Differential Geometry of the Laplace Equation

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Given any $n$-dimensional Euclidean space in general co-ordinates, with a metric $ds^2 = g_{ij}dx^i dx^j$, Laplace’s equation for the given system of co-ordinates becomes

\[
\text{div. grad. } u \equiv g^{ij}u_{,1j} = g^{ij}u_{,ij} - g^{ij}u_{,r}\Gamma_{ij}^r = 0.
\]  

(1)

For a general Riemannian space with a non-vanishing curvature tensor, (1) may still be taken as the generalized Laplace equation. I propose to deal with the simple inverse problem:

*Given a linear partial differential equation of the second order

\[
a^{ij}u_{,ij} + b^i u_{,i} = 0;
\]  

(2)

under what conditions may it be regarded as the Laplace equation associated with some Riemann space?*

In this connection, the term Laplace equation includes such types as the classical wave equation $\frac{\partial^2 u}{c^2 \partial t^2} - \Delta_2 u = 0$. The space here will obviously be that of special relativity:

\[
ds^2 = c^2 dt^2 - \Sigma (dx^i)^2.
\]

(3)

It is clear that the equation (2) must be tensor invariant, and if associated with a Riemann space can differ at most from (1) by a factor $\lambda$. The space is, therefore, conformal to that with a fundamental tensor $a_{ij}$ obtained from the equations $a^{ir}_{,rj} = \delta_j^i$, or, what is the same thing, the tensor obtained by dividing the cofactor of $a^{ij}$ in $|a^{ij}|$ by the determinant itself. To this end, a first condition is that $|a^{ij}| \neq 0$. That $a^{ij}$ must be a contravariant tensor is seen from the law of transformation of $u_{,ij}$,

\[
-u_{,i,j} = u_{,r,s} \frac{\partial x^r}{\partial x^i} \frac{\partial x^s}{\partial x^j} + u_{,r} \frac{\partial^2 x^r}{\partial x^i \partial x^j},
\]

(4)

and from the tensor invariance postulated for (2). In fact, we must have
\[ a_{ij} = W a_{rs} \frac{\partial x^i}{\partial x^r} \frac{\partial x^s}{\partial x^j}. \]  

(5)

The weighting factor \( W \) may be assimilated to the transformed \( \lambda \), or, assumed to be unity.

Let \( \Gamma^i_{jk} \) be the usual Christoffel symbols calculated for a tensor \( g_{ij} \) and \( \{i, j \} \) the corresponding expressions for the tensor \( a_{ij} \). If we put \( \lambda a_{ij} \) for \( g_{ij} \) in (1), the equation (1) now has the form

\[ \lambda a_{ij} u_{i,j} - \lambda a_{ik} \left( \{ i, j \} - \frac{a_{ir}}{2\lambda} \left[ a_{jr} \lambda_{,k} + a_{kr} \lambda_{,j} - a_{jr} \lambda_{,k} \right] \right) u_{i,=} = 0, \]  

(6)

since

\[ g_{ij} = \frac{1}{\lambda} a_{ij}. \]

As we require this to be of type (2) but for the factor \( \lambda \) we have the condition:

\[ b^i + a^{ik} \{ i, k \} + \frac{n-2}{2\lambda} a^{ir} \lambda_{,r} = 0. \]  

(7)

Putting \( \mu = \log \lambda \), and \( \beta_i = a_{ir} \left( b^r + a^{rk} \{ i, k \} \right) \) this is equivalent to

\[ \frac{n-2}{2} \mu, i + \beta_i = 0. \]  

(8)

Though indeterminate for \( n=2 \), this system of partial differential equations for the unknown \( \mu \) has a solution provided the conditions \( \beta_{i,j} - \beta_{i,;} = 0 \) are satisfied; the metric of the space is then of the form

\[ ds^2 = e^{-\mu} a_{ij} dx^i dx^j. \]

All these results may be summed up in the form of

**Theorem I.** The equation \( a_{ij} u_{i,j} + b^i u_{i,=} = 0 \) may be regarded as the Laplace equation for a Riemann space if and only if: (a) \( a_{ij} \) is a symmetric contravariant tensor with non-vanishing determinant, \( |a_{ij}| \neq 0 \), (b) \( n=2 \) and \( \beta_i = 0 \), or (b') \( n>2 \), \( \beta_{i,j} - \beta_{i,;} = 0 \). Under these conditions, the space for \( n=2 \) is any conformal to that of the fundamental tensor \( a_{ij} \); for \( n>2 \), the metric of the space is given by

\[ ds^2 = e^{-\mu} a_{ij} dx^i dx^j \]

and determined to a constant factor by the partial differential equation \( \frac{n-2}{2} \mu, i + \beta_i = 0 \).

It is clear that for \( n=1 \), the problem is trivial. The theory of functions of a complex variable and the fact that any surface with
a regular positive-definite groundform may be represented con-
formally on the Euclidean plane lead us to expect the result given
for \( n=2 \).

Equations of a more general type than (2) cannot be treated
without recourse to some kind of a transformation. For instance,
the type

\[ a^{ij}u_{,i,j} + b^i u_{,i} + cu = R \]  

must be reduced to the homogeneous form \((R=0)\) by the usual
methods, and the substitution \( u=fv \) must be employed to get rid
of the last term. For this, \( f \) must be a known solution of (9) with
\( R=0 \), and we have a reduced form:

\[ a^{ij}v_{,i,j} + (2a^{ij} f_j + b^i)v_{,i} = 0. \]

This is much the same equation as (2), but with \( a^{ij} \phi_{,i} \) added to
\( b^i \), or, a term \( \psi_i \) added to \( \beta_i \), where \( \psi = 2 \log f \) and \( f \) is any parti-
cular non-trivial solution of (9) with \( R=0 \). This allows us to
generalize the theorem stated, as (2) is a special case of (9); we
obtain a relaxation of condition (b) for \( n=2 \), and also a new
space for every solution of the given equation, though a space
conformal to that of the tensor \( a_{ij} \). We sum these up in

**Theorem II.** The conditions and conclusions of Theorem I
are applicable to the equation

\[ a^{ij}u_{,i,j} + b^i u_{,i} + cu = 0 \]

when the transformation \( v=fu \) is allowed; but condition (b)
becomes:

\( n=2 \), \( \beta_{,i} - \beta_{,i} = 0 \); \( a^{ij}(\beta_{i} \beta_{,j} - 2\beta_{,i,j} - 2b^{j} \beta_{i} + 4c = 0. \)

And for \( n>2 \), the metric is given by

\[ ds^2 = f^{4/(n-2)} e^{-\nu} a_{ij} dx^i dx^j, \]

where \( f \) is a solution of the given equation, \( \nu \) being, as before, a
solution if any, of

\[ \frac{n-2}{2} \mu_{,i} + \beta_{i} = 0. \]

This says nothing about equations of a more general type, and
transformations which are less simple. It would seem, however,
that the entire problem is better approached from a different point
of view. One should discuss necessary and sufficient conditions
for the given equations to be deducible from a variational principle
(self-adjoincy) and then see whether some sort of geometry may
be associated with the integrand of the variational principle so
obtained,