Arthurs–Kelly joint measurements and applications

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Originally devised as an extension of von Neumann measurement Hamiltonian to joint measurement of conjugate variables, the Arthurs–Kelly Hamiltonian has been found to have many other practical applications. I summarize in particular, experimental bounds on von Neumann entropy, noiseless quantum tracking of conjugate observables, remote tomography, entanglement swapping and exact measurement of correlation between conjugate observables.

Keywords: Conjugate variables, joint measurements, quantum tracking, remote tomography, teleportation.

Introduction

THE idea of 'quantum tracking' for a single observable first occurred in the measurement theory of von Neumann[1], and generalized to two canonically conjugate observables by Arthurs and Kelly Jr (refs 2 and 3). The nomenclature was probably used first by Arthurs and Goodman[4], who discovered the joint measurement uncertainty relation which implies that the minimum uncertainty product is twice that in Heisenberg's preparation uncertainty relation. Recently, several other applications of the Arthurs–Kelly interaction Hamiltonian have been discovered. They include an experimental bound on von Neumann entropy, noiseless quantum tracking of conjugate variables, remote tomography and exact measurements of correlations between conjugate observables such as position and momentum. I shall review these applications, many of which require initial states of the apparatus or tracker particles which are more general than those used by Arthurs and Kelly.

von Neumann and Arthurs–Kelly measurement theories

The initial values of a system observable are to be inferred from the final values of pointer observables of the apparatus after a measurement interaction with the system. To measure an observable of the system described by a Hilbert space $\mathbf{H}_1$, it is allowed to interact with an apparatus in a Hilbert space $\mathbf{H}_2$ for a time interval T. The system–apparatus states are in $\mathbf{H}_1 \otimes \mathbf{H}_2$ and have a unitary time evolution,

$$|\psi(T)\rangle = U(T)|\psi(0)\rangle, \text{ i.e. } |T\rangle = U(T)|0\rangle, \tag{1}$$

if we denote for brevity, $|\psi(T)\rangle = |T\rangle$. The system–apparatus initial state is assumed to be factorizable

$$|0\rangle = |\phi(0)\rangle|\chi(0)\rangle, \tag{2}$$

where ϕ, χ denote the system and apparatus states respectively. More generally, in case of impure states, the density operator $\rho(T)$ for the system–apparatus combine obeys

$$\rho(T) = U(T)\rho(0)U(T)^{\dagger}, \quad \rho(0) = \rho_1 \otimes \rho_2. \tag{3}$$

In the Heisenberg representation, observables have the time evolution

$$X(T) = U(T)^{\dagger} X U(T), \quad \mathrm{Tr}\rho(0)X(T) = \mathrm{Tr}\rho(T)X. \tag{4}$$

Tracking (Arthurs and Goodman[4]). The apparatus observable X at $t = T$ 'tracks' the system observable A at $t = 0$, if

$$\mathrm{Tr}\rho(T)X = \mathrm{Tr}\rho(0)A \Leftrightarrow \mathrm{Tr}\rho(0)(X(T) - A) = 0, \tag{5}$$

for all initial states ρ_1 of the system. Denoting averages of any operator in the initial state by an overline

$$\overline{X(T)} \equiv \mathrm{Tr}\rho(0)X(T),$$

we write,

$$\overline{X(T) - A} = 0, \text{ for every } \rho_1. \tag{6}$$

Noiseless tracking. If, in addition, X^2 at T tracks A^2 at $t = 0$, i.e.

$$\overline{(X^2(T) - A^2)} = 0,$$

then the r.m.s. deviations of the observables also agree

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$$\sigma_{X(T)} = \sigma_A, \tag{7}$$

and we say that the tracking is 'noiseless'. Let me introduce another definition.

Faithful tracking. The apparatus observable X at T tracks the system observable at $t = 0$ faithfully, if X and A have the same spectrum

$$(A - x)|x\rangle_1 = 0, (X - x)|x\rangle_2 = 0,$$

and if for all functions $f(A), f(X)$ defined by

$$f(A) = \Sigma_x f(x)|x\rangle_1\langle x|_1, \ f(X) = \Sigma_x f(x)|x\rangle_2\langle x|_2, \tag{8}$$

$$\mathrm{Tr}\rho(T)f(X) = \mathrm{Tr}\rho(0)f(A) = 0, \text{ for all } \rho_1.$$

Equivalently, the tracking is faithful if all projectors are tracked

$$\overline{(|x\rangle_2\langle x|_2\ (T) - |x\rangle_1\langle x|_1)} = 0, \text{ for all } \rho_1.$$

It is clear that if the tracking is faithful, it is also noiseless.

von Neumann measurement

von Neumann found an interaction, an initial apparatus state, and an apparatus observable X which at $t = T$ tracks any chosen system observable A at $t = 0$ faithfully. For $A = Q$, the system position operator, the von Neumann interaction Hamiltonian is,

$$H = KQP_1, \tag{9}$$

where P_1 is the pointer momentum operator and K is a constant so large that the free Hamiltonians can be neglected during the time T of interaction. The Schrödinger equation yields

$$\psi(q, x_1, T) = \phi(q)\chi(x_1 - KqT), \tag{10}$$

where x_1 is the pointer coordinate. If we choose

$$\chi(x_1) = \pi^{-1/4}b^{-1/2}\exp[-x_1^2/(2b^2)], \tag{11}$$

and $KT = 1$, we see that in the limit $b \to 0$, the centre of a narrow pointer wave packet shifts after the measurement by an amount which yields the system position q in the individual measurement.

The apparatus density operator is obtained by taking a trace of the system–apparatus density operator over the system coordinates. The diagonal elements are

$$\begin{aligned}[\rho_{\mathrm{APP}}(T)]_{x_1,x_1} &= \pi^{-1/2}b^{-1}\int \mathrm{d}q\,|\phi(q)|^2 \\ &\quad \times \exp(-(x_1 - q)^2/b^2) \\ &\quad \to |\phi(q = x_1)|^2, \text{ for } b \to 0,\end{aligned} \tag{12}$$

Thus, if $X_1(T)$, $(f(X_1))(T)$ denote Heisenberg operators, von Neumann obtains

$$\langle (f(X_1))(T) - f(Q)\rangle \to 0, \text{ for } b \to 0, \tag{13}$$

i.e. faithful tracking of position in the limit of narrow initial pointer states. However, the initial system density operator cannot be deduced from the final apparatus density operator, e.g. probability density of the initial system momentum cannot be deduced.

Arthurs–Kelly joint measurements of conjugate variables

The uncertainty principle does not allow joint measurements of conjugate variables to arbitrary accuracy. Arthurs and Kelly, and later Arthurs and Goodman extended von Neumann's idea to the best permissible joint measurements of conjugate observables and discovered a generalized Heisenberg uncertainty relation. Their idea is that the system interacts with an apparatus which has two commuting observables X_1, X_2 and approximate values of system position and momentum are extracted from accurate observation of X_1, X_2. The von Neumann–Arthurs–Kelly interaction during the time interval $(0, T)$ is

$$H = K(QP_1 + PP_2), \tag{14}$$

where K is a constant, with $KT = 1$ and the other symbols denote the respective operators. During interaction time, H is so strong that the free Hamiltonians of the system and apparatus are neglected. Arthurs and Kelly start with the system–apparatus initial state,

$$\psi(q, x_1, x_2, t = 0) = \phi(q)\chi_1(x_1)\chi_2(x_2), \tag{15}$$

where

$$\chi_1(x_1) = \pi^{-1/4}b^{-1/2}\exp(-x_1^2/(2b^2)), \tag{16}$$

$$\chi_2(x_2) = \pi^{-1/4}(2b)^{1/2}\exp((-2b^2x_2^2), \tag{17}$$

and $b/\sqrt{2}$ is the uncertainty of x_1 in the initial apparatus state. The uncertainty of x_2 is chosen as above for optimum results. The commutator of the two terms in H in fact commutes with each of the terms. Hence

$$\exp(-\mathrm{i}Ht) = \exp(-\mathrm{i}Ktqp_1)\exp(-\mathrm{i}Ktpp_2) \times \exp(-\mathrm{i}K^2t^2 p_1 p_2/2). \quad (18)$$

If we work in the q, x_1, p_2 representation, the three exponentials on the right-hand side successively translate x_1, q, x_1 acting on the initial wavefunction. Hence the exact solution of the Schrödinger equation is

$$\langle q, x_1, p_2 \mid t\rangle = \chi_1(x_1 - qKt + (1/2)p_2K^2t^2) \times \bar{\chi}_2(p_2)\phi(q - p_2Kt), \quad (19)$$

where $\bar{\chi}_2$ denotes a Fourier transform of χ_2.

The coordinate space wave function is given by a Fourier transform. The final joint probability density of the apparatus variables given by the diagonal elements in the x_1, x_2 representation of the final reduced density matrix ρ_A^{final} of the apparatus turns out to be just the Husimi function[2] of the initial system density operator $\rho_S = |\phi\rangle\langle\phi|$

$$P(x_1, x_2) = \langle x_1 x_2 \mid \rho_A^{\text{final}} \mid x_1, x_2\rangle = \frac{|\langle\phi_{b,x_1,x_2} \mid \phi\rangle|^2}{(2\pi)} = \frac{\langle\phi_{b,x_1,x_2} \mid \rho_{\text{S}} \mid \phi_{b,x_1,x_2}\rangle}{(2\pi)},$$

where

$$\langle q \mid \phi_{b,x_1,x_2}\rangle = \phi_{b,x_1,x_2}(q) = (2\pi)^{-1/4} \times b^{-1/2}\exp(\mathrm{i}qx_2 - (x_1 - q)^2/(4b^2)) \quad (20)$$

is a minimum uncertainty system state centred at $q = x_1$, $p = x_2$. These are normalized non-orthogonal states. Their completeness relation

$$\int \mathrm{d}x_1 \mathrm{d}x_2 \mid \phi_{b,x_1,x_2}\rangle\langle\phi_{b,x_1,x_2} \mid /(2\pi) = \mathbf{1}, \quad (21)$$

implies that,

$$\operatorname{Tr} A = \int \mathrm{d}x_1 \mathrm{d}x_2 \langle\phi_{b,x_1,x_2} \mid A \mid \phi_{b,x_1,x_2}\rangle/(2\pi). \quad (22)$$

We then obtain $\langle X_1\rangle = \langle Q\rangle$, and $\langle X_2\rangle = \langle P\rangle$, but the dispersions in X_1, X_2 are larger than for the system

$$(\Delta X_1)^2 = (\Delta Q)^2 + b^2,$$
$$(\Delta X_2)^2 < (\Delta P)^2 + \frac{1}{4b^2}. \quad (23)$$

By varying b, we obtain the 'measurement or noise' uncertainty relation (in units $\hbar = 1$),

$$\Delta X_1 \Delta X_2 \geq 1, \quad (24)$$

where the minimum uncertainty is twice the 'preparation uncertainty'. This was later proved by Arthurs and Goodman, as well as Gudder, Hagler and Stulpe to be independent of any particular measurement Hamiltonian, and a special case of the following theorem[4].

Theorem. If Heisenberg operators $R(T)$, $S(T)$ corresponding to commuting apparatus observables R, S track the non-commuting system observables A, B after interaction for time T, then

$$\sigma_R^2 \sigma_S^2 \geq |\overline{[A, B]}|^2, \quad (25)$$

where all expectation values are taken in the initial factorized state of the system–apparatus combine.

This fundamental uncertainty relation for simultaneous measurement of non-commuting observables is distinct from the more well-known preparation uncertainty relation and the minimum value on its right-hand side is four times the usual value. The extra uncertainty has been ascribed to inherent and unavoidable extra noise in joint quantum measurements.

We now discuss some other applications.

Bound on von Neumann entropy

I have noticed recently that an upper bound on the von Neumann entropy $S(\rho)$ of a system with density operator ρ can be obtained from Arthurs–Kelly joint measurements using the Wehrl entropy bound[5] in terms of coherent states and its generalizations using generalized coherent states[6]. Using the Arthurs–Kelly results for a pure initial state and the linearity of the Schrödinger equation, it follows that the relation

$$\langle x_1, x_2 \mid \rho_A^{\text{final}} \mid x_1, x_2\rangle = \langle\phi_{b,x_1,x_2} \mid \rho_S \mid \phi_{b,x_1,x_2}\rangle/(2\pi)$$

also holds for impure initial states ρ_S.

Recall first that the von Neumann density operator for any state ρ is,

$$S(\rho) = -\operatorname{Tr}\rho \ln \rho = \operatorname{Tr} f(\rho), \; f(x) \equiv -x \ln x, \quad (26)$$

and that for c-numbers $x \geq 0$, $f(x)$ is a concave function (i.e. $f''(x) < 0$). In the basis $|k\rangle$ of eigen functions of ρ, if $\rho = \sum_i \lambda_i \rho_i$, where $0 \leq \lambda_i \leq 1$, $\sum_i \lambda_i = 1$, then

$$S(\rho) = \sum_k f(\langle k \mid \rho \mid k\rangle) \geq \sum_k \sum_i \lambda_i f(\langle k \mid \rho_i \mid k\rangle)$$
$$\geq \sum_i \lambda_i \sum_k (\langle k \mid f(\rho_i) \mid k\rangle) = \sum_i \lambda_i S(\rho_i), \quad (27)$$

i.e. $S(\rho)$ is a concave function[5]. The concavity of $f(x)$ directly gives the first inequality above, and the second inequality follows from the lemma

$$f(\langle\phi\,|\,A\,|\,\phi\rangle) \geq \langle\phi\,|\,f(A)\,|\,\phi\rangle$$

valid for any concave function $f(A)$ of a self-adjoint operator A.

In the continuum case, the lemma also yields

$$S(\rho) = \int \mathrm{d}x_1 \mathrm{d}x_2 \langle\phi_{b,x_1,x_2}\,|\,f(\rho)\,|\,\phi_{b,x_1,x_2}\rangle/(2\pi\hbar)$$

$$\leq \int \mathrm{d}x_1 \mathrm{d}x_2 f(\rho^{\mathrm{cl}}(x_1,x_2))/(2\pi\hbar),$$

where

$$\rho^{\mathrm{cl}}(x_1,x_2) = \langle\phi_{b,x_1,x_2}\,|\,\rho\,|\,\phi_{b,x_1,x_2}\rangle, \quad (28)$$

which is the Wehrl bound on von-Neumann entropy. Notice now that $\langle\phi_{b,x_1,x_2}\,|\,\rho\,|\,\phi_{b,x_1,x_2}\rangle$ is just what is determined by the Arthurs–Kelly experiment if $\rho = \rho_S$. Hence the Arthurs–Kelly experiment yields an upper bound on the von Neumann entropy. I find that a Wehrl-type bound can also be derived in terms of generalized coherent states, and exploited in conjunction with Arthurs–Kelly experiments with such states as initial tracker states.

Bound on von Neumann entropy in terms of generalized coherent states

The generalized coherent states (see Roy and Singh[6]) are

$$|\,n,\alpha\rangle = U(\alpha)\,|\,n\rangle, \quad U(\alpha) = \exp(\alpha a^\dagger - \alpha^* a), \quad (29)$$

where a is the annihilation operator for the oscillator and α is a complex number, which is a linear combination of q_{cl} and p_{cl} as for the usual coherent states ($n = 0$)

$$\alpha = q_{\mathrm{cl}}\sqrt{M\omega/(2\hbar)} + \mathrm{i}p_{\mathrm{cl}}/\sqrt{2\hbar M\omega}, \quad (30)$$

(M = mass, ω = angular frequency of oscillator). The generalized coherent states are normalized over complete states, and obey the completeness relation

$$\int (\mathrm{d}^2\alpha/\pi)\,|\,n,\alpha\rangle\langle n,\alpha\,| = \mathbf{1}. \quad (31)$$

Using the concavity of the function ($-x\ln x$) and the above completeness relation, it follows as in the paper by Wehrl that the von Neumann entropy S obeys

$$S \leq -\int (\mathrm{d}^2\alpha/\pi)\langle n,\alpha\rangle\,|\,\rho\,|\,n,\alpha\rangle \ln(\langle n,\alpha\,|\,\rho\,|\,n,\alpha\rangle). \quad (32)$$

Note that

$$\mathrm{d}^2\alpha = \mathrm{d}(\mathrm{Re}\,\alpha)\mathrm{d}(\mathrm{Im}\,\alpha) = \mathrm{d}q_{\mathrm{cl}}\mathrm{d}p_{\mathrm{cl}}/(2\hbar),$$

and that the ground state $|0\rangle$ as well as the coherent states depend on the parameter $M\omega$. We may thus optimize the entropy bound for a given density operator ρ by varying the parameter $M\omega$ and the integer n. Different density operators may be tested and may need different values of n for an optimum result. Experimental evaluation of this bound requires Arthurs–Kelly experiments with generalized coherent states as initial tracker states.

Noiseless tracking, remote tomography and entanglement swapping via von Neumann–Arthurs–Kelly interaction

Before discussing these new results[7], it is useful to recall the usual teleportation protocol.

Teleportation (Figure 1) for discrete[8] and continuous variables[9], already realized experimentally[10,11], usually involves four steps. (i) An EPR-pair E_1, E_2 is shared by observers Alice (A) and Bob (B) at distant locations. (ii) The system particle P is received by A, who makes a Bell-state measurement on the joint state of that particle and E_1, and (iii) communicates the result via a classical channel to B, (iv) B then makes a unitary transformation depending on the classical information on E_2 to replicate the unknown system state.

We report here a method for remote tomography based on noiseless tracking which replaces the four technologies in usual teleportation by two steps (Figure 2): an interaction between the system particle and two apparatus particles, and quantum transmission of the apparatus particles to a remote location. At Alice's location A, a system particle P with unknown state interacts via an Arthurs–Kelly interaction with two apparatus particles A_1, A_2 in a known state. When the particles are photons, the interaction can easily be generated (see ref. 3). The particles A_1, A_2 are

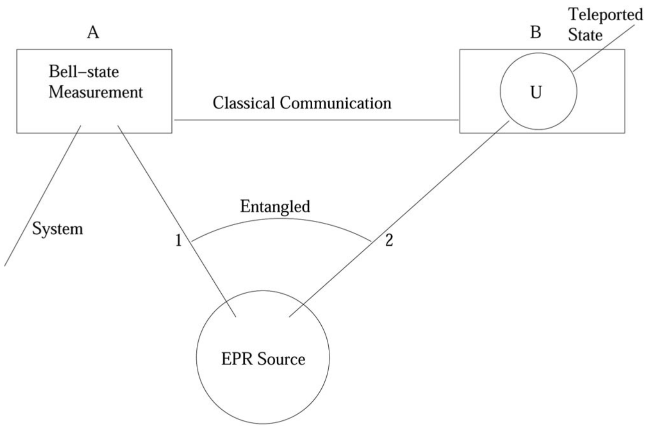

Figure 1. Usual protocol of the Bennett *et al.*[8] for teleportation.

then sent to a distant observer Bob (B). B makes quantum tomographic measurements on them (quadrature measurements in the case of photons) and reconstructs the exact initial density matrix of the system particle without ever having received that particle. Further, if another particle P' in Alice's hands is EPR-entangled with P, it will be EPR-entangled with the distant pair A_1, A_2. Practical implementation will require a quantum channel to send the two apparatus particles from location A to the distant location of B followed by tomographic measurements by B: for photons, a generalization of single photon optical homodyne tomography (see refs 12–14) to two photons, which seems feasible and worthwhile.

From the 'application point of view', why is it practically useful to transport the apparatus particles with the system state imprinted on them? Why can't Alice directly send the system particle to Bob? There can be several reasons. For example, the system particle might be unstable; or in the case of a photon, it might have a frequency unsuitable for optical fibre transmission. The apparatus photons can be chosen to have frequency in the telecom windows around 1300 or 1550 nm, where optical fibres have very low absorption facilitating long-distance transmission. The scheme we propose exploits the entanglement between the system photon and the apparatus photons generated by the three-particle Arthurs–Kelly interaction. Multiparticle interactions to generate entanglement have been previously exploited for quantum enhanced metrology[15].

A symmetry property

The Arthurs–Kelly system–apparatus interaction Hamiltonian is invariant under a class of simultaneous transformations on the system and apparatus specified below

$$H = K(\hat{q}\hat{p}_1 + \hat{p}\hat{p}_2) = K(\hat{q}_\theta \hat{p}_{1,\theta} + \hat{p}_\theta \hat{p}_{2,\theta}), \quad (33)$$

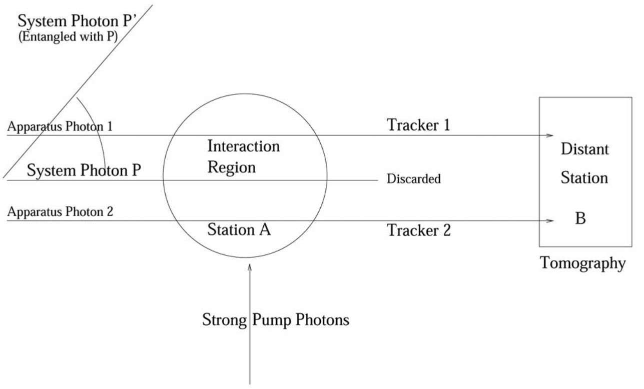

Figure 2. Remote tomography and entanglement swapping via von Neumann–Arthurs–Kelly interaction between system photon P and tracker photons. If the photon P' is EPR-entangled with P, the tracker photons become entangled with P'.

where the rotated quadrature operators with subscript θ are defined using the rotation matrix R

$$\begin{pmatrix} \hat{q}_\theta \\ \hat{p}_\theta \end{pmatrix} = R\begin{pmatrix} \hat{q} \\ \hat{p} \end{pmatrix}, \begin{pmatrix} \hat{p}_{1,\theta} \\ \hat{p}_{2,\theta} \end{pmatrix} = R\begin{pmatrix} \hat{p}_1 \\ \hat{p}_2 \end{pmatrix},$$

$$R = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}, \quad (34)$$

The operators $\hat{p}_{j,\theta}$ are seen to be just the commuting momentum operators of the apparatus particles corresponding to rotated coordinates $x_{j,\theta}$, for $j = 1, 2$

$$x_{1,\theta} + \mathrm{i}x_{2,\theta} = \exp(-\mathrm{i}\theta)(x_1 + \mathrm{i}x_2), \hat{p}_{j,\theta} = -\mathrm{i}\partial / \partial x_{j,\theta}. \quad (35)$$

We also define

$$\hat{x}_{1,\theta} + \mathrm{i}\hat{x}_{2,\theta} = \exp(-\mathrm{i}\theta)(\hat{x}_1 + \mathrm{i}\hat{x}_2). \quad (36)$$

Then, in the case of the apparatus being two photons with annihilation operators a_i, $i = 1, 2$

$$\hat{x}_{i,\theta} = a_i \exp(-\mathrm{i}\theta)/\sqrt{2} + \text{h.c.}, \hat{p}_{i,\theta} = \hat{x}_{i,\theta+\pi/2}. \quad (37)$$

The Arthurs–Goodman theorem on joint measurement mentions that noiseless tracking of non-commuting system observables is impossible using only commuting apparatus observables (such as the apparatus variables x_1, x_2) as trackers. We ask whether noiseless tracking is possible if we allow that the apparatus observables tracking the conjugate system observables may not commute. Let us generalize the initial apparatus states of Arthurs and Kelly and choose

$$\chi_1(x_1) = \pi^{-1/4} b_1^{-1/2} \exp[-x_1^2/(2b_1^2)] \quad (38)$$

and

$$\chi_2(x_2) = \pi^{-1/4}(2b)_2^{1/2} \exp[-2b_2^2 x_2^2)]. \quad (39)$$

The apparatus state contains two independent parameters b_1, b_2 whereas Arthurs and Kelly chose $b_2 = b_1 = b$. They obtained approximate q, p measurements by postulating their correspondence with the joint probability distribution of the commuting variables x_1, x_2 given by the diagonal elements of the reduced density matrix of the apparatus. This led to (i) arbitrarily accurate position probability density for $b \to 0$, (ii) arbitrarily accurate momentum probability density for $b \to \infty$ and for other values of b to approximate measurements of both subject to the joint measurement uncertainty relation.

In fact, we find non-commuting apparatus observables to do 'noiseless tracking' of all quadrature operators $\hat{q}_\theta$, provided that (i) we also exploit the off-diagonal elements of the apparatus density matrix, and (ii) move away drastically from the Arthurs–Kelly choice of parameters $b_2 = b_1 = b$ for the initial state of the apparatus and instead choose

$$b_2 b_1 = 1/2,\ b_1 \to 0,\ b_2 \to \infty, \tag{40}$$

which makes the initial state invariant under rotations in the x_1, x_2 plane.

Exact solution of the Schrödinger equation with the new initial conditions gives

$$\psi(q, x_1, x_2) = \int \psi(q, x_1, x_2, \xi)\, \mathrm{d}\xi, \tag{41}$$

where

$$\psi(q, x_1, x_2, \xi) = \frac{\phi(\xi)\exp(\mathrm{i}(q-\xi)x_2)}{2\pi\sqrt{b_1 b_2}} \times \exp\left(-\frac{(2x_1 - q - \xi)^2}{8b_1^2} - \frac{(q-\xi)^2}{8b_2^2}\right). \tag{42}$$

Tracing the system–apparatus density matrix over the system coordinate, we obtain the apparatus density matrix at time T

$$\langle x_1, x_2 \mid \rho_{\mathrm{APP}} \mid x_1' x_2' \rangle = \int \psi(q, x_1, x_2, \xi)\psi^*(q, x_1', x_2', \xi')\mathrm{d}q\mathrm{d}\xi\mathrm{d}\xi'. \tag{43}$$

Integration over x_2, x_2' gives

$$\int \langle x_1, x_2 \mid \rho_{\mathrm{APP}} \mid x_1' x_2' \rangle \mathrm{d}x_2 \mathrm{d}x_2' = \frac{1}{b_1 b_2}\int |\phi(q)|^2 \exp\left(-\frac{(x_1 - q)^2 + (x_1' - q)^2}{2b_1^2}\right)\mathrm{d}q. \tag{44}$$

This yields one of our key results. We can extract the exact initial system position probability density from the final apparatus density matrix as an expectation value of an apparatus observable.

$$|\phi(q = x_1)|^2 = \lim_{b_1\to 0}\frac{b_2}{\sqrt{\pi}}\int \mathrm{d}x_2 \mathrm{d}x_2' \langle x_1, x_2 \mid \rho_{\mathrm{APP}} \mid x_1 x_2' \rangle = \lim_{b_1 \to 0} \mathrm{Tr}\rho_{\mathrm{APP}} Y(x_1), \tag{45}$$

where $Y(x_1)$ is the apparatus observable

$$Y(x_1) = \frac{b_2}{\sqrt{\pi}} | x_1 \rangle\langle x_1 | \int | x_2' \rangle\langle x_2'' | \mathrm{d}x_2' \mathrm{d}x_2'' = 2b_2\sqrt{\pi}\, | x_1 \rangle\langle x_1 \| p_2 = 0 \rangle\langle p_2 = 0 |. \tag{46}$$

We have faithful tracking since $Y(x_1)$ exactly tracks the system position projector with $q = x_1$. Interestingly, $Y(x_1)$ equals the Arthurs–Kelly projector $|x_1\rangle\langle x_1|$ (involving only the first pointer), times an operator involving only the second pointer. Similarly, the exact initial system momentum probability density is an expectation value of an apparatus observable in the final apparatus density matrix

$$|\tilde{\phi}(p = x_2)|^2 = \lim_{b_2\to\infty}\frac{1}{2b_1\sqrt{\pi}}\int \mathrm{d}x_1 \mathrm{d}x_1' \langle x_1, x_2 \mid \rho_{\mathrm{APP}} \mid x_1' x_2 \rangle = \lim_{b_2\to\infty} \mathrm{Tr}\rho_{\mathrm{APP}} Z(x_2), \tag{47}$$

where $Z(x_2)$ is the apparatus observable

$$Z(x_2) = \frac{\sqrt{\pi}}{b_1} | x_2 \rangle\langle x_2 \mid p_1 = 0 \rangle\langle p_1 = 0 |. \tag{48}$$

Again, we have faithful tracking since $Z(x_2)$ exactly tracks the system momentum projector with $p = x_2$. $Z(x_2)$ equals the Arthurs–Kelly projector $|x_2\rangle\langle x_2|$ (involving only the second pointer) times an operator involving only the first pointer.

The Wigner function of the initial system state can also be reconstructed exactly from off-diagonal elements of the final apparatus density matrix

$$W(x, x_2) = \lim_{b_1\to 0, b_2\to\infty}\frac{b_2}{2\pi b_1} \times \int \mathrm{d}x_1' \mathrm{d}x_2' \langle x_1, x_2 \mid \rho_{\mathrm{APP}}(T) \mid x_1' x_2' \rangle. \tag{49}$$

Further, because of the noted symmetry property of the Hamiltonian and the invariance of the initial conditions under rotations in the x_1, x_2 plane, we can recover exactly not only the above q and p probability densities, but also the probaility densities of arbitrary Hermitian linear combinations $\hat{q}_\theta$ as expectation values of Hermitian operators in the final state of the apparatus after the interaction.

Thus, we obtain for arbitrary θ

$$|\langle \hat{q}_\theta = u \mid \phi \rangle|^2 = \lim_{b_1\to 0} \mathrm{Tr}\rho_{\mathrm{APP}}(T) Y_\theta(u), \tag{50}$$

$$Y_\theta(u) \equiv \frac{\sqrt{\pi}}{b_1} | \hat{x}_{1,\theta} = u \rangle\langle \hat{x}_{1,\theta} = u | \times | \hat{p}_{2,\theta} = 0 \rangle\langle \hat{p}_{2,\theta} = 0 |. \tag{51}$$

Since $\hat{p}_\theta = \hat{q}_{\theta+\pi/2}$, the initial system probability densities for it are obtained from above just by replacing $\theta \to \theta + \pi/2$. Since we recover exactly the initial system probability densities of arbitrary Hermitian linear combinations $\hat{q}_\theta$

$$\langle \hat{q}_\theta = u \,|\, \rho_S \,|\, \hat{q}_\theta = u \rangle = |\langle \hat{q}_\theta = u \,|\, \phi \rangle|^2, \quad (52)$$

we can also obtain the initial Wigner function in terms of these observables measured in the same final state of the apparatus.

Reconstruction of initial density matrix of the system from the final apparatus density matrix

Quantum tomography is completed by calculating the Wigner function $W(q, p)$ as an inverse Radon transform

$$W(q,p) = (2\pi)^{-2} \int_0^\infty \eta \mathrm{d}\eta \int_0^{2\pi} \mathrm{d}\theta \int_{-\infty}^\infty \mathrm{d}u$$
$$\times \exp(\mathrm{i}\eta(u - (q\cos\theta + p\sin\theta)))\langle \hat{q}_\theta = u \,|\, \rho_S \,|\, \hat{q}_\theta = u \rangle, \quad (53)$$

and from that the density operator

$$\langle q \,|\, \rho_S \,|\, q' \rangle = (2\pi)^{-1} \int_0^\pi |q - q'| \,\mathrm{d}\theta (\sin\theta)^{-2}$$
$$\times \exp((-\mathrm{i}(q^2 - q'^2)\cot\theta)/2) \int_{-\infty}^\infty \mathrm{d}u$$
$$\times \exp(\mathrm{i}u(q - q')/\sin\theta)\langle \hat{q}_\theta = u \,|\, \rho_S \,|\, \hat{q}_\theta = u \rangle. \quad (54)$$

Accounting for time evolution of the apparatus photons during transit time τ to distant location B

Note that

$$\mathrm{Tr}\rho_{\mathrm{APP}}(T) Y_\theta(u) = \mathrm{Tr}\rho_{\mathrm{APP}}(T + \tau)$$
$$\times \exp(-\mathrm{i}H_0\tau) Y_\theta(u) \exp(\mathrm{i}H_0\tau),$$

where the Hamiltonian

$$H_0 = \omega(a_1{}^\dagger a_1 + a_2{}^\dagger a_2 + 1),$$

if the photons have the same frequency ω. Hence the $\langle \hat{q}_\theta = u \,|\, \rho_S \,|\, \hat{q}_\theta = u \rangle$ are equivalently given by replacing

$$\rho_{APP}(T), \hat{x}_{1,\theta}, \hat{p}_{2,\theta}$$

by

$$\rho_{\mathrm{APP}}(T+\tau), \cos(\omega\tau)\hat{x}_{1,\theta} - \sin(\omega\tau)\hat{p}_{1,\theta},$$
$$\cos(\omega\tau)\hat{p}_{2,\theta} + \sin(\omega\tau)\hat{x}_{2,\theta}$$

respectively. We just have to measure different quadratures for the apparatus photons depending on the transit time τ.

Our exact theorems are for the limit $b_1 \to 0$. The purpose here is to estimate how small this parameter has to be for reasonably accurate reconstruction of the initial state which, in this example, is chosen to be the highly non-classical third excited state (Figure 3) of the oscillator. The wave function in the position basis is

$$\phi(q) = (2q^3 - 3q)\exp\left(-\frac{q^2}{2}\right) \bigg/ (\sqrt{3}\pi^{1/4}). \quad (55)$$

The Wigner function is a function of $q^2 + p^2 \equiv d$

$$W(d) = \exp(-d)[4d^3 - 18d^2 + 18d - 3]/(3\pi). \quad (56)$$

In Figure 4 we make quantitative comparisons between the Wigner function, our reconstructed Wigner function with $2b_1b_2 = 1$ (for $b_1 = \{0.1, 0.3\}$) and the Arthurs–Kelly probability distribution. It is worth noting that for $b_1 = \frac{1}{\sqrt{2}}$, the reconstructed Wigner function is equal to

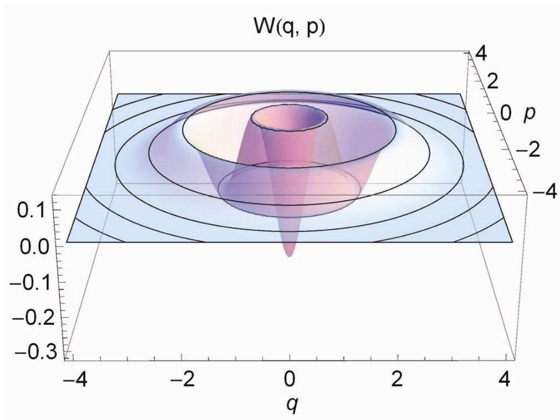

Figure 3. The Wigner function for the third excited state of the harmonic oscillator.

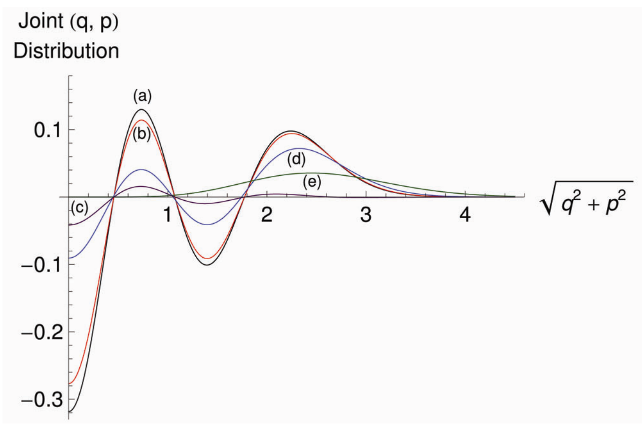

Figure 4. Joint distributions in (q, p) for the third excited state of the oscillator as a function of $\sqrt{q^2 + p^2}$: (a) Wigner function, (b) reconstructed Wigner function with $b_1 = 0.1$, (c) difference between curves (a) and (b), (d) reconstructed Wigner function with $b_1 = 0.3$, (e) Arthurs–Kelly probability distribution.

the Arthurs–Kelly distribution, which differs greatly from the true Wigner function. Towards practical utility, note that for $b_1 = .1$ the reconstructed Wigner function and the position probability derived from it are already very close to the actual.

A well-known measure of the distance between two probability distributions is given by the Kolmogorov–Smirnov distance, $D(K - S) = \max_x|F_1(x) - F_2(x)|$, where $F_i(x)$ is the cumulative probability for the variable $X \leq x$ for the ith probability distribution. This distance between the pseudo-probabilities given by the Wigner function and the reconstructed Wigner function, as well as for the corresponding position probabilities derived from them are plotted in Figures 5 and 6. The distance (especially for the position probability) is very small even up to $b_1 = 0.2$, though the theorem of exact equality is only in the limit $b_1 \to 0$.

Teleportation of entanglement

If the photon P with coordinate q is EPR-entangled with another photon P' with coordinate q' with initial wave function $\phi(q, q')$, the density matrix for particles 1, 2, P' after interaction can be shown to obey analogues of eqs (14) and (15) with $\langle q = x_1|\phi\rangle$ replaced by $\langle q = x_1, q'|\phi\rangle$, and $Y(x_1)$ replaced by $Y(x_1)|q'\rangle\langle q'|$

$$|\langle q = x_1, q' | \phi\rangle|^2 = \lim_{b_1 \to 0} \mathrm{Tr}\rho_{\mathrm{APP}}(T)Y(x_1)|q'\rangle\langle q'|.$$

Thus the apparatus photons after interaction with P become entangled with P' achieving interaction-based teleportation of EPR-entanglement. The exact initial probability densities for q, q' (and similarly for p, p'), i.e. the exact EPR-correlations can be retrieved from this final entangled state.

Remark. The theorem on impossibility of simultaneous noiseless tracking of position and momentum by commuting apparatus observables is not violated, but circumvented.

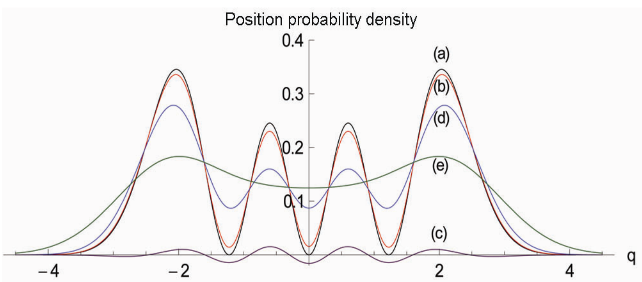

Figure 5. Position probability densities for the third excited state: (a) Quantum probability density of the state, (b) obtained from reconstructed Wigner function with $b_1 = 0.1$, (c) difference between curves (a) and (b), (d) obtained from reconstructed Wigner function with $b_1 = 0.3$ and (e) obtained from Arthurs–Kelly probability distribution.

The secret of success is that, although we have the same final state, the tracking observables do not commute

$$|Y(x_1), Z(x_2)] = \frac{b_2}{b_1}(|x_1\rangle\langle x_2 || p_2 = 0\rangle\langle p_1 = 0| - |x_2\rangle\langle x_1 || p_1 = 0\rangle\langle p_2 = 0|) \neq 0. \quad (57)$$

Exact quantum correlations of conjugate observables from Arthurs–Kelly measurements

Correlations between conjugate observables, being rather different from Bell-type correlations among commuting observables, are a largely unexplored area with possible fundamental importance. I present a method for the exact measurement of local and global correlations between conjugate observables in quantum mechanics[16].

We noted that the exact position and momentum probability densities of the system are recovered by the usual Arthurs–Kelly measurement ($b_1 = b_2 = b$) in the limits $b \to 0$ and $b \to \infty$ respectively, i.e. in two experiments with very different initial apparatus states. It is a pleasant surprise that the joint measurement can nevertheless give local and global correlations between $\hat{q}$ and $\hat{p}$ exactly. We define the conditional expectation values of momentum at a given position and of position at a given momentum as expectation values of self-adjoint operators

$$\langle\hat{p}\rangle(q) \equiv \frac{\langle\Lambda(q)\hat{p} + \hat{p}\Lambda(q)\rangle}{2\langle\Lambda(q)\rangle};\ \langle\hat{q}\rangle(p) \equiv \frac{\langle\Lambda(p)\hat{q} + \hat{q}\Lambda(p)\rangle}{2\langle\Lambda(p)\rangle}, \quad (58)$$

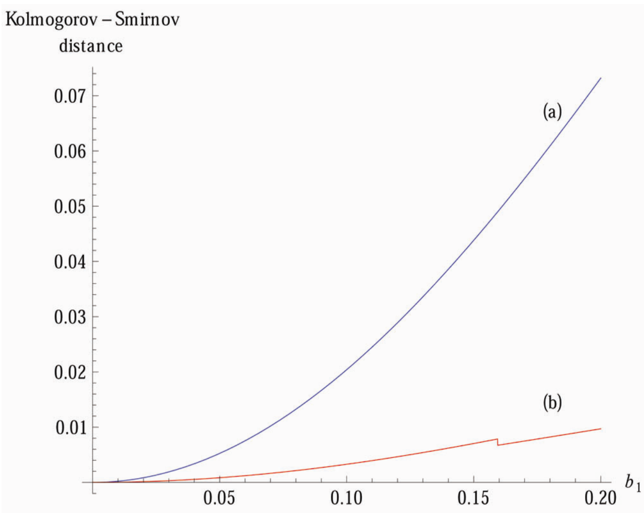

Figure 6. Plots for the Kolmogorov–Smirnov (K–S) distance between (a) the Wigner function and the reconstructed Wigner function and (b) the position probability density and the reconstructed density versus b_1. Even when b_1 is as large as 0.2, the K–S distance in case (a) reaches a value of only 0.072. The agreement is even better in case (b) (the small discontinuity in the K–S distance at $b_1 = 0.16$ is due to the shifting of the position where the maximum K–S distance is reached).

where $\langle A \rangle$ denotes the quantum expectation value of a self-adjoint operator A, and the projection operators $\Lambda(q)$, $\Lambda(p)$ are defined by,

$$\Lambda(q) = |q\rangle\langle q|, \ \Lambda(p) = |p\rangle\langle p|. \tag{59}$$

For a pure state $|\phi\rangle$, we have the explicit expressions

$$\langle \hat{p} \rangle(q) = \frac{\mathrm{Re}(\phi^*(q)(-i)\partial\phi(q)/\partial q)}{|\phi(q)|^2}, \tag{60}$$

$$\langle \hat{q} \rangle(p) = \frac{\mathrm{Re}(\tilde{\phi}^*(p)(i)\partial\tilde{\phi}(p)/\partial p)}{|\tilde{\phi}(p)|^2}. \tag{61}$$

We shall see that the local correlations $\langle \hat{p}\rangle(q) - \langle \hat{p}\rangle$ and $\langle \hat{q}\rangle(p) - \langle \hat{q}\rangle$ can be measured exactly for arbitrary q and p respectively, for appropriate values of b. The global correlation $\langle \hat{q}\hat{p} + \hat{p}\hat{q}\rangle - 2\langle \hat{q}\rangle\langle \hat{p}\rangle$ is in fact exactly measurable for any value of b.

For the Arthurs–Kelly measurement we define as for a classical distribution

$$\langle x_2 \rangle_{A-K}(x_1) \equiv \int x_2 P(x_1, x_2)\mathrm{d}x_2 / P_1(x_1), \tag{62}$$

$$\langle x_1 \rangle_{A-K}(x_2) \equiv \int x_1 P(x_1, x_2)\mathrm{d}x_1 / P_2(x_2), \tag{63}$$

$$\langle x_1 x_2 \rangle_{A-K} \equiv \int x_1 x_2 P(x_1, x_2)\mathrm{d}x_1\mathrm{d}x_2. \tag{64}$$

Substituting the value of $P(x_1, x_2)$, and doing the integral over x_2 we obtain

$$\int x_2 P(x_1, x_2)\mathrm{d}x_2 = (b\sqrt{2\pi})^{-1}\int \mathrm{d}q\mathrm{d}q'\phi(q)\phi^*(q')$$

$$\times \exp\left(-\frac{(x_1-q)^2 + (x_1-q')^2}{4b^2}\right) i\frac{\partial\delta(q-q')}{\partial q}$$

$$= \mathrm{Re}\int \frac{\mathrm{d}q}{b\sqrt{2\pi}} \exp\left(-\frac{(x_1-q)^2}{2b^2}\right)\phi^*(q)(-i)\frac{\partial\phi(q)}{\partial q}, \tag{65}$$

where $\delta(q - q')$ is the Dirac delta function. Similarly, we obtain

$$\int x_1 P(x_1, x_2)\mathrm{d}x_1$$

$$= b\sqrt{2\pi}\ \mathrm{Re}\int \mathrm{d}p \exp(-2b^2(x_2 - p)^2)\tilde{\phi}^*(p) i\frac{\partial\tilde{\phi}(p)}{\partial p}. \tag{66}$$

Taking the limits of b going to 0 and ∞ yield respectively

$$\langle x_2 \rangle_{A-K}(x_1) \to_{b\to 0} \langle \hat{p}\rangle(q = x_1), \tag{67}$$

$$\langle x_1 \rangle_{A-K}(x_2) \to_{b\to\infty} \langle \hat{q}\rangle(p = x_2). \tag{68}$$

Thus we have proved that the quantum position probability density and the local correlation $\langle \hat{p}\rangle(q) - \langle \hat{p}\rangle$ can be measured exactly with the initial condition $b \to 0$; the quantum momentum probability density and the local correlation $\langle \hat{q}\rangle(p) - \langle \hat{q}\rangle$ can be measured exactly with the very different initial condition $b \to \infty$. A similar calculation shows that for any value of b

$$\langle 2x_1x_2 \rangle_{A-K} = \langle \hat{q}\hat{p} + \hat{p}\hat{q}\rangle, \tag{69}$$

the global correlation is exactly measured in the Arthurs–Kelly measurement. Thus, the Arthurs–Kelly measurements with $b \to 0$ and $b \to \infty$ equip us with exact probability densities of position and momentum as well as their exact local and global correlations.

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