Pramāṇa – J. Phys., Vol. 35, No. 2, August 1990, pp. 115–125. © Printed in India.

A new formalism for the statistical dynamics of planar spin systems

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MS received 2 March 1990; revised 11 May 1990

Abstract. The relaxational dynamics of a classical planar Heisenberg spin system is studied using the Fokker-Planck equation. A new approach is introduced in which we attempt to directly calculate the eigenvalues of the Fokker-Planck operator. In this connection a number space representation is introduced, which enables us to visualize the eigenvalue structure of the Fokker-Planck operator. The mean field approximation is derived and a systematic method to improve the mean field approximation is presented.

Keywords. Relaxational dynamics; planar Heisenberg model; Fokker-Planck equation; mean field approximation; critical slowing-down.

PACS Nos **05·70; 05·40; 75·40**

1. Introduction

The dynamical critical behaviour of the macroscopic systems is a well studied subject. Starting with the pioneering work of Van Hove (1954) and Landau and Khalatinkov (1954), the subject has now reached a mature stage, in which the advanced theoretical techniques like renormalization group have been brought to bear upon the problem (Hohenberg and Halperin 1977; Enz 1979). However, most of the recent theoretical developments have been confined to basically two kinds of models. The first kind are the so-called 'kinetic Ising models' while the second kind are a variety of field theoretic models, known more familiarly as time-dependent Landau-Ginzburg models. Though these models bring out very fascinating and germane physics, both these classes of models are either somewhat idealized abstractions of the physical systems they intend to describe, or are semi-microscopic approximations of the microscopic equations of motion (Kawasaki 1970). For example, in the kinetic Ising models the dynamics is described in terms of a master equation for the joint probability distribution of spins, as the spins as such have no dynamics of their own (Glauber 1963). While in Landau-Ginzburg models (Ma 1976) the order parameter field and the other relevant fields are allowed to assume values over an infinite range, and the basic nonlinearities of the problem, viz interaction between fields and restrictions on their range are simplified by taking them to be the simplest polynomials.

Given the complexity involved in the study of the dynamics of macroscopic systems it is not surprising that so much attention has been given to these models, which are

*To whom all correspondence should be addressed particularly amenable to treatment by renormalization group methods (Halperin *et al* 1972) and field-theoretic method (Martin *et al* 1973; De Dominicis *et al* 1975). However, the main rationale for the study of these models does not arise from their simplicity, but rather from the concept of universality, which has been rather firmly established for static critical phenomena. If one accepts the notion of the universality classes for the dynamical critical phenomena, it suffices to determine the behaviour, i.e. exponents, scaling functions etc. of the theoretically simplest member of each class (Hohenberg and Halperin 1977). It should, of course, be noted that the dynamical universality classes are not expected to be as wide as the universality classes for the static critical phenomena. There can be a number of models having different dynamic behaviours but with the same equilibrium properties. The universality classes for the dynamic behaviour are thought to depend upon the conservation principles obeyed by the equations of motion of the basic variables (conserved densities and order parameter field) involved, and the couplings amongst them determined by their poisson-bracket relations, in addition to factors that decide the static critical behaviour. Though the principle of universality classes for dynamic critical phenomena has received impressive justification through the use of renormalization group methods, these concepts have been tested only within the context of Landau-Ginzburg models. It is clear that since the macroscopic systems can have different kinds of dynamics and constraints, decided by their set of relevent vaiables, the dynamic behaviour is much richer and needs to be explored over a much wider class of models, before the universality principles can be firmly believed.

This has motivated us to study the statistical dynamics of the fixed-length vector spin systems. Such models, eg. classical Heisenberg model and planar spin model have been extensively studied in statistical mechanics, as the classical spins form rather good models for microscopic degrees of freedom in many situations. However, their relaxational dynamics have only been inferred from the study of their TDGL analogs. The purpose of the present paper is to develop a method to study the relaxational dynamics of a planar spin system. Our method is based on the Fokker-Planck equation (FPE) for the joint distribution function of the spins. The basic idea of our approach is as follows. The FPE is a linear equation, which can be generally written as

$$\partial P/\partial t = -\mathscr{L}P, \tag{1}$$

where P is the joint distribution function and $\mathscr{L}$ is a linear operator to be called the Fokker–Planck Operator (FPO). The essential problem of dynamics can be reduced to determining the eigenvalues of the operator. To do this, it is necessary to find an appropriate representation. We present here a natural representation, which allows a simple visualization of the nature of the eigenvalues and the eigenstates of the FPO of the planar spin system. Additionally, this representation allows for a systematic perturbation theory to be built around the mean field result of dynamics. In fact, the method described here closely parallels the methods in quantum mechanics. Our method is not just confined to planar spins, though it admittedly is the simplest case. In a subsequent paper, we shall extend the method to three dimensional spins.

The paper is organized as follows. In §2, we present the basic formalism of the Fokker-Planck equation for the planar spins. We show how a Heisenberg like picture for dynamics can be obtained and obtain convenient expressions for the correlation functions. In §3, expressions for the response functions are derived and the fluctuation-response relations are obtained in this formalism. Section 4 is devoted to developing a number space representation for the FPO and its adjoint. We further show how a mean field approximation can be very naturally made in this representation. Section 5 takes upon the task of developing systematic improvements over the mean field approximation. This section is of demonstrative nature, as the detailed analysis of the improved approximation and its physical consequences are not presented. We conclude with some remarks regarding the potential of the present work for further studies in critical dynamics.

2. Relaxational dynamics in the Fokker-Planck approach

We consider a d-dimensional lattice of N two-dimensional spins, which are unit vectors. The Hamiltonian of the systems can be written as

$$H=\tfrac{1}{2}I\sum_i \dot{\theta}_i^2-\tfrac{1}{2}\sum_{i,j} J_{ij}\cos(\theta_i-\theta_j), \tag{2}$$

where θ_i denotes the angle the spin vector at site i makes with a fixed axis in the plane, $\dot{\theta}_i$ is the time derivative of θ_i, I denotes the moment of inertia and J_{ij} is the exchange interaction between spins at sites i and j. The Langevin equation for the spins can be written as

$$I\ddot{\theta}_i+\eta\dot{\theta}_i=-\sum_j J_{ij}\sin(\theta_i-\theta_j)+f_i(t), \tag{3}$$

where $\eta\dot{\theta}_i$ denotes the viscous damping due to heat bath and $f_i(t)$ is the usual random force due to heat bath fluctuations. $f_i(t)$ is assumed to be Gaussian random process, characterized by zero mean and correlation function, given by

$$\langle f_i(t_1)f_j(t_2)\rangle=(2/\eta\beta)\delta_{ij}\delta(t_1-t_2), \tag{4}$$

where $\beta=1/k_BT$, and the brackets denote the average over the distribution of f_i. In this work we are only interested in the overdamped regime and at times t, such that $t\gg I/\eta$. Under these conditions the first term in (3) can be dropped and a Fokker-Planck equation for the time-dependent joint probability distribution of θ_i's, to be denoted by $P(\{\theta_i\},t)$ can be written to be (Edwards and Auderson 1976)

$$\frac{\partial P}{\partial t}=\frac{1}{\eta\beta}\sum_i\left[\frac{\partial}{\partial\theta_i}\left\{\sum_j \beta J_{ij}\sin(\theta_i-\theta_j)P\right\}+\frac{\partial^2 P}{\partial\theta_i^2}\right]. \tag{5}$$

Comparing it with (1), we note that the Fokker-Planck evolution operator $\mathscr{L}$ has the form

$$\mathscr{L}=-\frac{1}{\eta\beta}\left[\frac{1}{2}\sum_{i,j}\beta J_{ij}\left(\frac{\partial}{\partial\theta_i}-\frac{\partial}{\partial\theta_j}\right)\sin(\theta_i-\theta_j)+\sum_i\frac{\partial^2}{\partial\theta_i^2}\right]. \tag{6}$$

Equation (5) describes the evolution of $P(\{\theta_i\},t)$ as $t\to\infty$ towards the equilibrium distribution function $P_{eq}(\{\theta_i\})$ given by

$$P_{eq}(\{\theta_i\})=\frac{1}{Z}\exp\left[\frac{1}{2}\sum_{i,j}\beta J_{ij}\cos(\theta_i-\theta_j)\right] \tag{7}$$

where Z is the partition function of the system. For the purpose of defining correlation and response function, it is convenient to consider the conditional probability distribution $G(\{\theta_i\}, t; (\theta_i'\}, t_0)$ for $t > t_0$, which is the solution of (5) with the initial condition

$$G(\{\theta_i\}, t_0; \{\theta_i'\}, t_0) = \prod_i \delta(\theta_i - \theta_i'). \tag{8}$$

Formally, G can be written as

$$G(\{\theta_i\}, t; \{\theta_i'\}, t_0) = \exp(-(t-t_0)\mathscr{L}) \prod_i \delta(\theta_i - \theta_i'). \tag{9}$$

We now define a correlation function C for any two observables $A_1(\{\theta_i\})$ and $A_2(\{\theta_i\})$ by the following equation

$$C_{A_1,A_2}(t) = \int \prod_k \mathrm{d}\theta_k\, \mathrm{d}\theta_k' A_1(\{\theta_i\}) G(\{\theta_i\}, t; \{\theta_i'\}, 0) A_2(\{\theta_i'\}) P_{\mathrm{eq}}(\{\theta_i'\}). \tag{10}$$

We first note that for $t=0$, C reduces to equilibrium average

$$C_{A_1,A_2}(0) = \int \prod_k \mathrm{d}\theta_k A_1(\{\theta_i\}) A_2(\{\theta_i\}) P_{\mathrm{eq}}$$
$$= \langle A_1 A_2 \rangle_0 \tag{11}$$

where $\langle \quad \rangle_0$ denotes the equilibrium average as defined by the integral in (11). Further, noting the fact that $G \to P_{\mathrm{eq}}$ like any other solution of the FPE,

$$C_{A_1,A_2}(\infty) = \langle A_1 \rangle_0 \langle A_2 \rangle_0. \tag{12}$$

For the calculation of correlation functions, (10) is not very convenient. So we can recast it in a Heisenberg representation, in which the dynamics is transferred from the conditional probability distribution G to the operators. Using (9), the formal solution for G, we can write (10) as

$$C_{A_1,A_2}(t) = \int \prod_k \mathrm{d}\theta_k\, \mathrm{d}\theta_k' [\exp(-\mathscr{L}^\dagger t) A_1(\{\theta_i\})] \prod_i \delta(\theta_i - \theta_i')$$
$$\times A_2(\{\theta_i'\}) P_{\mathrm{eq}}(\{\theta_i'\}) \tag{13}$$
$$= \langle A_1(t) A_2 \rangle \tag{14}$$

where we define

$$A(t) = \exp(-\mathscr{L}^\dagger t) A \tag{15}$$

and $\mathscr{L}^\dagger$ denotes the adjoint of $\mathscr{L}$ over a space of functions periodic in θ_i's. $\mathscr{L}^\dagger$ given by

$$\mathscr{L}^\dagger = -\frac{1}{\eta\beta}\left[-\frac{1}{2}\sum_{i,j} \beta J_{ij} \sin(\theta_i - \theta_j)\left(\frac{\partial}{\partial\theta_i} - \frac{\partial}{\partial\theta_j}\right) + \sum_i \frac{\partial^2}{\partial\theta_i^2}\right]. \tag{16}$$

Further note

$$\langle A_1(t) A_2 \rangle = \langle A_1 A_2(t) \rangle \tag{17}$$

where (17) follows by noting that

$$\mathscr{L}(A_2P_{\text{eq}})=(\mathscr{L}^\dagger A_2)P_{\text{eq}} \tag{18}$$

which can be explicitly verified. In the following, we shall work with the Laplace transforms of the correlation functions defined as

$$\begin{aligned}\tilde{C}_{A_1,A_2}(z)&=\int_0^\infty \exp(izt)C_{A_1,A_2}(t)\,\mathrm{d}t\\&=\left\langle A_1\frac{1}{-iz+\mathscr{L}^\dagger}A_2\right\rangle.\end{aligned} \tag{19}$$

Equation (15) enables us to write the equation of motion of the correlation functions, given by

$$\begin{aligned}\frac{1}{\Gamma_0}\frac{\partial}{\partial t}C_{A_1,A_2}(t)=&\sum_i\left\langle\left(\frac{\partial^2}{\partial\theta_i^2}A_1(t)\right)A_2(0)\right\rangle\\&-\frac{1}{2}\sum_{i,j}\beta J_{ij}\left\langle\sin(\theta_i-\theta_j)\left(\frac{\partial A_1(t)}{\partial\theta_i}-\frac{\partial A_1(t)}{\partial\theta_j}\right)A_2(0)\right\rangle\end{aligned} \tag{20}$$

where $\Gamma_0=(\eta\beta)^{-1}$. Equation (20) is a basic equation which would be utilized to arrive at the various physical conclusions in this method. At this point, it may be useful to recall that the first term on the r.h.s. of (20) represents the action of diffusion caused by the heat bath, whereas the second term is the time change due to the Hamiltonian of the system. At $T=\infty(\beta=0)$, only the diffusion term survives.

3. Response functions and fluctuation-dissipation theorem

In this section, we study the linear response of the system to an external potential and thereby establish the form of the response functions and the fluctuation-dissipation theorem in the present formalism. We add to the Hamiltonian a time-dependent perturbation

$$V(t)=-\lambda V(\{\theta_i\})F(t). \tag{21}$$

Now the Fokker-Planck operator becomes

$$\mathscr{L}(t)=\mathscr{L}+\mathscr{L}_{\text{ext}}(t)$$

with

$$\mathscr{L}_{\text{ext}}P=-\frac{\lambda}{\eta\beta}\sum_i\frac{\partial}{\partial\theta_i}\left[\frac{\partial(\beta V)}{\partial\theta_i}P\right]F(t). \tag{22}$$

If we now assume that the perturbation is switched on adiabatically at $t=-\infty$, and $P(t=-\infty)=P_{\text{eq}}$, the solution of the FPE to the first order in $\mathscr{L}_{\text{ext}}$ is

$$P(\{\theta_i\},t)=\left[1-\int_{-\infty}^{t}\mathrm{d}t'\exp(-(t-t')\mathscr{L})\mathscr{L}_{\text{ext}}(t')\right]P_{\text{eq}}. \tag{23}$$

Substituting for $\mathscr{L}_{\text{ext}}$ yields

$$P(\{\theta_i\},t)=P_{\text{eq}}+\frac{\lambda}{\eta}\int_{-\infty}^{t}\mathrm{d}t'F(t')\exp(-(t-t')\mathscr{L})\sum_i\left[\frac{\partial^2 V}{\partial\theta_i^2}P_{\text{eq}}+\frac{\partial V}{\partial\theta_i}\frac{\partial P_{\text{eq}}}{\partial\theta_i}\right]$$

$$=\left[1-\beta\lambda\int_{-\infty}^{t}\mathrm{d}t'F(t')\exp(-(t-t')\mathscr{L})(\mathscr{L}^{\dagger}V)\right]P_{\text{eq}}. \tag{24}$$

Let us now consider the expectation value of an operator B at time t

$$\langle B(t)\rangle=\langle B\rangle_0-\beta\lambda\int_{-\infty}^{t}\mathrm{d}t'F(t')\langle B\exp(-(t-t')\mathscr{L})(\mathscr{L}^{\dagger}V)\rangle_0 \tag{25}$$

$$=\langle B\rangle_0-\beta\lambda\int_{-\infty}^{t}\mathrm{d}t'F(t')\langle B(t-t')\dot{V}\rangle_0 \tag{26}$$

(26) can be used to obtain an expression for the frequency dependent susceptibility, $\chi(\omega)$, to be

$$\chi(\omega)=-\beta\int_0^{\infty}\mathrm{d}\tau\exp(i(\omega+i\varepsilon)\tau)\langle B(\tau)\dot{V}\rangle_0. \tag{27}$$

This is the familiar fluctuation-response relation which shows how a response function is related to the correlation function. Another useful way to write (27) is

$$\chi(\omega)-\chi(0)=\beta\left\langle V\frac{i\omega}{-i\omega+\varepsilon+\mathscr{L}^{\dagger}}B\right\rangle_0. \tag{28}$$

4. Number space representation for $\mathscr{L}^{\dagger}$ and mean field theory

In order to evaluate the correlation and response functions discussed above, we need to find a convenient representation for $\mathscr{L}^{\dagger}$. Since all the functions of interest are periodic in θ in the interval $(0,2\pi)$, we construct a basis set in which at each site k, the basis is the set $\{\exp(in_k\theta_k)\}$ with n_k being an integer ranging from $-\infty$ to $+\infty$. Another advantage of this basis is that the diffusion part of $\mathscr{L}^{\dagger}$ is diagonal in this representation. A typical member of this basis is denoted as

$$\langle\theta_1,\ldots\theta_N|n_1\ldots.n_N\rangle=(2\pi)^{-N/2}\exp\left(\sum_k i\theta_k n_k\right). \tag{29}$$

Noting that

$$\frac{\partial}{\partial\theta_k}|n_1\ldots,n_k,\ldots\rangle=in_k|n_1,\ldots,n_k,\ldots\rangle. \tag{30}$$

and

$$\exp(\pm i\theta_k)|n_1,\ldots,n_k\ldots\rangle=|n_1,\ldots,n_k\pm1,\ldots\rangle. \tag{31}$$

it is straightforward to write the action of $\mathscr{L}^{\dagger}$ on a typical member of the basis. This is given by

$$\mathscr{L}^{\dagger}|\{n_k\}\rangle=\Gamma_0\Bigg[\sum_k n_k^2|\{n_k\}\rangle+\tfrac{1}{4}\sum_{k,j}\beta J_{kj}(n_k-n_j)$$
$$\times\{|\ldots n_k+1,\ldots n_j-1,\ldots\rangle-|\ldots n_k-1,\ldots n_j+1,\ldots\rangle\}\Bigg]. \tag{32}$$

This representation is not only mathematically convenient, but is also very helpful in a physical visualization of the action of $\mathscr{L}^{\dagger}$. First note that $\mathscr{L}^{\dagger}$ connects states only within a given subspace defined by $C=\sum_k n_k$ i.e. C is a conserved quantity of the above dynamics. This conservation principle is a reflection of the rotational invariance of the Hamiltonian, though note that the magnetization is not a conserved quantity in the above model. The conservation of C allows one to seek eigenvalues in a subspace of fixed C.

To obtain further physical insight, let us turn off the interaction term (i.e. $\beta=0$). Then we have a discrete set of diffusion excitations at each site. For example, for $C=1$ these excitations have eigenvalues 1, 3, 5, etc (in units of Γ_0). Thus for a lattice we have an N-fold degeneracy at each of these eigenvalues. When the interaction is switched on, the diffusion excitations at different sites are coupled. One result of this coupling is that a collective diffusion mode arises, which also results in the lifting of the N-fold degeneracy and formation of bands about each of the discrete diffusion eigenvalues. From this it is evident that for a given values of C the eigenvalue structure of $\mathscr{L}^{\dagger}$ consists of bands of excitations, which are well separated at high temperatures at least. As the temperature is lowered, the band widths increase being governed by βJ, allowing for the possibility of overlap. The present problem is not linear, as part of the coupling causes interactions between different bands of these modes. Nevertheless, from the point of view of kinetics, this feature is very useful, as it allows us to distinguish between fast and slow modes. For understanding the long time relaxation, one can integrate out the higher bands corresponding to the fast modes successively, thereby building a renormalization group scheme.

The above remarks are best illustrated by considering the action of $\mathscr{L}^{\dagger}$ on the simplest of $C=1$ state in which $n_j=\delta_{jk}$.

$$\mathscr{L}^{\dagger}|0,\ldots n_k=1,\ldots\rangle=\Gamma_0[|0,\ldots n_k=1,\ldots\rangle+\tfrac{1}{2}\sum_j\beta J_{kj}$$
$$\{|0,\ldots n_k=2,\ldots n_j=-1,\ldots\rangle-|0,\ldots n_k=0,\ldots n_j=1,\ldots\rangle\}]. \tag{33}$$

Now note that the second term has the diagonal expectation value, $K=\sum_k n_k^2=5$, which implies that its relaxation band is around $5\Gamma_0$. So at long times the second term may justifiably be dropped. The equation is now linear and the eigenvalues for a uniform system are given to be

$$r_q=\Gamma_0(1-\tfrac{1}{2}\beta\tilde{J}(q)) \tag{34}$$

where

$$\tilde{J}(q)=\sum_j J_{ij}\exp(i\boldsymbol{q}\cdot(\boldsymbol{r}_i-\boldsymbol{r}_j)) \tag{35}$$

Within this approximation, the value of the correlation function $C_{jk}=$

$\langle \cos(\theta_j(t)-\theta_k(0))\rangle$ is given by

$$C_{jk}=\mathrm{Re}\,\langle \exp(i\theta_j(t)-i\theta_k(0))\rangle$$
$$=\frac{1}{N}\mathrm{Re}\sum_q \exp(i\boldsymbol{q}\cdot(\boldsymbol{R}_j-\boldsymbol{R}_k)\exp[-\Gamma_0 t(1-\tfrac{1}{2}\beta\tilde{J}(q))]. \quad (36)$$

This is equivalent to the mean field approximation of Edwards and Anderson (1976) and also that of Suzuki and Kubo (1968) for the kinetic Ising model. We would like to emphasize that the present derivation provides its justification in a somewhat different way than the earlier work. Moreover, the range of validity of the approximation is clearly identified to be (i) high temperatures (ii) long times.

5. Beyond mean field theory

The main strength of the above formalism is that it enables us to improve upon the mean field approximation in a systematic way. There are various possibilities in this direction and here we present preliminary result of one approach. The idea behind this approach is to obtain successive approximations for the long time relaxation by including successively higher bands of relaxation rates. In other words, we want to incorporate the effects of coupling of higher bands on the lowest lying band, which is the one responsible for long time behaviour in a set of controlled approximations.

For this purpose, we define a set of Green's functions, the first one of which is

$$G_1(\boldsymbol{R}_i,t|\boldsymbol{R}_j)=\langle \exp[i\theta_i(t)]\exp[-i\theta_j(0)\rangle_0 \quad t>0$$
$$=0 \quad t<0. \quad (37)$$

We also define its spatial fourier transform and time-Laplace transforms through the following relations

$$G_1(\boldsymbol{q},t)=\sum_i \exp[i\boldsymbol{q}\cdot(\boldsymbol{R}_j-\boldsymbol{R}_i)G_1(R_i;t|R_j) \quad (38)$$

$$\tilde{G}_1(\boldsymbol{q};z)=\int_0^\infty \exp[izt]G_1(\boldsymbol{q},t)\,\mathrm{d}t, \quad \mathrm{Im}\,z>0. \quad (39)$$

The other Green's functions are defined generally by the following equations

$$G_{n_1,n_2},\ldots(\boldsymbol{R}_1,\boldsymbol{R}_2,\ldots;t|\boldsymbol{R}_j)=\left\langle \exp\left(i\sum_k n_k\theta_k(t)\right)\exp(-i\theta_j(0))\right\rangle \quad t>0$$
$$=0 \quad t<0 \quad (40)$$

$$G_{n_1,n_2},\ldots(\boldsymbol{q}_1,\boldsymbol{q}_2,\ldots;z)=\sum_{\boldsymbol{R}_1,\boldsymbol{R}_2\ldots}\int_0^\infty \mathrm{d}t\exp(izt)\exp\left[i\sum_k \boldsymbol{q}_k\cdot(\boldsymbol{R}_j-\boldsymbol{R}_k)\right]$$
$$G_{n_1,n_2},\ldots(\boldsymbol{R}_1,\boldsymbol{R}_2,\ldots;t|\boldsymbol{R}_j). \quad (41)$$

A further point about notation is that if any of the n's is negative, we denote it by putting a bar over it, i.e. $G_{2,-1}$ is written as $G_{2,\bar{1}}$. The equations of motion of G's also involve the equilibrium correlation functions, which we denote as

$$C_{n_1,n_2,\ldots}(\boldsymbol{R}_i,\boldsymbol{R}_j,\ldots)=\left\langle \exp\left(\sum_k i\theta_k n_k\right)\right\rangle_0 . \tag{42}$$

The fourier transform of the correlation functions are defined in the same way as in (38). We now write down the equation of motion for $G_1(\boldsymbol{R}_i;t|\boldsymbol{R}_j)$ for $t>0$.

$$\begin{aligned}\frac{\partial}{\partial t}G_1(\boldsymbol{R}_i;t|\boldsymbol{R}_j) &= -\langle(\mathscr{L}^\dagger \exp[i\theta_i(t))\exp[-i\theta_j(0)\rangle \\ &= -\Gamma_0[G_1(\boldsymbol{R}_i;t|\boldsymbol{R}_j)+\tfrac{1}{2}\sum_k \beta J_{ik} \\ &\quad \times\{G_{2,\bar{1}}(\boldsymbol{R}_i,\boldsymbol{R}_k;t|\boldsymbol{R}_j)-G_1(\boldsymbol{R}_k;t|\boldsymbol{R}_j)\}].\end{aligned} \tag{43}$$

Taking its Fourier and Laplace transforms yields

$$\tilde{G}_1(\boldsymbol{q},z)=\frac{1}{(-iz/\Gamma_0)+r_q}\left[C_{11}(\boldsymbol{q})-\frac{1}{2N}\sum_{\boldsymbol{q}_1}K(\boldsymbol{q}_1)\tilde{G}_{2,\bar{1}}(\boldsymbol{q}-\boldsymbol{q}_1,\boldsymbol{q}_1;z)\right] \tag{44}$$

where $K(\boldsymbol{q})=\beta\tilde{J}(\boldsymbol{q})$. To go beyond mean field theory, we now write the equation of motion for $G_{2,\bar{1}}$, which follows by considering the equation

$$\begin{aligned}&\Gamma_0^{-1}\mathscr{L}^\dagger|\ldots n_i=2,\ldots n_k=-1,\ldots\rangle=5|\ldots n_i=2,\ldots n_k=-1,\ldots\rangle \\ &+\tfrac{1}{2}\beta\Bigg[\sum_{l(\neq k)}2J_{il}\{|\ldots n_i=3,\ldots n_l=-1,\ldots n_k=-1,\ldots\rangle \\ &-|\ldots n_i=1,\ldots n_l=1,\ldots n_k=-1,\ldots\rangle\} \\ &+3J_{ik}\{|\ldots n_i=3,\ldots n_k=-2,\ldots\rangle-|\ldots n_i=1,\ldots\rangle\} \\ &-\sum_{l(\neq i)}J_{kl}\{|\ldots n_i=2,\ldots n_l=-1,\ldots\rangle \\ &-|\ldots n_i=2,\ldots n_k=-2,\ldots n_l=1,\ldots\rangle\}\Bigg].\end{aligned} \tag{45}$$

From this equation, it is clear that the equation of motion for $G_{2,\bar{1}}$ will introduce a number of higher order Green's functions, and to make things mathematically manageable, we must introduce a procedure to truncate these equations. Such a procedure can be based by generalizing the considerations discussed in the previous section. The different terms occurring here correspond to different bands whose mean position is approximately given by the diagonal expectation value K, of that term. The mean field approximation was obtained by retaining terms up to $K=1$. The next order approximation is obtained by keeping terms with $K\leqslant 5$. Retaining such terms reduces (45) to

$$\begin{aligned}&\Gamma_0^{-1}\mathscr{L}^\dagger|\ldots n_i=2,\ldots n_k=-1,\ldots\rangle=5|\ldots n_i=2,\ldots n_k=-1,\ldots\rangle \\ &-\tfrac{1}{2}\sum_{l\neq i}\beta J_{kl}|\ldots n_i=2,\ldots n_l=-1\rangle \\ &-\tfrac{1}{2}\sum_{l(\neq k)}2\beta J_{il}|\ldots n_i=1,\ldots n_l=1,\ldots n_k=-1,\ldots\rangle \\ &-\tfrac{3}{2}\beta J_{ik}|\ldots n_i=1,\ldots\rangle.\end{aligned} \tag{46}$$

Using this truncated equation, one obtains the following equation for $\tilde{G}_{2,1}(q_1, q_2; z)$

$$\tilde{G}_{2,\bar{1}}(\boldsymbol{q}_1, \boldsymbol{q}_2; z) = \frac{1}{(-iz/\Gamma_0)+4+r_q}\Big[C_{2,\bar{1},\bar{1}}(\boldsymbol{q}_1, \boldsymbol{q}_2)$$

$$+\tfrac{3}{2}K(\boldsymbol{q}_2)\tilde{G}_1(\boldsymbol{q}_1+\boldsymbol{q}_2; z)+\frac{1}{N}\sum_{q_3}K(\boldsymbol{q}_3)G_{1,\bar{1},1}(\boldsymbol{q}_1-\boldsymbol{q}_3, \boldsymbol{q}_2, \boldsymbol{q}_3; z)\Big]. \tag{47}$$

Next, in order to write the equation for $G_{1\bar{1}1}$ we consider the action of $\mathscr{L}^\dagger$ on such states. Omitting states with $K>5$, one has

$$\Gamma_0^{-1}\mathscr{L}^\dagger|\ldots n_i=1,\ldots n_l=-1,\ldots n_k=1\ldots)$$

$$=3|\ldots n_i=1,\ldots n_l=-1,\ldots n_k=1\ldots\rangle$$

$$-\tfrac{1}{2}\Big[\sum_{m\neq l,k}\{K_{im}|\ldots n_m=1,\ldots n_l=-1,\ldots n_k=1\ldots\rangle$$

$$+\sum_{m\neq i,k}K_{lm}|\ldots\ldots. n_i=1,\ldots n_m=-1,\ldots n_k=1,\ldots\rangle$$

$$+\sum_{m\neq i,l}K_{km}|\ldots n_i=1,\ldots n_l=-1,\ldots n_m=1\ldots\rangle\}\Big]$$

$$-K_{il}|\ldots n_k=1\ldots\rangle-K_{lk}|\ldots n_i=1\ldots\rangle.$$

Using (48), one obtains the following equation for $G_{1\bar{1}1}(\boldsymbol{q}_1, \boldsymbol{q}_2, \boldsymbol{q}_3; z)$

$$\tilde{G}_{1\bar{1}1}(q_1, q_2, q_3; z) = \frac{1}{(-iz/\Gamma_0)+r_{q_1}+r_{q_2}+r_{q_3}}[C_{1\bar{1}1\bar{1}}(q_1, q_2, q_3)]$$

$$+N\{\delta_{q_1+q_2,0}K(\boldsymbol{q}_1)\tilde{G}_1(\boldsymbol{q}_3; z)+\delta_{q_2+q_3,0}K(\boldsymbol{q}_2)\tilde{G}_1(\boldsymbol{q}_1, z)\}]. \tag{49}$$

We now see a closure is obtained within our scheme, which includes the relaxation rates of order $5\Gamma_0$. Clearly till the temperature is high enough to keep this band well removed from the lowest band, this should be an excellent approximation. Equation (49) can be substituted into (47) which in turn can be substituted into (44) to yield the following expression for $\tilde{G}_1(\boldsymbol{q}, z)$

$$\tilde{G}_1(\boldsymbol{q}, z) = \frac{A(\boldsymbol{q}, z)}{(-iz/\Gamma_0)+r_q+\Sigma(\boldsymbol{q}, z)} \tag{50}$$

with

$$\Sigma(\boldsymbol{q}, z) = +\frac{1}{N}\sum_{q_1}\Big\{\frac{3/4K^2(\boldsymbol{q}_1)}{(-iz/\Gamma_0)+4+r_{q_1}}$$

$$+\frac{K^3(\boldsymbol{q}_1)}{[(-iz/\Gamma_0)+4+r_{q_1})][(-iz/\Gamma_0)+r_q+2r_{q_1})]}\Big\} \tag{51}$$

and

$$A(q, z) = C_{1\bar{1}}(q) - \frac{1}{2N}\sum_{q_1}\frac{K(\boldsymbol{q}_1)C_{2\bar{1}\bar{1}}(\boldsymbol{q}-\boldsymbol{q}_1, \boldsymbol{q}_1)}{(-iz/\Gamma_0)+4+r_{q_1}} - \frac{1}{2N^2}$$

$$\times\sum_{q_1,q_3}\frac{K(\boldsymbol{q}_1)K(\boldsymbol{q}_3)C_{1\bar{1}1\bar{1}}(q_1, q_2, q_3)}{[(-iz/\Gamma_0)+4+r_{q_1})][[(-iz/\Gamma_0)+r_{q_1}+r_{q_3}+r_{q-q_1-q_3})]} \tag{52}$$

These equations have interesting implications which can be fully understood only by their numerical evaluation. However, we do not present such results here as in a later paper we intend to present a systematic study of the perturbation expansion and its consequences for the critical dynamics.

6. Concluding remarks

To summarize, we have introduced in this paper a new method to deal with the statistical dynamics of planar systems. Since the method is different from the usual field-theoretic methods, we expect that further studies by this method will lead to elucidation of the notion of universality in dynamic critical behaviour. The method also reveals interesting qualitative features about the hierarchy in the relaxation spectrum of planar spin systems. This hierarchical feature which we believe has been pointed out for the first time, should provide new methods to implement renormalization group ideas in the dynamic studies of the spin systems.

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