THE CONCEPT OF ISOTROPY IN GENERALIZED PATH-SPACES

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1. This note attempts a generalization of the concept of isotropy for spaces defined by "paths", i.e. the solution curves of

$$\ddot{x}^i + \alpha^i(x, \dot{x}, t) = 0; \; i = 1, .., n. \quad \dot{x}^i = dx^i/dt \text{ etc.} \tag{I}$$

Besides the tensor summation convention for repeated indices, we shall use the notations

$$A^{\cdot}_{.,i} = \frac{\partial A^{\cdot}}{\partial x^i}; \; A^{\cdot}_{.;i} = \frac{\partial A^{\cdot}}{\partial \dot{x}^i}; \; \frac{dA}{dt} = -\alpha^r A_{;r} + \dot{x}^r A_{,r} + \frac{\partial A}{\partial t}.$$

The fundamental tensorial operations for the path-space are now:

$$T^{\cdot}_{.;i} \; ; \; \mathbf{D}T^{\cdot}_{.} = \frac{dT^{\cdot}_{.}}{dt} + \gamma^i_r T^r_{.} - \gamma^r_j T^{\cdot}_{\sigma}$$

$$T^{\cdot}_{./i} = T^{\cdot}_{.,i} - \gamma^r_i T^{\cdot}_{.;r} + \gamma^j_{ri} T^r_{.} - \gamma^r_{ik} T^{\cdot}_{r} \ldots, \tag{1.1}$$

where $\quad \gamma^i_j = \frac{1}{2}\alpha^i_{;j} \; ; \; \gamma^i_{jk} = \frac{1}{2}\alpha^i_{;j;k} \; ; \; \gamma^i_{jkl} = \frac{1}{2}\alpha^i_{;j;k;l}.$

The fundamental differential invariants, besides $\dot{x}^i$ and γ^i_{jkl} are

$$\varepsilon^i = -\mathbf{D}\dot{x}^i = \alpha^i - \gamma^i_r \dot{x}^r \; ; \; \frac{\partial \alpha^i}{\partial t};$$

$$P^i_j = -\alpha^i_{,j} + \frac{d}{dt}\gamma^i_j + \gamma^i_r \gamma^r_j \; ; \; R^i_{jk} = \tfrac{1}{3}(P^i_{j;k} - P^i_{k;j}), \tag{1.2}$$

where $R^i_{jk;l}$ corresponds to the usual mixed Riemann-Christoffel tensor provided the suffixes are taken in the proper order.

By a metric for the space defined by (I), we shall mean the integrand f of any regular problem of the calculus of variations whose extremals are defined by (I). For the existence of such a metric, a set of necessary and sufficient conditions are[1]:

$$\mathbf{D}f_{;i;j} = 0; \; f_{;i;r}P^r_j - f_{;r;j}P^r_i = 0; \; |f_{;i;j}| \neq 0. \tag{1.3}$$

If, however, the space admits a metric such that any non-trivial function thereof $\phi(f)$ is also a metric, the conditions above reduce to

$$\mathbf{D}f = 0; \; f_{/i} = 0; \text{ along with } |f_{;i;j}| \neq 0. \tag{1.4}$$

In particular, the Finsler metric and all homogeneous metrics come under this special case. If the α^i are homogeneous of degree two, i.e. ε^i vanishes, we have $\mathbf{D}f \equiv \dot{x}^r f/_r + \partial f/\partial t$. Hence, in view of (1.4), the *metric cannot contain the parameter explicitly, and therefore, the* α^i *also cannot contain t explicitly, if* $\varepsilon^i = 0$.

The equations of variation of (I) are most important and can be written in the invariant form:

$$\mathbf{D}^2 u^i = P^i_r u^r. \tag{II}$$

It is to be noted that the covariant form of these equations can be written[1] as

$$\mathbf{D}^2 u_i = P^r_i u_r, \tag{1.5}$$

without any device for raising or lowering the indices, merely by seeking the integrating factor for (II). As u^i is a contravariant vector u_i will be covariant; but there is no other connection between the two forms unless a metric exists.

2. The equations (II) and (1.5) are the sole means of exploring our path-space, the paths themselves being regarded, *im Kleinen* as the fundamental entities of the space. The usual mechanism for Riemannian spaces which leads to the theorem of Schur is missing. But the equations (I) always admit a dynamical interpretation, and for isotropy it would be necessary that every direction transverse to any path (trajectory) should be a principal direction. One should expect the equations (1.5) to reduce to a canonical form $\mathbf{D}^2 u_i = \phi u_i$ provided $\dot{x}^r u_r = 0$. This, however, is a restriction on P^i_j, which must, according to well-known results in tensor algebra* have the structure $\phi\delta^i_j - \dot{x}^i q_j$. All this motivates our first

DEFINITION 1. *A path-space will be called quasi-isotropic if* $P^i_j = \phi\delta^i_j - \dot{x}^i q_j$.

The implications of this definition for Riemann spaces and for those with a symmetric affine connection can be summed up in

THEOREM I. *For* $n > 2$, *a Riemann space is quasi-isotropic if and only if it is isotropic in the ordinary sense. A space with a symmetric affine connection is quasi-isotropic if and only if its Weyl tensor vanishes.*

For $n = 2$, nothing particular can be said. For the rest, I shall prove only the second part of the theorem, as it includes

* cf. Schouten, *Ricci-Kalkül* p. 59, ex. 5.

the first. We have $\alpha^i = \Gamma^i_{jk}\dot{x}^j\dot{x}^k$, and $P^i_j = B^i_{jkl}\dot{x}^k\dot{x}^l$ (using the notation of Eisenhart's *Non-Riemannian Geometry*). For quasi-isotropy, we must have

$$B^i_{jkl}\dot{x}^k\dot{x}^l = \phi\delta^i_j - \dot{x}^i q_j, \text{ i. e. } \phi = a_{ij}\dot{x}^i\dot{x}^j, \; q_j = q_{jr}\dot{x}^r, \tag{2.1}$$

where $a_{ij} = a_{ji}$ and q_{ij} are functions of x alone. As $B^i_{jkl}\dot{x}^j\dot{x}^k\dot{x}^l = 0$ identically, we have $a_{ij} = \frac{1}{2}(q_{ij}+q_{ji})$, and the result follows by differentiation with respect to $\dot{x}$,

$$B^i_{jkl} = \delta^i_l a_{kj} - \delta^i_k a_{jl} + \tfrac{1}{3}(\delta^i_l \pi_{kj} + \delta^i_k \pi_{jl} + 2\delta^i_j \pi_{kl}),$$

where $\pi_{ij} = \frac{1}{2}(q_{ij} - q_{ji})$. (2.2)

But the projective curvature tensor for a space with symmetric affine connection is given by

$$W^i_{jkl} = B^i_{jkl} + \frac{2}{n+1}\,\delta^i_j\beta_{kl} + \frac{1}{n-1}\left\{\delta^i_k B_{jl} - \delta^i_l B_{jk}\right\} + \frac{2}{n^2-1}\left\{\delta^i_l\beta_{jk} - \delta^i_k\beta_{jl}\right\}, \tag{2.3}$$

where $B_{jk} = B^i_{jki}$, $\beta_{jk} = \frac{1}{2}(B_{jk} - B_{kj})$.

By direct substitution from (2.2), it follows here that

$$W^i_{jkl} = 0.$$

For the converse, we see that for $W^i_{jkl} = 0$, the tensor P^i_j is given by

$$B^i_{lkj}\dot{x}^k\dot{x}^l = \frac{1}{n-1}\,\delta^i_j B_{lk}\dot{x}^k\dot{x}^l - \dot{x}^i\left\{\frac{2}{n+1}\beta_{kj} + \frac{1}{n-1}\,B_{kj} - \frac{2}{n^2-1}\beta_{kj}\right\}\dot{x}^k, \tag{2.4}$$

which is precisely of the form required by our definition. The proposed definition, therefore, covers the classical Riemannian isotropy, and its generalization, the projectively flat space of symmetric affine connection for $n > 2$.

3. To deal with the general α^i, we shall require a certain number of identities, in particular,

$$\dot{x}^i_{/j} = -\varepsilon^i_{;j} \;;\; \mathbf{D}\,\varepsilon^i = -P^i_r\dot{x}^r + \partial\alpha^i/\partial t. \tag{3.1}$$

Then, the analogue of the Bianchi identities

$$R^i_{kj/l} + R^i_{kl/j} + R^i_{lj/k} = 0. \tag{3.2}$$

To clarify the relation of this with the usual form of the identity, we can restate it in the 4-index tensor:

$$R^i_{jk;m/l} + R^i_{kl;m/j} + R^i_{lj;m/k} = R^r_{kj}\gamma^i_{rml} + R^r_{lk}\gamma^i_{rmj} + R^r_{jl}\gamma^i_{rmk}. \tag{3.3}$$

These identities hold for all sufficiently differentiable α^i, and each of (3.2) and (3.3) can be derived from the other.

For quasi-isotropy, with $P^i_j = \phi\delta^i_j - \dot{x}^i q_j$, we have, as a consequence of (3.2),

$$\delta^i_j\{\phi_{;k/l}-\phi_{;l/k}+q_{k/l}-q_{l/k}\}+\delta^i_k\{\phi_{;l/j}-\phi_{;j/l}+q_{l/j}-q_{j/l}\}$$
$$+\delta^i_l\{\phi_{;j/k}-\phi_{;k/j}+q_{j/k}-q_{k/j}\}$$
$$-2\{(\dot{x}^i\pi_{jk})_{/l}+(\dot{x}^i\pi_{kl})_{/j}+\dot{x}^i\pi_{lj})_{/k}\}=0, \tag{3.4}$$

where
$$2\pi_{ij}=q_{i;j}-q_{j;i}.$$

As expected, this vanishes identically for $n=2$. For $n=3$, it reduces to

$$\phi_{;k/l}-\phi_{;l/k}+q_{k/l}-q_{l/k}=2\{(\dot{x}^j\pi_{jk})_{/l}+(\dot{x}^j\pi_{kl})_{/j}+(\dot{x}^j\pi_{lj})_{/k}\}. \tag{3.5}$$

The odd appearance of this expression is accounted for by the fact that for $n=3$, and for no other case, it is possible to display an anti-symmetric two index covariant tensor as a contravariant vector. In (3.5), it is immaterial whether the repeated index i is summed or not, as (3.4) is identically satisfied if any two subscripts are the same, and the superscript i must therefore be one of j, k, l, which can be taken as distinct among themselves. From this point of view, (3.5) says that the curl-vector on the left is twice the divergence of the contravariant tensor formed by the product of $\dot{x}^i$ with the bi-vector π_{kl}. We shall not follow this any further.

For $n\geqslant 4$, (3.3) can be broken up by first taking $i\neq j\neq k\neq l$ which implies the vanishing of the last bracket by itself for those indices. Then take, say, $i=j\neq k\neq l$, and finally, sum by contracting i and j. A comparison of the various results shows that we must have

$$(a)\ \phi_{;k/l}-\phi_{;l/k}+q_{k/l}-q_{l/k}=0;$$
$$(b)\ (\dot{x}^i\pi_{jk})_{/l}+(\dot{x}\,\pi_{kl})_{/j}+(\dot{x}\,\pi_{jl})_{/k}=0. \tag{3.6}$$

Of these, the second has further consequences obtained by writing it in the form—by making use of (3.1)

$$(b)\ \dot{x}^i(\pi_{jk/l}+\pi_{kl/j}+\pi_{lj/k})=\varepsilon^i_{;l}\pi_{jk}+\varepsilon^i_{;j}\pi_{kl}+\varepsilon^i_{;k}\pi_{lj}. \tag{3.61}$$

Hence, either, each side of this vanishes separately, or $\dot{x}^i$ is a factor of each side. This would mean $\varepsilon^i_{;j}=\dot{x}^i\sigma_j$. Differentiating, we get $\varepsilon^i_{;j;k}=\delta^i_k\sigma_j+\dot{x}^i\sigma_{j;k}$. By alternating these and subtracting, we have $\dot{x}^i(\sigma_{k;j}-\sigma_{j;k})=\delta^i_k\sigma_j-\delta^i_j\sigma_k$. For $i\neq j\neq k$, we have $\sigma_{k;j}-\sigma_{j;k}=0$, hence $\delta^i_k\sigma_j-\delta^i_j\sigma_k=0$ for all i, j, k and by contracting

i, j in the last step, we see that $\sigma_j = 0$. This means $\varepsilon^i = v^i(x, t)$, $\alpha^i = \alpha^{*i} + v^i$, where α^{*i} is homogeneous of degree two in $\dot{x}$. Moreover, in any case, each side of (2.6) vanishes separately.

If the right side of (3.61) vanishes, and π_{ij} is not identically zero, and for even values of n, the determinant π_{ij} does not vanish, and we can solve $\pi_{ij}\pi^{jk} = \delta^k_i$. Contracting the right side of (3.61), with π^{jk}, we get $(n-2)\,\varepsilon^i_{;l} = 0$, which is again the conclusion of the last paragraph. For odd n, the determinant of π_{ij} will vanish identically.

THEOREM II. *For quasi-isotropy, $n = 3$, the equations* (3.5) *must be satisfied. For $n \geqslant 4$, we must have*

$$\phi_{;k/l} - \phi_{;l/k} + q_{k/l} - q_{l/k} = 0$$

and

$$\pi_{jk/l} + \pi_{kl/j} + \pi_{jl/k} = \varepsilon^i_{;j}\pi_{kl} + \varepsilon^i_{;k}\pi_{lj} + \varepsilon^i_{;l}\pi_{jk} = 0.$$

I shall make one more general remark in this section about quasi-isotropic spaces. If we perform the analogue of a projective change of connection by replacing α^i with $\alpha^i + \dot{x}^i\psi$, we get

$$\bar{P}^i_j = P^i_j + \tfrac{1}{2}\delta^i_j\left\{D\psi + \tfrac{1}{2}\psi^2 - \psi\psi_{;r}\dot{x}^r\right\} - \tfrac{1}{2}\varepsilon^i\psi_{;j} + \psi\varepsilon^i_{;j}$$

$$- \dot{x}^i\left\{-\tfrac{1}{2}D\psi_{;j} + \psi_{/j} + \tfrac{1}{2}\psi\psi_{;j;r}\dot{x}^r - \tfrac{1}{4}\psi_{;j}\psi_{;r}\dot{x}^r\right\}. \tag{3.7}$$

As in the case of projectively flat spaces with a symmetric affine connection, it is possible to make P^i_j vanish if the space is quasi-isotropic, and $\varepsilon^i = \partial\alpha^i/\partial t = 0$, but not in general.

4. We now consider quasi-isotropic spaces for which $\varepsilon^i = \partial\alpha^i/\partial t = 0$, and a metric exists. In view of (3.1), we must have $\phi = q_r\dot{x}^r$. The metric can, without loss of generality, be taken as homogeneous[2] in $\dot{x}$, and there are two cases to be distinguished: homogeneity of degree zero, and homogeneity of any other degree. For the non-zero case, the metric can always be replaced[2] by one of degree two; we could demand a metric of degree one, but then the determinant $|f_{;i;j}|$ will vanish; we have merely followed the device used in Finsler spaces of replacing the metric f by f^2. In either case, we apply the second part of (1.3), or the integrability conditions of (1.4), which are $P^r_i f_{;r} = R^r_{ij} f_{;r} = 0$. We shall have to use the condition of homogeneity $\dot{x}^r f_{;r} = 0$, or $= 2f$, respectively. The results are:

$$\text{case 1}: P^i_j = \mu \dot{x}^i f_{;j} \quad ; \quad \mu_{;r}\dot{x}^r = 2\mu,$$

$$\text{case 2}: P^i_j = \lambda(f\delta^i_j - \tfrac{1}{2}\dot{x}^i f_{;j}) \text{ , } \lambda_{;r}\dot{x}^r = 0. \tag{4.1}$$

The homogeneity restrictions on μ, λ follow from the fact that in any case, the curvature tensor P^i_j must be homogeneous of degree two with $\dot{x}^i$. To apply (3.6), we keep in mind the fact that $f_{/i} = f_{;i/j} = 0$ are consequences of (1.4). We then obtain:

$$\text{case 1}, \quad \mu_{/i}f_{;j} - \mu_{/j}f_{;i} = 0,$$

$$\text{case 2}, \quad 2(\lambda_{;i/j} - \lambda_{;j/i})f + 3(\lambda_{/j}f_{;i} - \lambda_{/i}f_{;j}) = 0. \tag{4.2}$$

In each case, (3.61) is identically fulfilled, as can be verified by differentiation of (4.2) with respect to $\dot{x}^k$, and addition after cyclic rotation of the subscripts. In case 1, we can contract (4.2) with $\dot{x}^i$, to arrive at the conclusion that $\mathbf{D}\mu = 0$. The equations of variation have the form $\mathbf{D}^2 u_i = 0$, a case of neutral equilibrium. To get any further, we shall have to formulate another

Definition 2. *A quasi-isotropic space will be called isotropic if the function ϕ can, by suitable choice of the parameter t, be made to have the same value for the entire sheaf of paths through each point.*

With Definition 1, this amounts to saying that the "curvature is locally constant". For case 2 of this section, we see that the function λ will, for full isotropy, have to be a function of position alone, not of direction, inasmuch as it is of degree zero in $\dot{x}$, and hence, a change of parameter will not affect it at all. Here, $\phi = \lambda f$, and, from (1.4), $\mathbf{D}f = 0$, which means that the integral $f = const.$ along the path exists, and choice of parameter will fix this constant in value. We have then, $\lambda_{;i} = 0$, and (4.2) (2) reduces to

$$\lambda_{/i}f_{;j} - \lambda_{/j}f_{;i} = 0 \quad ; \quad \lambda_{/i} = \lambda_{,i}. \tag{4.3}$$

This can be possible only if $\lambda_{/i} = 0$, or $f_{;i} = \sigma\lambda_{/i}$. The latter would lead to $f_{;i;j} = \lambda_i \sigma_{;j}$, and to $|f_{;i;j}| = 0$, a contradiction of our hypothesis. Hence, $\lambda_{;i} = \lambda_{/j} = 0$, or λ is a constant, exactly as for the Riemannian case.

Theorem III. *Isotropy for a space admitting a metric f homogeneous of degree two in $\dot{x}$ is possible if and only if $P^i_j = c(f\delta^i_j - \frac{1}{2}\dot{x}^i f_{;j})$ where c is an absolute constant. Unless $P^i_j = 0$, the isotropic space cannot admit two essentially distinct metrics.*

The latter part of the theorem follows from (4.1). For, if there were two distinct metrics, one not being a function of the other, there would be two distinct metrics of degree two, and their ratio, satisfying (1.4), would also be a metric as the discriminant will not vanish in general. Then, P^i_j admits a metric of degree zero, and so has *both* the forms in (4.1), and hence vanishes.

5. For the general non-homogeneous metric and general α^i, we shall have $P^i_j = \phi\delta^i_j - \lambda \dot{x}^i f_{;j;r} \dot{x}^r$ Rather than deal with this, I consider an allied question: When can a given space be "immersed" into a quasi-isotropic space by taking the parameter t as an additional cordinate x^0 with a new parameter s, so as to make the the new $\varepsilon^i = \partial\alpha^i/\partial s = 0$? The new path cquations become[3]

$$x''^i + x'^{0\,2}\left\{ \alpha^i + \beta \dot{x}^i \right\} = 0 \qquad \begin{cases} i = 1, .., n, \\ \beta(x, \dot{x}, t) \text{ arbitrary,} \end{cases}$$

$$x''^0 + x'^{0\,2}\beta = 0 \qquad\quad x'^i = dx^i/ds\,;\ \dot{x}^i = dx^i/dt = x'^i/x'^0. \quad \text{(III)}$$

The new curvature tensor is given by

$$\frac{1}{x'^{0\,2}}\bar{P}^i_j = P^i_j + \tfrac{1}{2}\delta^i_j\left\{ \mathbf{D}\beta - \tfrac{1}{2}\beta^2 \right\} - \dot{x}^i\left\{ \beta_{/j} - \tfrac{1}{2}\mathbf{D}\beta_{;j} - \tfrac{1}{4}\beta\beta_{;j} \right\};$$

$$\frac{1}{x'^{0\,2}}\bar{P}^0_j = -\beta_{/j} + \tfrac{1}{2}\mathbf{D}\beta_{;j} + \tfrac{1}{4}\beta\beta_{;j}\ ;\ \ \bar{P}^0_0 = -\bar{P}^0_r\dot{x}^r;$$

$$\bar{P}^i_0 = \dot{x}^r\bar{P}^i_r\,, \tag{5.1}$$

where the summations for repeated indices are from 1 to n. For the new space to be quasi-isotropic, $\bar{P}^0{}_j = -x'^0\bar{q}_j$ which means, from (5.1), that $q_j = 0$, in addition, of course, to quasi-isotropy for the original space.

THEOREM IV. *A path-space may be immersed in a quasi-isotropic space with* $\bar{\varepsilon}^i = \partial\bar{\alpha}^i/\partial s = 0$ *only if its original curvature tensor has the form* $P^i_j = \phi\delta^i_j$.

If the original space had a metric, the new metric (of degree 2) will be $f^2 x'^{0\,2}$, and the function of immersion[3] is given by $\beta = \mathbf{D}f/f$. The condition for quasi-isotropy of the $n+1$ space can now be expressed according to (4.1) and (5.1) as

$$\left\{ \tfrac{1}{2}\mathbf{D}\beta - \tfrac{1}{4}\beta^2 + \phi \right\} f_{;j} - f\left\{ \beta_{/j} - \tfrac{1}{2}\mathbf{D}\beta_{;j} - \tfrac{1}{4}\beta\beta_{;j} \right\} = 0;\ \beta = \frac{\mathbf{D}f}{f}. \tag{5.2}$$

For full isotropy according to Theorem III, these can be further reduced in an obvious manner to give

$$\left\{\tfrac{1}{2}\mathbf{D}\beta-\tfrac{1}{4}\beta^2+\phi\right\}_{;j}=2\left\{\beta_{1j}-\tfrac{1}{2}\mathbf{D}\beta_{;j}-\tfrac{1}{4}\beta\beta_{;j}\right\};\ cf^2=\tfrac{1}{2}\mathbf{D}\beta-\tfrac{1}{4}\beta^2+\phi,$$
$$\beta=\mathbf{D}f/f. \tag{5.3}$$

The interest of these results lies in their applicability to the relationship between a space and a space-time, as required by modern physical theories. For example, an M_3 occurs in Milne's cosmology, defined after taking the velocity of light as unity by[4]:

$$p^i=x^i,\qquad g_{ii}=1,\ g_{ij}=0,\ i\neq j,\qquad Y=1-g_{ij}x^ix^j,$$
$$X=t^2-g_{ij}p^ip^j,\qquad Z=t-g_{ij}p^i x^j,\qquad \xi=Z^2/XY,$$
$$\alpha^i=(t\dot{x}^i-p^i)\frac{Y}{X}G(\xi). \tag{5.4}$$

$$X^2P^i_j=\delta^i_j\left\{\tfrac{3}{2}XYG-\tfrac{3}{4}t^2Y^2G^2+tYZ[-G+G^2+(1-\xi)(1+G)G']\right\}$$
$$-\Gamma(p^i-t\dot{x}^i)(Yp_j-Z\dot{x}_j)$$
$$\Gamma=-G(2+G)+(2\xi-1)(1+G)G'+\xi(\xi-1)\left\{2(1+G)G''-G'^2\right\}. \tag{5.5}$$

As the "fundamental quantities" X, Y, ξ, are not to be restricted, the condition for immersibility into a quasi-isotropic space is the same as that for quasi-isotropy of the M_3 itself, i. e. $\Gamma=0$.

The space into which this can be immersed is[4] the K_4 defined by

$$g_{00}=-g_{11}=-g_{22}=-g_{33}=1,\ g_{ij}=0,\ i\neq j,\ p_j=g_{jr}p^r\text{ etc., } p^i=x^i$$
$$\mathbf{X}=p_rp^r\qquad \mathbf{Y}=\dot{x}^r\dot{x}_r\qquad \mathbf{Z}=\dot{x}^rp_r\qquad \xi=\mathbf{Z}^2/\mathbf{XY}$$
$$\alpha^i=-\frac{p^i\mathbf{Y}}{\mathbf{X}}G(\xi)+\frac{2\dot{x}^i\mathbf{Z}}{\mathbf{X}}\gamma(\mathbf{X},\xi). \tag{5.6}$$

$$\mathbf{X}^2P^i_j=A\mathbf{XY}\delta^i_j+\dot{x}^i\left\{(C\xi-A)\mathbf{X}x_j-C\mathbf{Z}p_j\right\}-p^i\Gamma\left\{\mathbf{Z}\dot{x}_j-\mathbf{Y}p_j\right\}$$
$$A=G\left\{1+\gamma+2\gamma_2\xi(1-\xi)\right\}+\gamma+2\xi(\mathbf{X}\gamma_1-\gamma)+2\gamma_2\xi(1-)\xi-\xi\gamma^2$$
$$C=G'\left\{1+\gamma+2\gamma_2\xi(1-\xi)\right\}+2(\mathbf{X}\gamma_1-\gamma)-\gamma^2-6\gamma_2(1-\xi+G)$$
$$-2\gamma\gamma_2\xi-4\gamma_{12}\mathbf{X}\xi+8G\gamma_2\xi-4\gamma_{22}\xi(1-\xi)(1+G)$$
$$\Gamma\text{ as in (5.5)};\quad \gamma_1=\partial\gamma/\partial\mathbf{X},\quad \gamma_2=\partial\gamma/\partial\xi\text{ etc.} \tag{5.7}$$

The condition for quasi-isotropy of K_4 is again $\Gamma=0$. The differential equation $\Gamma=0$ has only two constant soutions for G, i.e. $G=0$, $G=-2$. For both of these, a metric exists in terms of the fundamental quantities: $\gamma=G=0$, $f=\mathbf{Y}$, $\gamma=G=-2$, $f=\mathbf{Y}/\mathbf{X}^2$. Both of these are actually flat spaces, which is obvious for the first from the definition of K_4 and is seen for the second by choosing the new coordinate system: $\bar{x}^i = x^i/\mathbf{X}$, which gives a Galilean metric again for the space, though the transformation is singular for the origin and for all light-tracks from it.

The condition for the existence of a metric homogeneous of degree zero for the K_4 has been given elsewhere[4] as

$$1+(1-\xi)(G-\xi G')+\xi\gamma=0. \tag{5.8}$$

In this case, the findings of (4.1) are confirmed by the fact that the coefficient of the δ^i_j term in P^i_j above becomes $\mathbf{XY}(\xi-1)\Gamma$, which vanishes for isotropy.

6. Our definition of isotropy allows one further analogue of the Riemannian case. For $n=2$, with homogeneous non-parametric α^i, and a metric homogeneous of degree zero or two we find the following results identically true:

$$P^r_1 f_{;r}=0,\ P^r_2 f_{;r}=0;\ P^1_r \dot{x}^r=0,\ P^2_r \dot{x}^r=0,\ \tfrac{1}{2} f_{;r}\dot{x}^r=kf\ (k=0, \text{ or } 1). \tag{6.1}$$

But these can be solved by elementary algebra, and lead at once to the result:

$P^i_j=\lambda(kf\delta^i_j-\frac{1}{2}\dot{x}^i f_{;j})$ identically, for $n=2$, and a homogeneous metric. (6.2)

Hence

If $n=2$ and a homogeneous metric exists, the "surface" is always identically quasi-isotropic.

In conclusion, I gratefully acknowledge the valuable aid given by Mr. V. Seetharaman in checking this note, particularly by his independent and direct verification of (3.2), (3.3), (5.5) and (5.7).

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