0 *let* W *be the* $(n+1) \times (n+1)$ *matrix*:

$$\begin{aligned} W = &\left(I_{n-1} \oplus (1/\sqrt{2})\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\right)(I_1 \oplus W_0 \oplus I_1) \\ &\times \left(I_{n-1} \oplus (1/\sqrt{2})\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\right). \end{aligned}$$

Then, an orthogonal matrix U *satisfies*

$$Uv_0 = y \quad and \quad U(v_{n-1} - v_n) = \left(\frac{2}{y_{n-1}^2 + y_n^2}\right)^{1/2} (0, \dots, 0, -y_n, y_{n-1}) \tag{3}$$

if and only if $U = U_0 W$, *where* U_0 *and* W *are as above, for some* $W_0 \in O(n-1, \mathbb{R})$.

Proof. First note that by construction U_0 satisfies

$$U_0 v_0 = y \quad \text{and} \quad U_0(v_{n-1} - v_n) = \left(\frac{2}{y_{n-1}^2 + y_n^2}\right)^{1/2} (0, \dots, 0, -y_n, y_{n-1}).$$

The matrix W is such that it leaves v_0 and $v_{n-1} - v_n$ unchanged. So if $U = U_0 W$, then U clearly satisfies Eq. (3).

Conversely, let the orthogonal matrix U be specified on just the two dimensional subspace spanned by v_0 and $v_{n-1} - v_n$ as in Eq. (3). Let $W = U_0^{-1} U$. Since both U_0 and U satisfy Eq. (3), the orthogonal matrix W leaves the vectors v_0 and $v_{n-1} - v_n$ unchanged. Consider the following orthogonal matrix:

$$\tilde{W} = \left(I_{n-1} \oplus (1/\sqrt{2})\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\right) W \left(I_{n-1} \oplus (1/\sqrt{2})\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\right).$$

Note that $\tilde{W} v_0 = v_0$ and $\tilde{W} v_n = v_n$. Thus $\tilde{W} = I_1 \oplus W_0 \oplus I_1$ for some $(n-1) \times (n-1)$ orthogonal matrix W_0. That gives the result. □

This lemma leads us to a converse of Lemma 2.2 in the following way.

Lemma 2.4. *Let* $\mathscr{C}$ *be a Cartan subalgebra of* $M_{n+1}(\mathbb{R})$ *arising from a fusion algebra through the isomorphism* φ. *Then* $\mathscr{C} = U^* \mathscr{C}_0 U$ *for some orthogonal matrix* U *of the form* $U_0 W$ *where* U_0 *and* W *are as above.*

Proof. Let $\mathscr{C}$ be a cyclic Cartan subalgebra of $M_{n+1}(\mathbb{R})$ with minimal projections $P_0, \dots, P_{n-2}, Q$ and cyclic vector v_0. Also let S be the unique (up to sign) skew-symmetric matrix in $\mathscr{C}$ such that $QSQ = S$ and $\|S\| = 1$. Then there is an orthogonal matrix U such that $\mathscr{C} = U^* \mathscr{C}_0 U$ and $Uv_0 = y = (y_0, y_1, \dots, y_n)$ is a cyclic vector for the standard Cartan algebra $\mathscr{C}_0$. If, moreover, we demand that $Sv_0/\|Sv_0\| = (v_{n-1} - v_n)/\sqrt{2}$, which we may in case if $\mathscr{C} = \varphi(\mathscr{A})$ for some fusion algebra $\mathscr{A}$, then $U(v_{n-1} - v_n) = \sqrt{2/(y_{n-1}^2 + y_n^2)}$. Lemma 2.3 then implies that $U = U_0 W$. Note that the Cartan algebra $\mathscr{C}$ exactly satisfies the requirements which would associate with it a fusion algebra according to the isomorphism φ. Thus if we fix a cyclic vector y for the standard Cartan algebra, then to any fusion algebra there corresponds to an $(n-1) \times (n-1)$ orthogonal matrix $U = U_0 W$ where U_0 and W are as in Lemma 2.3. □

The results of this section are summarised in the following theorem.

Theorem 2.5. *Let $\mathscr{A}$ be an $(n+1)$-dimensional non-hermitian fusion algebra with exactly two non-selfadjoint elements in the basis. Let $\mathscr{C}$ be the Cartan subalgebra of $M_{n+1}(\mathbb{R})$ which is isomorphic to $\mathscr{A}$ via the map φ. Then $\mathscr{C} = U^*\mathscr{C}_0 U$ if and only if U is an orthogonal matrix satisfying Eq.* (3).

3. A family of non-hermitian fusion algebras

Now we use the theorems of the last section to construct non-hermitian fusion algebras of all finite dimensions with exactly two non-hermitian elements in their bases.

Theorem 3.1. *Let $y_0, y_1, \ldots, y_n$ be real numbers satisfying Eq.* (1) *and $y_0^2 + y_1^2 \cdots + y_n^2 = 1$. Let $c_k = (y_0^2 + y_1^2 + \cdots + y_k^2)^{(1/2)}$ for $k = 1, \ldots, n-2$. The following set of $(n+1)$ matrices forms the basis of a (signed) fusion algebra:*

$$L_0 = I_{n+1},$$

$$L_k = \begin{bmatrix} 0 & 0 & \cdots & 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & \frac{y_{k+1}}{c_k c_{k+1}} & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & \frac{y_{k+1}}{c_k c_{k+1}} & 0 & 0 & \cdots & 0 & 0 \\ 1 & 0 & \cdots & 0 & \frac{y_{k+1}^2 - c_k^2}{y_{k+1} c_k c_{k+1}} & \frac{y_{k+2}}{c_{k+1} c_{k+2}} & \cdots & \frac{y_{n-2}}{c_{n-3} c_{n-2}} & \frac{(1-c_{n-2}^2)^{1/2}}{c_{n-2}} \\ 0 & 0 & \cdots & 0 & \frac{y_{k+2}}{c_{k+1} c_{k+2}} & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & \frac{y_{n-2}}{c_{n-3} c_{n-2}} & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & \frac{(1-c_{n-2}^2)^{1/2}}{c_{n-2}} & 0 & \cdots & 0 & 0 \end{bmatrix}$$

$$\oplus \frac{y_k}{2c_k c_{k+1}} I_2 \quad \textit{for } k = 1, \ldots, n-3,$$

$$L_{n-2} = \begin{bmatrix} 0 & 0 & \cdots & 0 & 1 \\ 0 & \frac{(1-c_{n-2}^2)^{1/2}}{c_{n-2}} & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & \frac{(1-c_{n-2}^2)^{1/2}}{c_{n-2}} & 0 \\ 1 & 0 & \ddots & 0 & \frac{(1-2c_{n-2}^2)}{c_{n-2}(1-c_{n-2}^2)^{1/2}} \end{bmatrix} \oplus \frac{(1-2c_{n-2}^2)}{c_{n-2}(1-c_{n-2}^2)^{1/2}} I_2,$$

$$L_{n-1} = \begin{bmatrix} 0 & 0 & \cdots & 0 & 0 & 0 & 1 \\ 0 & 0 & \cdots & 0 & 0 & \frac{y_2}{2c_1c_2} & \frac{y_2}{2c_1c_2} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & \frac{y_{n-2}}{2c_{n-3}c_{n-2}} & \frac{y_{n-2}}{2c_{n-3}c_{n-2}} \\ 0 & 0 & \cdots & 0 & 0 & \frac{1}{2c_{n-2}(1-c_{n-2}^2)^{1/2}} & \frac{1-2c_{n-2}^2}{2c_{n-2}(1-c_{n-2}^2)^{1/2}} \\ 1 & \frac{y_2}{2c_1c_2} & \cdots & \frac{y_{n-2}}{2c_{n-3}c_{n-2}} & \frac{(1-2c_{n-2}^2)}{2c_{n-2}(1-c_{n-2}^2)^{1/2}} & \frac{y_1^2-y_0^2}{2\sqrt{2}y_0y_1c_1} & \frac{y_1^2-y_0^2}{2\sqrt{2}y_0y_1c_1} \\ 0 & \frac{y_2}{2c_1c_2} & \cdots & \frac{y_{n-2}}{2c_{n-3}c_{n-2}} & \frac{1}{2c_{n-2}(1-c_{n-2}^2)^{1/2}} & \frac{y_1^2-y_0^2}{2\sqrt{2}y_0y_1c_1} & \frac{y_1^2-y_0^2}{2\sqrt{2}y_0y_1c_1} \end{bmatrix};$$

$$L_n = L_{n-1}^*.$$

Proof. We take the orthogonal matrix U_0 defined earlier and conjugate the basis $\{F_0, \ldots, F_n\}$ of the standard Cartan algebra defined in Eq. (2) with it to get the basis $\{A_0, \ldots, A_n\}$. We form the matrix B, whose jth column is the 1st column of A_j. It is as follows.

$$\begin{aligned} B &= \left[U_0^* F_1 U_0 v_0, U_0^* F_2 U_0 v_0, \ldots, U_0^* F_n U_0 v_0\right] \\ &= U_0^*[F_1 y, F_2 y, \ldots, F_n y] = U_0^*\left(\operatorname{diag}(y_0, y_1, \ldots, y_{n-2}) \oplus \begin{pmatrix} y_{n-1} & -y_n \\ y_n & y_{n-1} \end{pmatrix}\right). \end{aligned}$$

Set $C = B^{-1}$. Then

$$C = \left(\operatorname{diag}(1/y_0, 1/y_1, \ldots, 1/y_{n-2}) \oplus (1/(1-c_{n-2}^2))\begin{pmatrix} y_{n-1} & y_n \\ -y_n & y_{n-1} \end{pmatrix}\right) U_0.$$

Then the columns of C are as follows:

$$Cv_0 = (1, \ldots, 1, 0, 0),$$

$$Cv_k = \frac{y_{k+1}}{c_k c_{k+1}}\left(1, \ldots, 1, \frac{-c_k^2}{y_{k+1}^2}, 0, \ldots, 0\right) \quad \text{for } k = 1, \ldots, n-3,$$

$$Cv_{n-2} = \frac{(1-c_{n-2}^2)^{1/2}}{c_{n-2}}\left(1, \ldots, 1, \frac{-c_{n-2}^2}{1-c_{n-2}^2}, 0\right),$$

$$Cv_{n-1} = \frac{1}{\sqrt{2}}\left(\frac{y_1}{y_0 c_1}, \frac{-y_0}{y_1 c_1}, 0, \ldots, 0, 0, (1-c_{n-2}^2)^{-1/2}\right),$$

$$Cv_n = \frac{1}{\sqrt{2}}\left(\frac{y_1}{y_0 c_1}, \frac{-y_0}{y_1 c_1}, 0, \ldots, 0, 0, -(1-c_{n-2}^2)^{-1/2}\right).$$

Now we form the matrices

$$L_k = \sum_{j=0}^{n} c_{jk} A_j = U_0^* \left(\sum_{j=0}^{n} c_{jk} F_j \right) U_0.$$

The matrices $L_0, L_1, \ldots, L_n$ are of the above form. They form a basis of the algebra $U_0^* \mathscr{C}_0 U_0$. $\square$

Remark 3.2. If we choose

$$y_0 > 0, \quad y_{k+1} > c_k \quad \text{for } k = 0, 1, \ldots, n-3, \quad \text{and} \quad c_{n-2}^2 < 1/2$$

we get a genuine fusion algebra.

Here are some numerical examples of the matricial fusion algebras of the above theorem for special values of the parameters for $n = 4$, i.e., in dimension five. Note that the $(n+1)$-dimensional ones which we obtained depend on $n-1$ parameters. Since $L_0 = I_{n+1}$ and $L_n = L_{n-1}^*$ always, we are concerned about $L_1, \ldots, L_{n-1}$ only, in this case, L_1, L_2, L_3. The parameters are y_0, y_1 and y_2 subject to $y_1 \geqslant y_0 > 0, y_2 \geqslant \sqrt{y_0^2 + y_1^2}$ and $y_0^2 + y_1^2 + y_2^2 < 1/2$. We further specialize by putting $y_2 = \sqrt{y_0^2 + y_1^2}$. Then $c_2 = c_1\sqrt{2}$ and $c = c_1 = c_2/\sqrt{2} < 1/2$. This can be made further simpler by the following substitution:

$$c\sqrt{2} = \sin\theta, \quad y_0 = c\cos\varphi, \quad y_1 = c\sin\varphi.$$

Then

$$L_1 = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & \cot\theta & 0 & 0 \\ 0 & \cot\theta & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{\sin\varphi}{\sin\theta} & \frac{\sin\varphi}{\sin\theta} \\ 0 & 0 & 0 & \frac{\sin\varphi}{\sin\theta} & \frac{\sin\varphi}{\sin\theta} \end{pmatrix},$$

$$L_2 = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & \cot\theta & 0 & 0 & 0 \\ 1 & 0 & \frac{1}{2}\cot 2\theta & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2}\cot 2\theta & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{2}\cot 2\theta \end{pmatrix},$$

$$L_3 = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & \frac{1}{2\sin\theta} & \frac{1}{2\sin\theta} \\ 0 & 0 & 0 & \frac{1}{\sin 2\theta} & \cot 2\theta \\ 1 & \frac{1}{2\sin\theta} & \cot 2\theta & -\frac{\cot 2\varphi}{\sin\theta} & -\frac{\cot 2\varphi}{\sin\theta} \\ 0 & \frac{1}{2\sin\theta} & \frac{1}{\sin 2\theta} & -\frac{\cot 2\varphi}{\sin\theta} & -\frac{\cot 2\varphi}{\sin\theta} \end{pmatrix}.$$

By the conditions on y_0, y_1, y_2, one has $0 < \theta < \pi/4$ and $\pi/4 < \varphi < \pi/2$. Putting $\theta = \pi/6$ and $\varphi = \pi/3$, we get

$$L_1 = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & \sqrt{3} & 0 & 0 \\ 0 & \sqrt{3} & 0 & 0 & 0 \\ 0 & 0 & 0 & \sqrt{3} & \sqrt{3} \\ 0 & 0 & 0 & \sqrt{3} & \sqrt{3} \end{pmatrix},$$

$$L_2 = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & \sqrt{3} & 0 & 0 & 0 \\ 1 & 0 & \frac{1}{2}\sqrt{3} & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{\sqrt{3}} & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{\sqrt{3}} \end{pmatrix},$$

$$L_3 = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & \frac{2}{\sqrt{3}} & \sqrt{3} \\ 1 & 1 & \sqrt{3} & 2\sqrt{3} & 2\sqrt{3} \\ 0 & 1 & \frac{2}{\sqrt{3}} & 2\sqrt{3} & 2\sqrt{3} \end{pmatrix}.$$

4. Four dimensional non-hermitian fusion algebras

Theorem 4.1. *Let $\mathscr{A}$ be a four dimensional signed fusion algebra with basis $\{x_0, x_1, x_2, x_3\}$ such that $x_1^* = x_1$ and $x_2^* = x_3$. Then the matrices L_i of multiplication by x_i with respect to the above basis are as follows: $L_0 = I_4, L_3 = L_2^*$ and L_1 and L_2 are either*

$$L_1(a,r,\theta) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & f_1(a,r,\theta) & f_2(a,r,\theta)\sin\theta & f_2(a,r,\theta)\sin\theta \\ 0 & f_2(a,r,\theta)\sin\theta & f_3(a,r,\theta)\cos\theta & \frac{1}{2}f_2(a,r,\theta)\cos\theta \\ 0 & f_2(a,r,\theta)\sin\theta & \frac{1}{2}f_2(a,r,\theta)\cos\theta & f_3(a,r,\theta)\cos\theta \end{bmatrix},$$
$$L_2(a,r,\theta) = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & f_2(a,r,\theta)\sin\theta & \frac{1}{2}f_2(a,r,\theta)\cos\theta & f_3(a,r,\theta)\cos\theta \\ 1 & f_3(a,r,\theta)\cos\theta & f_4(a,r,\theta) & f_4(a,r,\theta) \\ 0 & \frac{1}{2}f_2(a,r,\theta)\cos\theta & f_5(a,r,\theta) & f_4(a,r,\theta) \end{bmatrix} \tag{4}$$

or

$$L_1(a,r,\theta)=\begin{bmatrix}0 & 1 & 0 & 0\\ 1 & -f_1(a,r,\theta) & f_2(a,r,\theta)\sin\theta & f_2(a,r,\theta)\sin\theta\\ 0 & f_2(a,r,\theta)\sin\theta & -f_3(a,r,\theta)\cos\theta & -\frac{1}{2}f_2(a,r,\theta)\cos\theta\\ 0 & f_2(a,r,\theta)\sin\theta & -\frac{1}{2}f_2(a,r,\theta)\cos\theta & -f_3(a,r,\theta)\cos\theta\end{bmatrix},$$

$$L_2(a,r,\theta)=\begin{bmatrix}0 & 0 & 0 & 1\\ 0 & f_2(a,r,\theta)\sin\theta & -\frac{1}{2}f_2(a,r,\theta)\cos\theta & -f_3(a,r,\theta)\cos\theta\\ 1 & -f_3(a,r,\theta)\cos\theta & f_4(a,r,\theta) & f_4(a,r,\theta)\\ 0 & -\frac{1}{2}f_2(a,r,\theta)\cos\theta & f_5(a,r,\theta) & e(a,r,\theta)\end{bmatrix}, \tag{5}$$

where

$$\begin{aligned}
&f_1(a,r,\theta)\\
&\quad=\Big(-(2-r^2)\cos^3\theta+3(1-r^2)\cos\theta+a\sqrt{1-r^2}\sin^3\theta\Big)\Big/r\sqrt{1-r^2},\\
&f_2(a,r,\theta)=\Big(r^2+(2-r^2)\cos2\theta+a\sqrt{1-r^2}\sin2\theta\Big)\Big/2\sqrt{2}r\sqrt{1-r^2},\\
&f_3(a,r,\theta)=\Big(-3r^2+(2-r^2)\cos2\theta+a\sqrt{1-r^2}\sin2\theta\Big)\Big/4r\sqrt{1-r^2},\\
&f_4(a,r,\theta)\\
&\quad=\Big(a\sqrt{1-r^2}\cos^3\theta+(2-r^2)\sin^3\theta-(3-4r^2)\sin\theta\Big)\Big/2\sqrt{2}r\sqrt{1-r^2},\\
&f_5(a,r,\theta)=\Big(a\sqrt{1-r^2}\cos^3\theta+(2-r^2)\sin^3\theta-3\sin\theta\Big)\Big/2\sqrt{2}r\sqrt{1-r^2}.
\end{aligned}$$

Proof. In dimension four, the matrix U_0 of Lemma 2.3 is given by,

$$\begin{bmatrix}
y_0 & y_0\sqrt{\frac{y_2^2+y_3^2}{y_0^2+y_1^2}} & \frac{y_1}{\sqrt{2(y_0^2+y_1^2)}} & \frac{y_1}{\sqrt{2(y_0^2+y_1^2)}}\\
y_1 & y_1\sqrt{\frac{y_2^2+y_3^2}{y_0^2+y_1^2}} & \frac{-y_0}{\sqrt{2(y_0^2+y_1^2)}} & \frac{-y_0}{\sqrt{2(y_0^2+y_1^2)}}\\
y_2 & -y_2\sqrt{\frac{y_0^2+y_1^2}{y_2^2+y_3^2}} & \frac{-y_3}{\sqrt{2(y_2^2+y_3^2)}} & \frac{y_3}{\sqrt{2(y_2^2+y_3^2)}}\\
y_3 & -y_3\sqrt{\frac{y_0^2+y_1^2}{y_2^2+y_3^2}} & \frac{y_2}{\sqrt{2(y_2^2+y_3^2)}} & \frac{-y_2}{\sqrt{2(y_2^2+y_3^2)}}
\end{bmatrix},$$

where $y=(y_0,y_1,y_2,y_3)$ is a unit vector such that $y_0\neq0, y_1\neq0$ and $y_0^2+y_1^2<1$. From the form of it, we see that the matrix U_0 depends on two parameters y_0 and y_1 only because $y_2^2+y_3^2$ can be replaced by $1-(y_0^2+y_1^2)$. The matrix W_0 is a 2×2 orthogonal matrix. So it is either

$$\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$$

or

$$\begin{pmatrix} -\cos\theta & \sin\theta \\ \sin\theta & \cos\theta \end{pmatrix},$$

where $\theta \in [0, 2\pi]$. Since W_0 depends on θ (only), let us call it $W_0(\theta)$. Accordingly, W, the matrix defined in the statement of Lemma 2.3, will be called $W(\theta)$. Thus the four dimensional non-hermitian fusion algebras depend on three real parameters. Two substitutions come in handy for simplifying the calculations. They are as follows.

$$a = \frac{y_1^2 - y_0^2}{y_0 y_1} \text{ (note that } y_0 y_1 \neq 0) \text{ and } r = \sqrt{y_0^2 + y_1^2}.$$

Thus L_1 and L_2 are functions of a, r and θ. Among the basis matrices, $L_0 = I_4$ and $L_3 = L_2^*$.

When $W_0(\theta) = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$, we have

$$W(\theta) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta/\sqrt{2} & -\sin\theta/\sqrt{2} \\ 0 & \sin\theta/\sqrt{2} & (1+\cos\theta)/2 & (-1+\cos\theta)/2 \\ 0 & \sin\theta/\sqrt{2} & (-1+\cos\theta)/2 & (1+\cos\theta)/2 \end{bmatrix}.$$

When $\theta = 0$ i.e., $W(\theta)$ is I_4, then L_1 and L_2 are as follows.

$$L_1(a, r, 0) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & \frac{(1-2r^2)}{r\sqrt{1-r^2}} & 0 & 0 \\ 0 & 0 & \frac{(1-2r^2)}{2r\sqrt{1-r^2}} & \frac{1}{2r\sqrt{1-r^2}} \\ 0 & 0 & \frac{1}{2r\sqrt{1-r^2}} & \frac{(1-2r^2)}{2r\sqrt{1-r^2}} \end{bmatrix},$$

$$L_2(a, r, 0) = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & \frac{1}{2r\sqrt{1-r^2}} & \frac{(1-2r^2)}{2r\sqrt{1-r^2}} \\ 1 & \frac{(1-2r^2)}{2r\sqrt{1-r^2}} & \frac{a}{2\sqrt{2}r} & \frac{a}{2\sqrt{2}r} \; h \\ 0 & \frac{1}{2r\sqrt{1-r^2}} & \frac{a}{2\sqrt{2}r} & \frac{a}{2\sqrt{2}r} \end{bmatrix}.$$

It is interesting to see the numerical form of these matrices for special values of r.

$$\text{For } r = 1/\sqrt{2},\ L_1 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \quad L_2 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & a & a \\ 0 & 1 & a & a \end{bmatrix}.$$

$$\text{For } r = 1/2,\ L_1 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 2/\sqrt{3} & 0 & 0 \\ 0 & 0 & 1/\sqrt{3} & 2/\sqrt{3} \\ 0 & 0 & 2/\sqrt{3} & 1/\sqrt{3} \end{bmatrix},$$

$$L_2 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 2/\sqrt{3} & 1/\sqrt{3} \\ 1 & 1/\sqrt{3} & a & a \\ 1 & 2/\sqrt{3} & a & a \end{bmatrix},$$

where a is any non-negative real number.

A long computation which consists of first forming the matrix $U(\theta) = U_0 W(\theta)$ and then conjugating the standard Cartan algebra with it, yields

$$L_1(a,r,\theta) = W(\theta)^*\left(\cos\theta\, L_1(a,r,0) + \frac{\sin\theta}{\sqrt{2}}(L_2(a,r,0) + L_2(a,r,0)^*)\right)W(\theta),$$

$$L_2(a,r,\theta) = W(\theta)^*\left(-\frac{\sin\theta}{\sqrt{2}}L_1(a,r,0) + \frac{1+\cos\theta}{2}(L_2(a,r,0) + \frac{-1+\cos\theta}{2}L_2(a,r,0)^*)\right)W(\theta).$$

After completing the calculation using the forms of $L_1(a,r,0)$ and $L_2(a,r,0)$ above, one gets the above form (4).

If, on the other hand, we had started with $W_0(\theta) = \begin{pmatrix} -\cos\theta & \sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$, then following a similar procedure as above we arrive at the form (5).

Since these are the only two possibilities, thus the matrices L_i associated with any four dimensional signed fusion algebra are bound to be of one of the above forms. □

The characterization of genuine four dimensional fusion algebras is as follows.

Theorem 4.4. *For* $0 < r < 1$ *and* $\theta \in [0, 2\pi]$, *let*

$$\alpha_1(r,\theta) = \frac{3\sin\theta - (2-r^2)\sin^3\theta}{\sqrt{1-r^2}\cos^3\theta},$$
$$\alpha_2(r,\theta) = \frac{3r^2 - (2-r^2)\cos 2\theta}{\sqrt{1-r^2}\sin 2\theta},$$
$$\alpha_3(r,\theta) = \frac{(2-r^2)\cos^3\theta - 3(1-r^2)\cos\theta}{\sqrt{1-r^2}\sin^3\theta},$$
$$\alpha_4(r,\theta) = \frac{-r^2 - (2-r^2)\cos 2\theta}{\sqrt{1-r^2}\sin 2\theta},$$
$$\alpha_5(r,\theta) = \frac{(3-4r^2)\sin\theta - (2-r^2)\sin^3\theta}{\sqrt{1-r^2}\cos^3\theta}.$$

Define

$$M_1(r,\theta) = \max\{\alpha_1, \alpha_2, \alpha_3\},$$
$$M_2(r,\theta) = \max\{\alpha_3, \alpha_4, \alpha_5\},$$
$$m_1(r,\theta) = \min\{\alpha_1, \alpha_2, \alpha_3\},$$
$$m_2(r,\theta) = \min\{\alpha_3, \alpha_4, \alpha_5\}.$$

Let $\mathscr{A}$ *be a four dimensional fusion algebra with basis* $\{x_0, x_1, x_2, x_3\}$ *such that* $x_1^* = x_1$ *and* $x_2^* = x_3$. *Let* L_i *be the matrices of multiplication by* x_i *with respect to the above basis. Then* $L_0 = I_4, L_3 = L_2^*$ *and* L_1 *and* L_2 *are necessarily among one of the following four types:*

Type I: L_1 *and* L_2 *as in Eq.* (4), $0 \leqslant \theta < \pi/2$ *and* $a \geqslant M_1(r,\theta)$,
Type II: L_1 *and* L_2 *as in Eq.* (4), $\pi \leqslant \theta < 3\pi/2$ *and* $a \leqslant m_2(r,\theta)$,
Type III: L_1 *and* L_2 *as in Eq.* (5), $\pi/2 < \theta < \pi$ *and* $a \leqslant m_1(r,\theta)$,
Type IV: L_1 *and* L_2 *as in Eq.* (5), $3\pi/2 < \theta < 2\pi$ *and* $a \geqslant M_2(r,\theta)$.

Proof. We have already seen that the matrices L_1 and L_2 for a four dimensional signed fusion algebra are of the form (4) or (5). For genuine fusion algebras, the entries have to be non-negative. This forces the above ranges where θ has to lie and the corresponding intervals for a. □

Acknowledgements

Most of this work was done at The Institute of Mathematical Sciences, Madras, India. I am thankful to Prof. V.S. Sunder for many helpful discussions and to the Institute for its hospitality.

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