PATH EQUATIONS ADMITTING THE LORENTZ GROUP—II

BY

D. D. KOSAMBI, *Poona*

[Received 25 January 1941]

In the first paper* with the same title, it was shown that the fundamental ideas of the theory of relativity could be applied directly to the trajectories of particles, without assuming the existence of a Riemann metric. The results of that paper are valid also for more than four dimensions, with the corresponding extended Lorentz group; in particular, to the Kaluza-Klein theory in five and the Proca-Goudsmit in six dimensions. From the analysis of Cartan and von Neumann dealing with the theory of spinors, it is not to be expected that anything of importance could be obtained from a manifold of more than eight dimensions, but the cases $n = 5$-8 will not be without some value. These path-spaces constitute the most general such extension of Einstein's *special* theory.

The results of my first paper cover somewhat more ground than is apparent therein. Consider that the path-spaces admitting Lorentz ($=\mathbf{L}$) and similitude ($=\mathbf{S}$) groups are derivable from three-dimensional observations, and have the trajectories of the "fundamental particles", $\dot{x}^i = cx^i$ as solutions:

$$\ddot{x}^i - p^i \frac{\mathbf{Y}}{\mathbf{X}} G(\xi) + 2\dot{x}^i \frac{\mathbf{Z}}{\mathbf{X}} \gamma(\xi) = 0 \, ; \; i = 0, 1, 2, 3 \, ;$$
$$2\gamma(1) - G(1) + 1 = 0. \tag{1}$$

Here, as throughout the rest of this paper, the notation of (1) is used, though later on, it will be found more

* Kosambi (1).

convenient to make direct use in formulæ of the quantities $g=1+G$, $v=1+\gamma$. These paths become singular, or indeterminate, for the origin of space-time $x^0, x^1, x^2, x^3=0$, for which reason I have called them elsewhere (4) cosmogonic instead of cosmological path-spaces. If however, the observer is fixed as the space origin and lets his time coordinate $x^0=ct$ vary in accordance with the equations, we find that the equations reduce to

$$\ddot{x}^0+\frac{\dot{x}^{0\,2}}{x^0}\left\{2\gamma(1)-G(1)\right\}\equiv\dot{x}^0\left(\frac{\ddot{x}^0}{\dot{x}^0}-\frac{\dot{x}^0}{x^0}\right)=0. \qquad (2)$$

This has the solution $x^0=ae^{b\tau}$, so that the relation between the observer's time and the parameter τ of the path-equations is precisely that between Milne's two time-scales. Of course, I do not make the claim that all of Milne's results are covered by the theory of path-equations admitting various special groups. For example, Milne feels bound to express certain views on creation and the deity*, whereas I am unable to venture upon theological applications of the theory of continuous groups.

The rest of this paper, then, will give rather elementary results in the theory of such path-spaces, unifying and illustrating some scattered work published in other papers.†

1. The following formulæ are handy in calculations:

$$-g_{00}=g_{11}=g_{22}=g_{33}=-1\ ;\ g_{ij}=0,\ i\neq j;\ p^i=\dot{x}^i,\ \dot{x}^i=\frac{dx^i}{d\tau}$$

$$p_i=g_{ir}p^r;\ \dot{x}_i=\dot{x}^r g_{ir};\ \mathbf{X}=p^r p_r;\ \mathbf{Y}=\dot{x}^r\dot{x}_r;\ \mathbf{Z}=p^r\dot{x}_r;\ \xi=\mathbf{Z}^2/\mathbf{XY}$$

$$\mathbf{X}_{,i}=2p_i;\ \mathbf{X}_{;i}=0;\ \mathbf{Y}_{;i}=2\dot{x}_i;\ \mathbf{Y}_{,i}=0;\ \mathbf{Z}_{,i}=\dot{x}_i;\ \mathbf{Z}_{;i}=p_i$$

$$\xi_{;i}=\frac{2\mathbf{Z}}{\mathbf{XY}^2}(\mathbf{Y}p_i-\mathbf{Z}\dot{x}_i)\ ;\ \xi_{,i}=\frac{2\mathbf{Z}}{\mathbf{X}^2\mathbf{Y}}(\mathbf{X}\dot{x}_i-\mathbf{Z}p_i)\ ;\ \xi_{;r}\dot{x}^r=\xi_{,r}p^r=0$$

$$\mathbf{X}\xi_{,r}\dot{x}^r=\mathbf{Y}\xi_{;r}p^r=2\mathbf{Z}(1-\xi),\ \text{where}\ F_{;r}=\frac{\partial F}{\partial \dot{x}^r};\ F_{,r}=\frac{\partial F}{\partial x^r}\qquad(1.1)$$

* Milne, (2), 138-40.

† Kosambi, (3) and (4).

and the tensor summation convention is used.

It is also convenient to know how to calculate the contravariant tensor corresponding to any given covariant tensor of rank two. This, of course, can always be done by taking the normalized cofactors of the elements in the original tensor matrix, but for our work the tensor will be found to have the particular form:

$$T_{ij} = Ag_{ij} + B\dot{x}_i\dot{x}_j + Cp_ip_j + D\dot{x}_ip_j + Ep_i\dot{x}_j. \qquad (1.2)$$

Assuming therefore that the associated contravariant tensor is of type

$$T^{ij} = ag^{ij} + b\dot{x}^i\dot{x}^j + cp^ip^j + d\dot{x}^i p^j + ep^i\dot{x}^j, \qquad (1.3)$$

we solve the equations $T^{ir}T_{rj} = \delta^i_j$, which must be true identically in the quantities concerned. We obtain therefore,

$$\begin{aligned} aA &= 1 \\ aB + b(A + B\mathbf{Y} + E\mathbf{Z}) + d(B\mathbf{Z} + E\mathbf{X}) &= 0 \\ aD + b(C\mathbf{Z} + D\mathbf{Y}) + d(A + C\mathbf{X} + D\mathbf{Z}) &= 0 \\ aC + c(A + C\mathbf{X} + D\mathbf{Z}) + e(C\mathbf{Z} + D\mathbf{Y}) &= 0 \\ aE + c(B\mathbf{Z} + E\mathbf{X}) + e(A + B\mathbf{Y} + E\mathbf{Z}) &= 0. \end{aligned} \qquad (1.4)$$

Eliminating a from the first of these, the remaining fall into two sets, which can be solved if and only if the determinant

$$\Delta = (A + B\mathbf{Y} + E\mathbf{Z})(A + C\mathbf{X} + D\mathbf{Z}) - (B\mathbf{Z} + E\mathbf{X})(C\mathbf{Z} + D\mathbf{Y}) \neq 0.$$

It is clear, of course that $A \neq 0$ is also a necessary restriction; it need not be added that the equations (1.4) can be solved for the coefficients of T_{ij} when T^{ij} is given. The explicit solutions are:

$$\begin{aligned} &a = 1/A; \Delta = A(A + B\mathbf{Y} + C\mathbf{X} + D\mathbf{Z} + E\mathbf{Z}) + (BC - DE)(\mathbf{XY} - \mathbf{Z}^2); \\ &A\Delta b = \mathbf{X}(DE - BC) - AB; \quad A\Delta d = \mathbf{Z}(BC - DE) - AD \\ &A\Delta c = \mathbf{Y}(DE - BC) - AC; \quad A\Delta e = \mathbf{Z}(BC - DE) - AE. \end{aligned} \qquad (1.5)$$

These formulæ become particularly important when we have T_{ij} as the fundamental tensor of a metric path-space, i.e. of the form $f_{;i;j}$, the condition $A\Delta \neq 0$ being then the condition for the metric to be non-degenerate, and the variational problem to be regular. For the special metric $f = \frac{\mathbf{Y}}{\mathbf{X}}\phi(\mathbf{X}, \xi)$ we have:

$$A = \phi - \xi\phi_2; \quad \mathbf{X}C = \phi_2 + 2\xi\phi_{22}; \quad \mathbf{Y}B = 2\xi^2\phi_{22}; \quad \mathbf{Z}D = \mathbf{Z}E = -2\xi^2\phi_{22}. \quad \phi_2 = \partial\phi/\partial\xi, \text{ etc.} \tag{1.6}$$

The condition $\Delta \neq 0$ reduces to

$$\phi^2 + (1-2\xi)\phi\phi_2 - \xi(1-\xi)\phi_2^2 + 2\xi(1-\xi)\phi\phi_{22} \neq 0. \tag{1.7}$$

This can be integrated for the cases when the expression vanishes, and the degenerate values of the metric are then given by

$$\sqrt{\phi} = P\sqrt{(\xi - 1)} + Q\sqrt{\xi}, \tag{1.8}$$

where P, Q are arbitrary functions of $\mathbf{X}$ alone. This includes the case $A = 0$ as well. Any other metric is permissible, if it gives the paths desired as extremals.

2. In the previous paper (1), it was shown that a path-space admitting **L** and **S** in addition to a Riemann metric was, if isotropic, necessarily flat. This result and its possible generalizations really illustrate a theorem in the projective change of connection for the classical path-spaces: that such a change of connection always exists if the space is projectively flat, so that the equations of the paths become those for an ordinary flat space.

Discarding the similitude group, the most general path-spaces with a symmetric affine connection admitting the Lorentz group can, as is obvious, always be put in the form:

$$\ddot{x}^i - p^i\frac{\mathbf{Y}}{\mathbf{X}}(A-1) + \frac{\mathbf{Z}^2}{\mathbf{X}^2}B + 2\dot{x}^i\frac{\mathbf{Z}}{\mathbf{X}}(C-1) = 0. \tag{2.1}$$

Here A, B, C are as yet arbitrary functions of $\mathbf{X}$ alone. The similitude group applies if and only if all the three functions are constants; the -1 is inserted here, as in later path-equations, to give the simplest ultimate formulæ. A Riemann metric must have the form:

$$f = \alpha\mathbf{Y}/\mathbf{X} + \beta\mathbf{Z}^2/\mathbf{X}^2 = \bar{g}_{ij}\dot{x}^i\dot{x}^j \text{ ; } \alpha,\ \beta, \text{ functions of } X \text{ alone.}$$

$$\bar{g}_{ij} = \frac{\alpha}{\mathbf{X}}g_{ij} + \frac{\beta}{\mathbf{X}^2}p_ip_j \text{ ; } |\bar{g}_{ij}| = -\alpha^3(\alpha+\beta) \neq 0. \tag{2.2}$$

Also the covariant derivative with respect to (2.1) of the tensor $\bar{g}_{ij}$ must vanish. These conditions can be calculated by the usual method, to give:

$$\alpha' = \alpha C \text{ ; } \beta' = \beta(2C-B) - \alpha B \text{ ; } (A-C)\alpha + A\beta = 0. \tag{2.3}$$

The dash indicates, as usual, differentiation with respect to the independent variable *taken here as* log X. As there are, for any given system of paths, only two unknown functions α, β to be determined, and three equations, we immediately obtain a compatibility condition:

$$A'/A - C'/C = A + B - C \text{ ; } AC \neq 0 \text{ (unless } A = C = 0). \tag{2.4}$$

The latter part of the formulæ is simply the non-degeneracy condition in (2.2).

Direct calculation shows that the projective curvature tensor of the path-space vanishes if and only if we have

$$A^2 + AB - 1 - 2A' = 0. \tag{2.5}$$

Now, for flatness, we must have in addition to (2.5), the conditions:

$$AC - 1 = 0 \text{ ; } 2C' + BC - C^2 + 1 = 0. \tag{2.6}$$

Of these, the condition $AC - 1 = 0$ is crucial, because it makes the other two conditions for flatness compatible. Moreover (if $AC = 1$), the condition for the existence of a non-degenerate metric reduces precisely to the condition for the space to be projectively flat. The relationship between our theory and that of a projective change of connection is furnished by the fact that the function $C(\mathbf{X})$ cannot be determined by three-dimensional (for the extended Lorentz group, $(n-1)$-dimensional) observations. Hence, if we utilize the indeterminacy to obtain the existence of a metric, we can always get projective flatness to coincide with ordinary flatness. This gives us:

THEOREM 1. *If A, B are prescribed, with $A \neq 0$, and C may be chosen at will, then the choice $C = 1/A$ in equations (2.1) gives a space which is both isotropic and flat; for the given choice of C, no projectively flat space can exist which is not also flat in the usual sense.*

The actual transformation carrying the metric of (2.2) into a flat space can be found by putting $\bar{x}^i = x^i\phi(\mathbf{X})$, calculating $g_{ij}\dot{\bar{x}}^i\dot{\bar{x}}^j$, and setting it equal to the original metric. The use of (2.3) and the condition $AC-1=0$ leads to the function ϕ at once, $\phi = \exp \int \frac{(C-1)}{2\mathbf{X}} d\mathbf{X}$.

If A, B, C are to be constants and a metric exists, then there is no other choice possible for isotropy except the one which also gives flat spaces. For the general case, it is quite clear that choices of C exist which allow a metric and isotropy, but do not then imply flatness. However, such a statement would mean that the function C has an intrinsic position of its own although it cannot be specified from $(n-1)$-dimensional observations.

3. To extend these results to more general types of spaces, we shall first have to discuss the existence of a metric under more general conditions. Any space whose paths are deducible from $(n-1)$-dimensional observations, and admit the Lorentz group is defined by the path-equations:

$$\ddot{x}^i - b^i\frac{\mathbf{Y}}{\mathbf{X}}\{ g(\mathbf{X}, \xi) - 1 \} + 2\dot{x}^i\frac{\mathbf{Z}}{\mathbf{X}}\{ v(\mathbf{X}, \xi) - 1 \} = 0. \quad (3.1)$$

These admit both **L** and **S** if g and v are functions of ξ alone. A metric exists for these spaces if and only if there exists a function f satisfying:

$$f_{|i} \equiv f_{,i} - \tfrac{1}{2}\alpha^r_{;i} f_{;r} = 0, \quad |f_{;i;j}| \neq 0, \quad (3.2)$$

where α^i is obtained by regarding the paths as $\ddot{x}^i+\alpha^i=0$. Now the condition of non-degeneracy has been discussed in the first section, so that it only remains to reduce the equations (3.2) to an amenable form. Taking $f=\frac{\mathbf{Y}}{\mathbf{X}}\exp H[\mathbf{X}, \xi]$, and recalling that with f, any function thereof such as $\log f$ will also be a solution of (3.2), we get the equations:

$$\begin{array}{ll} H_1=PH_2+Q & P=\xi(1-\xi)g_2+\xi v; \\ H_2=S/R & Q=-\xi g_2+v+2\xi v_2; \\ H_1=\partial H/\partial \log X & R=(1-\xi)(g-\xi g_2)+\xi v; \\ H_2=\partial H/\partial \xi \text{ etc.} & S=-(g-\xi g_2)+v-2\xi v_2. \end{array} \quad (3.3)$$

The notation has again been changed from that of my previous work to give the simplest final calculations. Solving the above equations explicitly for H_1, H_2 we have a simple first order partial differential system which is immediately integrable if and only if $H_{12}-H_{21}\equiv(PS/R+Q)_2-(S/R)_1=0$. This is a differential equation, from our present point of view, for the unobservable v in terms of the observed function g, and inasmuch as a solution exists in general, the metric exists unless the only possible solution does not satisfy the condition for non-degeneracy. If the metric be wanted directly, without troubling ourselves as to the choice of v, it can be obtained by regarding (3.3) as linear equations which can be solved for the unknowns v, v_2. We have:

$$\begin{aligned} v(\xi H_2-1)+2\xi v_2 &= (g-\xi g_2)\{(\xi-1)H_2-1\}; \\ v(\xi H_2+1)+2\xi v_2 &= H_1-\xi(\xi-1)g_2H_2+\xi g_2. \end{aligned} \quad (3.4)$$

These give at once

$$2v=H_1+g\{1-(\xi-1)H_2\}. \quad (3.5)$$

Substitution of this value in the solution for v_2 leads to

$$\begin{aligned} 2\xi H_{12}-2\xi(\xi-1)gH_{22}+\xi H_1H_2-\xi(\xi-1)gH_2^2-H_1 \\ +(1-2\xi)gH_2+g=0. \end{aligned} \quad (3.6)$$

The integration of this can be performed by the standard methods of Monge, but it is simplified by the transformation $H = \log \xi + 2 \log \phi$. The new equation in ϕ has then the form

$$\phi_{12} + (1 - \xi) g \phi_{22} + (3/2\xi - 2) g \phi_2 = 0. \tag{3.7}$$

This equation has been given* for g, v functions of ξ alone, but our derivation shows it to be valid whenever the path-space admits **L** and possesses a Finsler metric. The integration is obvious, by the standard methods such as that of Charpit, treating (3.7) as a first order linear differential equation in ϕ_2. For $g(= 1 + G$ in Milne's notation) not zero, the result is equivalent to that of Walker (**2**, 166). For $g = 0$, we have at once $\phi = \alpha(\xi) + \beta(X)$, and the second term can be discarded because it amounts to an additive perfect differential, such as is admissible in any variational problem. The metric for $g = 0$ (Milne's kinematic case) is then $\mathbf{Y}\phi(\xi)/\mathbf{X}$, the function ϕ being arbitrary, subject only to the condition of non-degeneracy.

When, however, the similitude group applies, the situation is better treated in another way, though the above methods are quite valid. Here, the metric, to admit both **L** and **S**, with relative invariance under the latter, must have the form $\frac{\mathbf{Y}}{\mathbf{X}}\mathbf{X}^a\phi(\xi)$, and ϕ is given as $\exp \int (S/R) d\xi$. The condition for this to be possible, i.e. the condition of integrability of (3.3) when g, v do not contain **X**, is $(PS/R + Q)_2 = 0$, which is $PS/R + Q = a - 1$, the constant a being the same as that which enters into the metric. In this case, we can determine the proper choice of v, for any given g, and the existence of a metric, as a solution of the Riccatian equation

* See (**1**), formula (12).

$$v' = \frac{a(g-\xi g')}{2\xi g} + \frac{v}{\xi g}\left(\xi g' - g - \frac{g+a\xi}{2(\xi-1)}\right) + \frac{v^2}{(\xi-1)g}\,. \quad (3.8)$$

The above equation always has the solution $v = \frac{1}{2}a + g/2\xi$, but this always leads to a degenerate metric, and can only be used to deduce the general solution:

$$v = a/2 + g/2\xi + 1/u,$$
$$u' + u\left\{ g'/g + 1/2(\xi-1) - 3/2\xi + a/2(\xi-1)g \right\} + 1/(\xi-1)g = 0. \quad (3.9)$$

As an application, it will be found that the metric spaces which have identically the relationship $2v = g$ (which need hold only for $\xi = 1$ to give us the solutions $\dot{x}^i = cx^i$ for the path-equations) have a g given by

$$g = \frac{a\xi}{\xi - 1 + b\sqrt{\xi - 1}}, \quad (3.10)$$

a, as in the metric; b, arbitrary constant.

So, the only sub-case which is also Riemannian is $v = g = 0$, the "kinematic" case again, metric $\mathbf{Y}/\mathbf{X}$.

4. Coming now to the question of isotropy, we note that the theorem of Schur admits of a partial extension to our general path-spaces, if the concept of isotropy is redefined (**3**). This amounts to the restriction that the first curvature tensor P^i_j of the space should reduce to the form $\lambda\delta^i_j - \dot{x}^i q_j$. This curvature tensor is, for the spaces (3.1) admitting the Lorentz group and deducible from three-dimensional observations, of the form

$$\mathbf{X}^2 P^i_j = A\mathbf{XY}\delta^i_j + \mathbf{X}B\dot{x}^i\dot{x}_j - C\mathbf{Z}\dot{x}^i p_j + Ep^i(\mathbf{Z}\dot{x}_j - \mathbf{Y}p_j);$$
$$A = g\left\{v + 2\xi(1-\xi)v_2\right\} + \xi(2v_1 - v^2) - 1; \quad B = C\xi - A;$$
$$C = 1 + vg_2 - v^2 + 2v_1 - 4\xi v_{12} + v_2\left\{2\xi(1-\xi)g_2 - 6g + 8\xi g - 2\xi v\right\} - 4\xi(1-\xi)gv_{22};$$
$$E = \xi(1-\xi)(2gg'' - g'^2) + (1-2\xi)gg' + g^2 - 1 + 2(g_1 - \xi g_{12});$$

where $v_1 = \dfrac{\partial v}{\partial \log \mathbf{X}}$, $v_2 = \dfrac{\partial v}{\partial \xi}$, etc. (4.1)

For simplicity, I consider the case where both **L** and **S** groups are admitted, and g, v are functions of ξ alone.

The condition of quasi-isotropy is, for these restricted path-spaces

$$\Gamma \equiv \xi(1-\xi)(2gg''-g'^2)+(1-2\xi)gg'+g^2-1=0. \qquad (4.2)$$

This differential equation has the integrating factor g'/g^2; the first integration gives

$$\xi(1-\xi)g'^2/g+g+1/g=\text{const.} \qquad (4.3)$$

The complete solution is best presented in the form

$$g=p\sqrt{\xi(\xi-1)}+q(\xi-1/2)\pm\tfrac{1}{2}\sqrt{(p^2-q^2+4)}. \qquad (4.4)$$

If we wish to make $P^i_j=0$, we get three more equations, of which only two are independent, and admit the common solution:

$$\left.\begin{aligned} v &= b(\xi-1)^{\frac{1}{2}}\xi^{-\frac{1}{2}}+c \\ g &= 2b\sqrt{\xi(\xi-1)}+\frac{b^2-1}{c}(\xi+1)+c\xi \end{aligned}\right\}, \; b, c, \text{ any constants.}$$

$$(4.4)$$

In spite of the apparent difference in form, it will be seen that the form of the solution for g is precisely that given in (4.4), with proper adjustment of the two sets of arbitrary constants. One further adjustment can be made by substitution of the above values of v, g in (3.8): the arbitrary constant c in (4.4) must be the same as the exponent in the metric; $c=a$. It is clear, then, that *a choice always exists for v which gives a metric, and whenever the g is such that the space is quasi-isotropic (the condition of quasi-isotropy being independent of v), we automatically have $P^i_j=0$.*

It does not follow, however, that the space is flat even then. To this end, it would be necessary and sufficient to have an additional condition $\alpha^i_{;j;k;l}=0$, which would bring us back to the symmetric affine connection discussed before. The facts of the matter here are as follows. For any system of paths, and for a sufficiently restricted piece of a given path thereof, it is possible to choose a coordinate system making $\alpha^i_{;j}=0$

along the path. When, as here, we have $\alpha^i - \frac{1}{2}\alpha^i_{;r}x^r = 0$, the path has the equation of a straight line $\ddot{x}^i = 0$. If, in addition, P^i_j is zero, the equations of variation admit along the chosen path as base, solutions for which the components of the vector variation are linear in the parameter τ. This means that the whole infinite sheaf of paths which can be obtained from the given path by giving it successive "small variations" all have the form of straight lines. Beyond this it is not possible to go unless $\alpha^i_{;j;k;l} = 0$, in which case alone is it possible to assert that *all* paths become straight lines in the chosen system of coordinates.

For non-homogeneous α^i, it is not possible to go even as far as the conclusions of the last paragraph. But the discussion of that case is beyond our scope here.

REFERENCES

1. D. D. KOSAMBI: Path Equations admitting the Lorentz Group, *Jour. Lond. Math. Soc.* 15 (1940), 86-91.

2. E. A. MILNE: *Relativity, Gravitation and World Structure*, (Oxford, 1935).

3. D. D. KOSAMBI: The Concept of Isotropy in Generalized Path-Spaces, *Jour. Ind. Math. Soc.* (2) 4 (1940), 80-8.

4. D. D. KOSAMBI: *Comptes Rendus* (Paris), tome 206, pp. 1086-8; 1538-41.