THE METRIC IN PATH-SPACE.

By D. D. Kosambi.

The paths defined by a suitably differentiable symmetric affine connection have the unique join property: that one and only one arc which is a path connects any two points within a sufficiently restricted n-dimensional neighbourhood. It can easily be shown that this property furnishes a map of the neighbourhood upon an open sphere in Euclidean R_n. Therefore, the path-space, according to a well-known topological result[1], may be endowed with a Riemann metric. But the relation of this metric to the paths is not clear. It is known that differentiability, no matter of how high an order, does not suffice by itself to make the path-equations equivalent to the Euler equations of a regular variational problem, let alone of a Riemann metric. Thus there is a gap between the global approach based upon the paths as curves, and the local which deals with their differential equations. This difference is reconciled by our main result:

Theorem 1. *By a suitable projective change of the (implicit) parameter t, the paths of a symmetric affine connection may be made the geodesics of a Riemann metric.*

1. The paths of our discussion are defined by

$$\ddot{x}^i + \Gamma^i_{jk}\dot{x}^j\dot{x}^k = 0, \quad i = 1, 2, \cdots, n, \quad \dot{x}^i = dx^i/dt, \tag{1.1}$$
$$\ddot{x}^i = d\dot{x}^i/dt, \quad \Gamma^i_{jk} = \Gamma^i_{kj}.$$

We assume these equations to be tensor invariant, which means that the coefficients of connection $\Gamma^i_{jk}(x)$ are endowed with a suitable law of transformation. It is known that the Euler equations of a variational problem are also tensor invariant (though covariant where (1.1) are contravariant) hence nothing can be done by mere change of coordinates towards obtaining a metric. If a metric is broken up into a sum of homogeneous polynomials of different degrees in x—which includes the case of a formal series expansion—the symmetric tensors furnished by coefficients of any given degree must have a vanishing covariant de-

Received by the editor July 18, 1953.

1) This is the theorem of Whitney (*Ann. Math.* **37** (1936), 645–680); for its application to group-spaces, see *Quart. J. Math.* (*Oxford*, 2), **3** (1952), 307–320, where most of the results of the present note have been outlined.

rivative with respect to the Γ's[2]. In particular, for a Riemann metric, we must have a tensor $g_{ij}=g_{ji}$ of non-vanishing determinant $g=|g_{ij}|$ such that the quadratic form $g_{ij}p^ip^j$ is definite in p (hence positive definite, by change of sign if necessary). Finally, we obtain the system of first order linear partial differential equations

$$(1.2)\qquad g_{ij|k}\equiv g_{ij,k}-g_{ir}\Gamma^r_{jk}-g_{rj}\Gamma^r_{ik}=0\ ,$$

where the vertical bar and comma in subscript denote covariant and simple partial differentiation respectively. These equations have the system of compatibility conditions[3]:

$$(1.3)\qquad \begin{aligned} &g_{ir}R^r{}_{jkl}+g_{rj}R^r{}_{ikl}=0\ ,\\ &g_{ir}R^r{}_{jkl|m}+g_{rj}R^r{}_{ikl|m}=0\ ,\cdots, \end{aligned}$$

where each set of equations is derived by partial differentiation of the preceding, and the curvature tensor $R^i{}_{jkl}$ has the formula

$$(1.4)\qquad R^i{}_{jkl}=\Gamma^i{}_{jk,l}-\Gamma^i{}_{jl,k}+\Gamma^i{}_{rl}\Gamma^r{}_{jk}-\Gamma^i{}_{rk}\Gamma^r{}_{jl}\ .$$

The problem is thus reduced to one of elementary algebra, in that a system of homogeneous linear equations for the unknowns g_{ij} is found explicitly; this must reduce to a finite number of independent equations, at least one less than the number of g_{ij}. Thus, for the most general case, no metric exists in this sense.

From the first of (1.3) follows a purely affine condition not involving g_{ij}, if we contract with the normalized cofactors g^{ij}:

$$(1.5)\qquad R^r{}_{rkl}=0\ .$$

This is only a necessary condition, for the first of (1.3) says that a certain number of matrices indicated by fixed values of indices k, l must be antisymmetric, whereas (1.5) is equivalent to the vanishing of corresponding traces.

By projective change of parameter, we mean replacing dt by (exp $\int\psi_i dx^i)dt$, which gives

$$(1.6)\qquad \Gamma^i{}_{jk}\longrightarrow\Gamma^i{}_{jk}+\frac{1}{2}\left(\delta^i{}_j\psi_k+\delta^i{}_k\psi_j\right).$$

Thus the general projective change of parameter depends upon the arc, except when ψ_i is a gradient vector, the integral then being independent of the curve of integration—though even here the unique

2) This result, which was undoubtedly known to the classical differential geometers, does not seem to appear in the available literature before *Quart. J. Math.* 3 (1952), p. 6.

3) L. P. Eisenhart, *Non-Riemannian Geometry* (Amer. Math. Soc. Coll. Publ. 8, 1927), sections 29, 30.

join property makes it a function of the path. One advantage of this projective change is that the contraction in (1.5) may thereby be made to vanish[4]. For it is easy to show that $R^r_{\ rjk}$, being antisymmetric, with a vanishing exterior derivative (by contraction of the Bianchi identities) is a curl. Thereafter, ψ_i in (1.6) is restricted to be a gradient vector (*i.e.* of vanishing curl). Thus our main theorem would be reduced to proving that *a metric always exists if* $R^r_{\ rkl}=0$ *and the paths are modified as in* (1.6) *by a gradient vector* ψ_i .

If proved, we would still have only an existence theorem, indefinite in the sense that the problem would again reduce to one of algebra. There is nothing in the method itself to make the tensor g_{ij} properly Riemannian; this depends upon initial values, which being suitably chosen, the metric form will be positive definite at the starting point, and by continuity for some neighbourhood thereof.

2. Rather than attempt a (necessarily clumsy) proof of the local theorem, we give the actual specification of the parameter, which simultaneously proves results otherwise derived from topology. If a new *implicit* parameter be chosen along the paths in terms of which each path has differentiable coordinates, the vector ψ_i of formula (1.6) is obviously determined by the choice, so that the proof of Theorem 1 will be furnished by the construction. Starting from a general point x_0, in any direction ξ, we get to a point x on the path by the formal series expansion:

$$x^i(x_0,t)\equiv x^i=x_0^i+t\xi^i-\frac{t^2}{2!}\Gamma^i_{\ jk}|_0\xi^j\xi^k-\frac{t^3}{3!}\frac{d}{dt}(\Gamma^i_{\ jk}\dot{x}^j\dot{x}^k)|_0+\cdots, \tag{2.1}$$

where

$$d/dt\equiv\dot{x}^r\frac{\partial}{\partial x^r}-(\Gamma^r_{\ jk}\dot{x}^j\dot{x}^k)\frac{\partial}{\partial\dot{x}^r}, \tag{2.2}$$

and the zero in subscript indicates replacement of x by x_0, $\dot{x}$ by ξ after the differentiation is finished. To every point distinct from x_0 corresponds just one direction ξ provided we sharpen the concept of direction and restrict ξ by

$$\delta_{ij}\xi^i\xi^j\equiv\sum(\xi^i)^2=1\,,\quad \delta_{ij}=\begin{cases}0, & i\neq j\\ 1, & i=j\,.\end{cases} \tag{2.3}$$

This has the important consequence of relating values of t along different directions, ensuring continuity in the functional relationship between t and x not only along any fixed path but as x is allowed to vary con-

4) Eisenhart, *loc. cit.* 32; J. A. Schouten, *Der Ricci-Kalkül*, p. 129.

tinuously in any direction. On the path, the formal expansion gives us a translation in t, i.e. an Abelian one-parameter group represented by the Lie-Taylor series. For questions of convergence, we note that the problem does not arise in the analytic case, while any Γ may be approximated by analytic connection coefficients. The limiting process would not affect the translation property of the parameter t. For that matter, we may replace the series by a finite expansion with remainder.

For the mapping *onto* R_n, we take new coordinates $y^i = t\xi^i$. This gives for $x(x_0, y)$ in (2.1),

$$x^i = x_0^i + y^i - \frac{1}{2!}\Gamma^i{}_{jk}|_0 y^j y^k - \frac{1}{3!}c^i{}_{jkl}|_0 y^j y^k y^l + \cdots . \tag{2.4}$$

There is then a 1–1 continuous (and differentiable) mapping of an x-neighbourhood onto the open sphere $\sum y^2 < t^2$. This holds provided there is no point on any path through x_0 which is conjugate[5] to x_0, the Jacobian $|\partial x^i/\partial y^j|$ then having the value unity at the origin, and not vanishing throughout the domain under consideration. That is, we have *connectedness, arcwise connectedness* (with path-arcs), and *orientability* in the neighbourhood mapped. The path-space is assumed to be covered by such overlapping neighbourhoods, or we restrict the argument to a connected component, which is so covered.

Though (2.4) makes all paths through x_0 into straight lines through the origin, nothing is said about any of the remaining paths of the neighbourhood. We have at any point just one path in a given direction. In R_n, the whole direction bundle at any point is to be obtained by parallel transfer of that at the origin. This is impossible for spaces which are not flat. However, the conditions are that 1) the tangent direction at any point on a path should correspond to that at any other on the same path; 2) the relation $\sum \xi^2 = 1$ should remain quadratic homogeneous in the local direction system: hence that *the correspondence between two local direction bundles should be linear.* This determines the structure of the fibre-space $(x, \dot{x})$, and solves the mapping problem.

Differentiating (2.1) with respect to the parameter t, we have

$$\begin{aligned} \dot{x}^i &= \xi^i - t\Gamma^i{}_{jk}|_0 \xi^j \xi^k - \frac{t^2}{2!}c^i{}_{jkl}|_0 \xi^j \xi^k \xi^l + \cdots \\ &= \xi^j (\delta^i{}_j - t\Gamma^i{}_{jk}|_0 \xi^k - \cdots) . \end{aligned} \tag{2.5}$$

That is, the correspondence for tangents at different points on the same

5) M. Morse: *Calculus of Variations in the Large* (Amer. Math. Soc. Coll. Publ. 18, 1934) particularly chapter V. The present note may therefore be regarded as a contribution to the affine and projective calculus of variations.

paths has the linear form

$$(2.6)\qquad \dot{x}^i = \frac{\partial x^i}{\partial y^j}\xi^j, \qquad\qquad \xi^i = \frac{\partial y^i}{\partial x^j}\dot{x}^j.$$

Our conditions above are satisfied by taking (2.6) as the general relationship between ξ^i at the origin and $\dot{x}^i$ at any point $x(y)$. We now note that t has been fixed as the parameter for all paths through the initial point, both for the x- and the y-space. Therefore there can be no other restriction upon $\dot{x}$ than that derived from $\sum\xi^2=1$. This leads to

Theorem 2. *The correspondence between two local direction bundles being given by* (2.6), *the parameter t is determined on every path by*

$$(2.7)\qquad \begin{aligned} dt^2 &= g_{ij}dx^i dx^j, \\ g_{ij} &= \delta_{rs}\frac{\partial y^r}{\partial x^i}\frac{\partial y^s}{\partial x^j} = \sum_r \frac{\partial y^r}{\partial x^i}\frac{\partial y^r}{\partial x^j}. \end{aligned}$$

With this metric tensor g_{ij} for the space, the paths become geodesics, t the arc distance. The relation (2.6) *is also necessary and sufficient to map all paths into straight lines of the y-space with the same t-values for corresponding arc-segments.*

Equations (2.7) are the local reflection of $\sum\xi^2=1$. It is clear that g_{ij} so defined gives a positive definite fundamental form. We may note that at the origin, directions in y are identical with those in x coordinates, $\|\partial x^i/\partial y^j\|$ becoming the unit matrix at O. Thus (2.6) may be restated in terms of the transformation between one local coordinate system and another instead of between the local system and the map onto R_n. We have only to prove the last part of the theorem for if the correspondence be set up, it will follow that t on any regular smooth arc in x-space is measured by the integral of dt along its image in y-space; further, that the paths, having straight lines as images, are lines of shortest distance.

Suppose now that $y=u+v$ where u^i, v^j are arbitrary displacement vectors in R_n. Our problem reduces to showing what is in effect the Weierstrass strong-variation condition, namely that $x(y)\equiv x(u+v)=x(x(u),\bar{v})=x(x(v),\bar{u})$. In this derivation, parameters and directions in y-space are determined, say, by $y=t\xi$, $u=t_1\mu$, $v=t_2\nu$. In the x-space, we have the same parametric values, but some correspondence between directions will then have to be determined, so that $\bar{v}$ depends upon $x(u)$, or $\bar{u}$ upon $x(v)$. This is obvious because the direction $\bar{v}$ necessary to reach the point $x(u+v)$ from $x(u)$ is not in general the same at $x(u)$ as the direction v needed to reach $y=u+v$ from the point $y=u$ in the y-space. We now expand in formal power series and demand term by term equality (taking $x_0=0$ without loss of generality).

$$(2.8)\quad \begin{aligned} x^i(y) \equiv x^i(u+v) &= u^i+v^i-\frac{1}{2!}\Gamma^i{}_{jk}|_0(u^j+v^j)(u^k+v^k)+\cdots \\ &= x^i(u)+\bar{v}^i-\frac{1}{2!}\Gamma^i{}_{jk}(x(u))\,\bar{v}^j\bar{v}^k+\cdots, \end{aligned}$$

with

$$x^i(u)=u^i-\frac{1}{2!}\Gamma^i{}_{jk}|_0 u^j u^k+\cdots$$

and

$$\Gamma^i{}_{jk}(x(u))=\Gamma^i{}_{jk}|_0+\Gamma^i{}_{jk,l}|_0 x^l(u)+\frac{1}{2!}\Gamma^i{}_{jk,l,m}|_0 x^l(u)x^m(u)+\cdots.$$

The last of these equations states that we have a topological change of space, not a transformation of coordinates. Equating the two sets of expansions gives us, with rather tedious calculations, precisely conditions (2.6), namely

$$t_2\bar{\nu}^i=\bar{v}^i=\frac{\partial x^i}{\partial u^r}v^r=\frac{\partial x^i}{\partial u^r}t_2\nu^r,$$

whence

$$\bar{\nu}^i=\frac{\partial x^i}{\partial u^r}\nu^r,\quad \nu^r=\frac{\partial u^r}{\partial x^i}\bar{\nu}^i.$$

This is both necessary and sufficient, for nothing else follows from any of the higher terms in the series. This completes the proof of both theorems. We could have used the values in (2.7) for g_{ij} to calculate Christoffel symbols, and shown that they differ from our original $\Gamma^i{}_{jk}$ by at most projective additions. The formidable calculation involved in inverting $x(y)$ to $y(x)$ as a formal series expansion makes such a differential method of little value for insight. Actually these Christoffel symbols have the values

$$(2.9)\quad \bar{\Gamma}^i{}_{jk}=\frac{\partial^2 y^r}{\partial x^j\partial x^k}\frac{\partial x^i}{\partial y^r}.$$

3. The symmetric affine connections are not essential. *Any system of curves with smoothly turning tangents and the unique join property may be used to map the neighbourhood into an open sphere in R_n by specification of the parameter on the curves, thereby endowing the space with a Riemann metric of which the given system of curves are geodesics.*

The proof is now obvious, setting up a correspondence between local direction bundles which is uniquely determined by and conversely determines the relation between local coordinate systems. The essential property common both to the general system of paths and to the geodesics is that if the unique join of two points P and Q be drawn, it is also the unique join of P with every point on the arc PQ. Thus we can set up the translation parameter. It is seen that there is no great generalisation in this result.

Our results have several applications, such as, for example, to more general path-equations or non-kinematic dynamical systems for which no variational principle is given *a priori*, but which can usually be included in the above by taking additional coordinates. Inasmuch as the automorphism group of a Riemann space is a Lie group, it is seen that the automorphism group of a path-space preserving the parameter is also a Lie group. Inasmuch as the one-parameter subgroups of a locally Euclidean topological group and their (left-) cosets form such general paths, a solution of the Hilbert fifth problem is made fairly simple. For Riemann spaces in general, *we get a whole series of groups*, such as that of rigid motions (preserving the metric), collineations (preserving path-equations but not the metric, which is then taken into some other metric), projective collineations (preserving the path-curves, but not the equations), and so on, *each of which is an invariant subgroup* of all that follow.

The geometry of the given path-space is not the same as that after projective transformation of the parameter. For example, all projectively flat spaces are equivalent to a flat space; but a projectively flat Riemann space is one of constant isotropic (Schur) curvature—necessarily positive or zero if the global property is given, while the geometries differ for different values of the said constant. To visualize the simplest case, we take a spherical surface in R_3, and project from its center upon the plane tangent at the south pole. The convention may be made that the upper hemisphere projects upon a different side of the plane than the lower. We get then the one-sided projective plane (or alternatively stop with the southern hemisphere). If we take the distance induced on the surface by the Euclidean metric in R_3, we obtain the usual spherical geometry. If the distance for any path-segment geodesic arc (great circles) is taken to be the same as that in its projection on the plane (which is a straight line), the metric geometry is that of the plane. In general, the original x–space need not be a topological group at all, whereas the y–map on R_n can always be regarded as an Abelian group with vector composition for the group law.

The reason for these differences is elucidated by the following considerations. The geometry of the path-space can be based upon the concept of parallelism. A vector λ^i is said to be parallel-displaced along a given path if

$$\frac{d\lambda^i}{dt} + \gamma^i_r \lambda^r = 0 \; ; \qquad \gamma^i_j = \Gamma^i{}_{jk} \dot{x}^k \, . \tag{3.1}$$

This gives us another method of setting up a linear correspondence between direction-bundles at two different points joined by a (unique) path, namely

$$\begin{aligned} \bar{\lambda}^i &= \lambda^i - t\gamma^i_r|_0 \lambda^r - \frac{t^2}{2!}\left(\frac{d}{dt}\gamma^i_r - \gamma^i_k \gamma^k_r\right)_0 \lambda^r + \cdots \\ &= \lambda^r \left\{ \delta^i_r - t\gamma^i_r|_0 - \frac{t^2}{2!}\left(\frac{d}{dt}\gamma^i_r - \gamma^i_k \gamma^k_r\right)_0 + \cdots \right\}. \end{aligned} \tag{3.2}$$

This agrees with (2.5) and (2.6) as to the first two terms, but the coefficients of t^2 differ by

$$\frac{1}{3!} \xi^k \xi^l R^i{}_{klr}|_0 \lambda^r \quad . \tag{3.3}$$

The paths themselves are autoparallel lines in both senses, for (3.1) become the path-equations when $\lambda^i = \dot{x}^i$. But only the direction vectors tangent to any given path correspond to themselves under both linear correspondences, unless the curvature tensor vanishes; in that case a coordinate system exists for which the paths become straight lines $\ddot{x}^i = 0$, so that the space is flat and endowable with a proper Euclidean metric.

In formula (3.2) the higher terms are clearly expressible by covariant derivatives of the curvature tensor. The meaning of (3.2)–(3.3) is that *the local holonomy group is completely determined by the curvature tensor, and conversely. Formula* (3.3) *gives the equations of variation.*

For our general case, the unique join property excludes such paths as the geodesics on a cylinder that go at least once around the cylinder. That is, local arcwise connectedness becomes a completely local property, the whole arc remaining in the same neighbourhood as the two points it connects. On the other hand, smoothness or some equivalent concept is necessary in order to specify the "direction" and obtain one path in each direction from the unique join of two points.

Tata Institute of Fundamental Research,
Bombay, India