DIFFERENTIAL GEOMETRY OF THE LAPLACE EQUATION

By D. D. KOSAMBI, Poona

[Received 15 July 1936]

Given any n—dimensional Euclidean space in general co-ordinates, with a metric $ds^2=g_{ij}dx^i dx^j$, Laplace's equation for the given system of co-ordinates becomes

$$\text{div. grad. } u \equiv g^{ij}u_{|i|j} \equiv g^{ij}u_{,i,j} - g^{ij}u_{,r}\Gamma_{ij}{}^{r}=0. \tag{1}$$

For a general Riemannian space with a non-vanishing curvature tensor, (1) may still be taken as the generalized Laplace equation. I propose to deal with the simple inverse problem:

Given a linear partial differential equation of the second order

$$a^{ij}u_{,i,j}+b^{i}u_{,i}=0; \tag{2}$$

under what conditions may it be regarded as the Laplace equation associated with some Riemann space?

In this connection, the term Laplace equation includes such types as the classical wave equation $\frac{\partial^2 u}{c^2\partial t^2}-\Delta_2 u=0$. The space here will obviously be that of special relativity:

$$ds^2=c^2dt^2-\Sigma(dx^i)^2. \tag{3}$$

It is clear that the equation (2) must be tensor invariant, and if associated with a Riemann space can differ at most from (1) by a factor λ. The space is, therefore, conformal to that with a fundamental tensor a_{ij} obtained from the equations $a^{ir}a_{rj}=\delta^i_j$, or, what is the same thing, the tensor obtained by dividing the cofactor of a^{ij} in $|a^{ij}|$ by the determinant itself. To this end, a first condition is that $|a^{ij}|\neq 0$. That a^{ij} must be a contravariant tensor is seen from the law of transformation of $u_{,i,j}$,

$$\bar{u}_{,i,j}=u_{,r\,s}\frac{\partial x^r}{\partial \bar{x}^i}\frac{\partial x^s}{\partial \bar{x}^j}+u_{,r}\frac{\partial^2 x^r}{\partial \bar{x}^i\partial \bar{x}^j}, \tag{4}$$

and from the tensor invariance postulated for (2). In fact, we must have

$$\bar{a}^{ij}=Wa^{rs}\frac{\partial\bar{x}^i}{\partial x^r}\frac{\partial\bar{x}^s}{\partial x^j}. \tag{5}$$

The weighting factor W may be assimilated to the transformed λ, or, assumed to be unity.

Let Γ^i_{jk} be the usual Christoffel symbols calculated for a tensor g_{ij} and $\left\{ {i \atop jk} \right\}$ the corresponding expressions for the tensor a_{ij}. If we put λa^{ij} for g^{ij} in (1), the equation (1) now has the form

$$\lambda a^{ij}u_{,i,j}-\lambda a^{jk}\left(\left\{ {i \atop jk} \right\}-\frac{a^{ir}}{2\lambda}[a_{jr}\lambda_{,k}+a_{kr}\lambda_{,j}-a_{jk}\lambda_{,r}]\right)u_{,i}=0, \tag{6}$$

since
$$g_{ij}=\frac{1}{\lambda}a_{ij}.$$

As we require this to be of type (2) but for the factor λ we have the condition:

$$b^i+a^{jk}\left\{ {i \atop jk} \right\}+\frac{n-2}{2\lambda}a^{ir}\lambda_{,r}=0. \tag{7}$$

Putting $\mu=\log\lambda$, and $\beta_i=a_{ir}\left(b^r+a^{jk}\left\{ {i \atop jk} \right\}\right)$ this is equivalent to

$$\frac{n-2}{2}\mu_{,i}+\beta_i=0. \tag{8}$$

Though indeterminate for $n=2$, this system of partial differential equations for the unknown μ has a solution provided the conditions $\beta_{i,j}-\beta_{j,i}=0$ are satisfied; the metric of the space is then of the form

$$ds^2=e^{-\mu}a_{ij}dx^idx^j.$$

All these results may be summed up in the form of

THEOREM I. *The equation $a^{ij}u_{,i,j}+b^iu_{,i}=0$ may be regarded as the Laplace equation for a Riemann space if and only if: (a) a^{ij} is a symmetric contravariant tensor with non-vanishing determinant, $|a^{ij}|\neq 0$, (b) $n=2$ and $\beta_i=0$, or (b′) $n>2$, $\beta_{i,j}-\beta_{j,i}=0$. Under these conditions, the space for $n=2$ is any conformal to that of the fundamental tensor a_{ij}; for $n>2$, the metric of the space is given by*

$$ds^2=e^{-\mu}a_{ij}dx^idx^j$$

and determined to a constant factor by the partial differential equation $\frac{n-2}{2}\mu_{,i}+\beta_i=0$.

It is clear that for $n=1$, the problem is trivial. The theory of functions of a complex variable and the fact that any surface with a regular positive-definite groundform may be represented conformally on the Euclidean plane lead us to expect the result given for $n=2$.

Equations of a more general type than (2) cannot be treated without recourse to some kind of a transformation. For instance, the type

$$a^{ij}u_{,i,j}+b^{i}u_{,i}+cu=R \tag{9}$$

must be reduced to the homogeneous form ($R=0$) by the usual methods, and the substitution $u=fv$ must be employed to get rid of the last term. For this, f must be a known solution of (9) with $R=0$, and we have a reduced form:

$$a^{ij}v_{,i,j}+\left(2a^{ij}\frac{f_{,j}}{f}+b^{i}\right)v_{,i}=0. \tag{10}$$

This is much the same equation as (2), but with $a^{ij}\phi_{,j}$ added to b^{i}, or, a term $\phi_{,i}$ added to β_i, where $\phi=2\log f$ and f is any particular non-trivial solution of (9) with $R=0$. This allows us to generalize the theorem stated, as (2) is a special case of (9); we obtain a relaxation of condition (b) for $n=2$, and also a new space for every solution of the given equation, though a space conformal to that of the tensor a_{ij}. We sum these up in

THEOREM II. *The conditions and conclusions of Theorem I are applicable to the equation*

$$a^{ij}u_{,i,j}+b^{i}u_{,i}+cu=0$$

when the transformation $v=fu$ *is allowed; but condition* (b) *becomes:*

$$n=2,\ \beta_{i,j}-\beta_{j,i}=0;\ a^{ij}(\beta_i\beta_j-2\beta_{i,j})-2b^{i}\beta_i+4c=0.$$

And for $n>2$, *the metric is given by*

$$ds^2=f^{4/(n-2)}e^{-\mu}a_{ij}dx^{i}dx^{j},$$

where f *is a solution of the given equation,* μ *being, as before, a solution if any, of*

$$\frac{n-2}{2}\mu_{,i}+\beta_i=0.$$

This says nothing about equations of a more general type, and transformations which are less simple. It would seem, however, that the entire problem is better approached from a different point of view. One should discuss necessary and sufficient conditions for the given equations to be deducible from a variational principle (self-adjoincy) and then see whether some sort of geometry may be associated with the integrand of the variational principle so obtained.