THE TENSOR ANALYSIS OF PARTIAL DIFFERENTIAL EQUATIONS*

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After the comprehensive works of Bortolotti(1) on partial differential equations of the second order from the differential geometer's point of view, and the equally comprehensive memoir of Kawaguchi and Hombu(2) on systems of higher order, the present note serves only to show that slightly different results can be obtained by keeping to the point of view that I have used in my former papers(3). The method, in particular, is to handle such systems as obey the following postulates and for the special transformation groups under which the postulates hold: (1) the system of equations transforms according to the tensor law; (2) the equations of variation of the given system are also tensorial when the variation itself is a vector; (3) there exists at least one operator which is vectorial in character and corresponds to total differentiation with respect to one of the independent variables.

To illustrate this, let us consider the second order system

$$\frac{\partial^2 x^i}{\partial u^\alpha \partial u^\beta} + H^i_{\alpha\beta}(u, x, p^r_\nu) = 0; \; p^i_\alpha = \frac{\partial x^i}{\partial u^\alpha}. \tag{1}$$

Here, the Latin indices refer to the coordinates x, and range over values $1, \ldots, n$; the Greek indices to the parameters u, and have the values $1, \ldots, m$. The functions $H^i_{\alpha\beta}$ must have the transformation law

$$-H'^i_{\alpha\beta} = -H^r_{\nu\delta} \frac{\partial x'^i}{\partial x^r} \frac{\partial u^\nu}{\partial u'^\alpha} \frac{\partial u^\delta}{\partial u'^\beta} + \frac{\partial^2 x'^i}{\partial x^r \partial x^s} p^r_\delta p^s_\nu \frac{\partial u^\delta}{\partial u'^\alpha} \frac{\partial u^\nu}{\partial u'^\beta} + \frac{\partial^2 u^\nu}{\partial u'^\alpha \partial u'^\beta} \frac{\partial x'^i}{\partial x^r} p^r_\nu. \tag{2}$$

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under the group

$$x'^i = F^i(x^1, \ldots, x^n)\,;\; u'^\alpha = \phi^\alpha(u^1, \ldots, u^m). \qquad (3)$$

But we can speak of x-transformations or u-transformations alone and of x-tensors or u-tensors accordingly. *Tensor* will mean, unless specialised, a geometric object which has the proper law of transformation for both sorts of indices. It is assumed that the conditions of integrability of the partial differential equations are identically satisfied, but no direct use will be made of them. We introduce the non-tensorial operator of differentiation with respect to u, viz.

$$\partial_a \equiv \frac{\partial}{\partial u^a} + p^i_a \frac{\partial}{\partial x^i} - H^i_{\alpha\beta} \frac{\partial}{\partial p^i_\beta}. \qquad (4)$$

It follows that for an x-vector λ^i, a vectorial operator must be of the type $D_\alpha \lambda^i \equiv \partial_\alpha \lambda^i + \gamma^i_{\alpha r}\lambda^r$. But this will not do for any tensor with Greek indices. Therefore, the $\gamma^i_{\alpha j}$ must behave like covariant u-vectors, and an additional term will have to enter the D_α. We may, therefore, take the general operator to be of the form

$$\left.\begin{aligned} D_\alpha T^{\cdot\,\cdot\,\cdot}_{\cdot\,\cdot\,\cdot} = \partial_\alpha T^{\cdot\,\cdot\,\cdot}_{\cdot\,\cdot\,\cdot} + \gamma^i_{\alpha r} T^{r\,\cdot\,\cdot}_{\cdot\,\cdot\,\cdot} - \gamma^r_{\alpha j} T^{\cdot\,\cdot\,\cdot}_{r\,\cdot\,\cdot} \\ + \Gamma^\nu_{\alpha\rho} T^{\cdot\,\cdot\,\rho}_{\cdot\,\cdot\,\cdot} - \Gamma^\rho_{\alpha\sigma} T^{\cdot\,\cdot\,\cdot}_{\cdot\,\cdot\,\rho} \end{aligned}\right\}. \qquad (5)$$

The laws of transformation for the two sets of coefficients must be as follows:

$$\gamma^s_{\sigma r} \frac{\partial x'^i}{\partial x^s} \frac{\partial u^\sigma}{\partial u'^\alpha} = \gamma'^i_{\alpha s} \frac{\partial x'^s}{\partial x^r} + p^s_\sigma \frac{\partial^2 x'^i}{\partial x^r \partial x^s} \frac{\partial u^\sigma}{\partial u'^\alpha}.$$

$$\Gamma'^\sigma_{\alpha\beta} \frac{\partial u^\nu}{\partial u'^\sigma} = \Gamma^\nu_{\sigma\tau} \frac{\partial u^\sigma}{\partial u'^\alpha} \frac{\partial u^\tau}{\partial u'^\beta} + \frac{\partial^2 u^\nu}{\partial u'^\alpha \partial u'^\beta}. \qquad (6)$$

Let the coordinates x^i now undergo a vector variation represented by $x^i = x^i + \varepsilon\lambda^i$. Neglecting the coefficients of ε^2 and higher powers as usual, we have the equations of variation

$$\partial_\alpha \partial_\beta \lambda^i + \partial_\nu \lambda^r \frac{\partial H^i_{\alpha\beta}}{\partial p^r_\nu} + \lambda^r \frac{\partial H^i_{\alpha\beta}}{\partial x^r} = 0. \qquad (7)$$

These can at once be put in the invariantive form

$$D_\alpha D_\beta \lambda^i + D_\nu \lambda^r T^{i\nu}_{r\alpha\beta} + \lambda^r P^i_{r\alpha\beta} = 0, \qquad (8)$$

where

$$T^{i\nu}_{j\alpha\beta} = \delta^i_j \Gamma^\nu_{\alpha\beta} - \gamma^i_{\alpha j}\delta^\nu_\beta - \gamma^i_{\beta j}\delta^\nu_\alpha + \frac{\partial H^i_{\alpha\beta}}{\partial p^j_\nu}\,; \qquad (8\text{-a})$$

$$P^i_{j\alpha\beta} = \frac{\partial H^i_{\alpha\beta}}{\partial x^j} - \frac{\partial H^i_{\alpha\beta}}{\partial p^r_\sigma} \gamma^r_{\sigma j} + \gamma^r_{\beta j}\gamma^i_{\alpha r} - \partial_\beta \gamma^i_{\alpha j}. \qquad (8\text{-b})$$

In my former work, the coefficients $\gamma^i_{\alpha j}$ were determined by taking the tensor corresponding to the first of these as equal to zero. This can no longer be done here, as it would be too restrictive, and would not even then serve to determine both $\gamma^i_{\alpha j}$ and $\Gamma^\nu_{\alpha\beta}$. $\Gamma^\nu_{\alpha\beta}$ need not be symmetrical in the subscripts, but the anti-symmetrical part (torsion) will be indeterminate; one may therefore assume $\Gamma^\nu_{\alpha\beta}$ to be symmetric. It is also clear that the derivatives $\frac{\partial^2 \gamma^i_{\alpha j}}{\partial p^k_\beta \partial p^l_\nu}$ and $\frac{\partial \Gamma^\nu_{\alpha\beta}}{\partial p^i_\sigma}$ are the components of tensors. The operator $\frac{\partial}{\partial p^i_\alpha}$ is also tensorial in character, adding a covariant x-component and a contravariant u-component.

The problem is now to determine the coefficients $\gamma^i_{\alpha j}$ and $\Gamma^\nu_{\alpha\beta}$ in a fashion which has some claim to be called intrinsic. Let us assume that all possible contractions of $T^{i\nu}_{j\alpha\beta}$ vanish. This gives

$$T^{r\nu}_{r\alpha\beta} \equiv n\Gamma^\nu_{\alpha\beta} - \delta^\nu_\beta \gamma^r_{\alpha r} - \delta^\nu_\alpha \gamma^r_{\beta r} + \frac{\partial H^r_{\alpha\beta}}{\partial p^r_\nu} = 0$$

$$T^{r\nu}_{r\alpha\nu} \equiv n\Gamma^\nu_{\alpha\nu} - (m+1)\gamma^r_{\alpha r} + \frac{\partial H^r_{\alpha\nu}}{\partial p^r_\nu} = 0, \text{ etc.} \tag{9}$$

The general solutions must then have the form:

$$\gamma^i_{\alpha j} = \frac{1}{m+1}\left[\Gamma^\sigma_{\alpha\sigma}\delta^i_j + \frac{\partial H^i_{\alpha\sigma}}{\partial p^j_\sigma}\right];$$

$$\Gamma^\nu_{\alpha\beta} = -\frac{1}{n}\frac{\partial H^r_{\alpha\beta}}{\partial p^r_\nu} + \delta^\nu_\beta \tau_\alpha + \delta^\nu_\alpha \tau_\beta. \tag{10}$$

The τ_α that enter into the second of these must have the law of transformation

$$\tau'_\alpha = \tau_\nu \frac{\partial u^\nu}{\partial u'^\alpha} - \frac{1}{n}\frac{\partial^2 x'^j}{\partial x^r \partial x^s}\frac{\partial x^r}{\partial x'^j} p^s_\nu \frac{\partial u^\nu}{\partial u'^\alpha}, \tag{11}$$

but are otherwise arbitrary so far as the present argument is concerned. This shows, in the first place, that *the logical types of connection for our systems are not affine but projective*; however we shall not pursue this further.

There exists another set of the equations of variation, namely those obtained by giving a vector variation to u. These are

$$p^i_\nu D_\alpha D_\beta \mu^\nu + D_\nu \mu^\sigma Q^{i\nu}_{\alpha\beta\sigma} + \mu^\sigma R^i_{\alpha\beta\sigma} = 0, \tag{12}$$

if $\bar{u}^\alpha = u^\alpha + \eta\mu^\alpha$, η being an infinitesimal,

where

$$Q^{i\nu}_{\alpha\beta\sigma} = \frac{\partial H^i_{\alpha\beta}}{\partial p^j_\nu} p^j_\sigma - H^i_{\alpha\sigma}\delta^\nu_\beta - H^i_{\beta\sigma}\delta^\nu_\alpha + \Gamma^\nu_{\alpha\beta} p^i_\sigma - p^i_\rho \Gamma^\rho_{\alpha\sigma}\delta^\nu_\beta - p^i_\rho \Gamma^\rho_{\beta\sigma}\delta^\nu_\alpha$$

$$R^i_{\alpha\beta\sigma} = p^i_\nu(\Gamma^\nu_{\delta\sigma}\Gamma^\delta_{\alpha\beta} - \Gamma^\nu_{\alpha\delta}\Gamma^\delta_{\beta\sigma} - \partial_\beta \Gamma^\nu_{\alpha\sigma}) - \frac{\partial H^i_{\alpha\beta}}{\partial u^\sigma} - \Gamma^\delta_{\nu\sigma} Q^{i\nu}_{\alpha\beta\delta}. \qquad (13)$$

The first of these might also be used to determine the connection $\Gamma^\nu_{\alpha\beta}$. But it is clear, from the manner in which p^i_ν enters, that this cannot be done without differentiation. In particular, inasmuch as $\frac{\partial \Gamma^\nu_{\alpha\beta}}{\partial p^j_\delta}$ is a tensor, we might set $\frac{\partial Q^{i\sigma}_{\alpha\beta\sigma}}{\partial p^i_\nu} + p^i_\sigma \frac{\partial \Gamma^\sigma_{\alpha\beta}}{\partial p^i_\nu} = 0$, which leads at once to

$$\Gamma^\nu_{\alpha\beta} = \frac{1}{n}\left[\frac{\partial^2 H^i_{\alpha\beta}}{\partial p^j_\sigma \partial p^i_\nu} p^j_\sigma - \frac{\partial H^i_{\alpha\beta}}{\partial p^i_\nu}\right]. \qquad (14)$$

This connection involves the *second* partial derivatives of $H^i_{\alpha\beta}$. With the first derivatives alone, we cannot go beyond (10). Since the difference of two sets of Γ's is a tensor, the differential invariants of the space may be calculated for any one connection, and are then obtained for any other by the use of this tensor-difference.

There is another operator besides D_ν and $\partial/\partial p^i_\alpha$, but it may be obtained by alternating the pair thus:

$$\left[\frac{\partial}{\partial p^j_\nu} D_\nu T^{i\alpha} - D_\nu \frac{\partial}{\partial p^j_\nu} T^{i\alpha}\right] = m\frac{\partial T^{i\alpha}}{\partial x^j} + \frac{\partial T^{i\alpha}}{\partial p^r_\beta}\left[\gamma^r_{\beta j} - \Gamma^\nu_{\nu\beta}\delta^r_j - \frac{\partial H^r_{\nu\beta}}{\partial p^j_\nu}\right] + T^{r\alpha}\frac{\partial \gamma^i_{\nu r}}{\partial p^j_\nu} + T^{i\beta}\frac{\partial \Gamma^\alpha_{\nu\beta}}{\partial p^j_\nu}. \qquad (15)$$

Discarding the additive tensorial terms and making use of (10), we get the simplified operator

$$\underset{j}{\nabla} T^{i\alpha} = \frac{\partial T^{i\alpha}}{\partial x^j} - \gamma^r_{\beta j}\frac{\partial T^{i\alpha}}{\partial p^r_\beta} + \frac{1}{m} T^{r\alpha}\frac{\partial \gamma^i_{\nu r}}{\partial p^j_\nu}. \qquad (16)$$

The differential invariants of the space are to be obtained by alternating the three operators given, as usual.

The usefulness of the foregoing discussion lies in its adaptability to the case of differential equations of higher order. Let, for instance, such a system be given by

$$\frac{\partial^{q+1} x^i}{\partial u^{\alpha_1}\partial u^{\alpha_2}\ldots\partial u^{\alpha_{q+1}}} + H^i_{\alpha_1\alpha_2\ldots\alpha_{q+1}}(u, x, p^i_\alpha, \ldots, p^i_{\alpha_1\ldots\alpha_q}) = 0. \quad (17)$$

The operator D_ν will be of the same type as before. The connection coefficients can again be determined from the two sets of equations of variation, in particular, by contraction of the corresponding coefficients of the varied equations. The remaining coefficients of the equations of variation give the "primary" differential invariants of the system. A new differential operator is obtained by alternating and contracting $\partial/\partial p^i_{\alpha_1 \cdots \alpha_q}$ and D_ν. This will give an operator with one covariant Latin index and $q-1$ contravariant Greek indices, viz. $\nabla_i^{\alpha_1 \cdots \alpha_{q-1}}$. Alternation and contraction of this $\nabla_i^{\alpha_1 \cdots \alpha_{q-1}}$ with D_ν will again get rid of another Greek index, and give a second operator $\nabla_i^{\alpha_1 \cdots \alpha_{q-2}}$. This can be continued till no Greek indices are left, and we obtain the operator which corresponds to the purely Latin index covariant derivative ∇_i. Further alternations will give only differential invariants. At each stage, additive tensorial terms can be discarded to obtain a reduced operator.

The complete set of differential invariants is not worked out here in view of the memoirs already cited. But it must consist, for the greater part, of those that enter into the equations of variation p^i_α, and such others as are to be obtained from these by the application of the tensorial operators.

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