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THE MAXIMUM MODULUS THEOREM

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A fundamental result in the theory of functions of a complex variable is represented by the theorem :

If $f(z)$ be analytic in a closed region R, then max $|f(z)|$ is assumed on the boundary of R.

As corollaries, by considering $1/f(z)$, we see that $|f(z)|$ cannot have a non-zero minimum in the interior of R ; an application of the theorem to $\exp f(z)$ shows that no harmonic function can have a maximum or a minimum within its region of definition.

The reach and bearing of the theorem has perhaps disguised the fact that it can easily be extended to non-analytic functions, *i.e.*, to transformations of the plane which are one to one and continuous, but need not be conformal. For instance, we may state a more general form :

THEOREM 1.

Hypothesis: $u(x,y)$, $v(x,y)$ are real functions of the real variables x,y, with continuous first partial derivatives and a non-vanishing Jacobian $J = \partial(u,v)/\partial(x,y)$ in some closed region R.

Conclusion: u^2+v^2 must assume its maximum value on the boundary and not within the interior of R.

Proof : There *is* a maximum value of $u^2 + v^2$, which, by the hypothesis, is a continuous function of the two variables in a closed region. This value cannot be taken on at an isolated interior point, for then, at that point, we should have

$$\frac{\partial}{\partial x}(u^2+v^2) \equiv 2u\frac{\partial u}{\partial x} + 2v\frac{\partial v}{\partial y} = 0$$

$$\frac{\partial}{\partial y}(u^2+v^2) \equiv 2u\frac{\partial u}{\partial y} + 2v\frac{\partial v}{\partial y} = 0$$

This gives $J=0$, contradictory to the assumption, or $u=v=0$ at the point. But the latter cannot represent a maximum value unless u, v, are identically null, which would again give $J=0$.

If the maximum be assumed along a curve (or for that matter a dense set of points), say the curve C, then $|\partial(u^2+v^2)/\partial s|_c=0$, and as the value is a maximum, we should also have the directional derivative along any curve cutting C vanish at the point of intersection.

But if these two derivatives vanish, then it is clear that the partial derivatives with respect to x, y, vanish all along the curve, which leads to the same contradiction as above. Therefore, the maximum is not only assumed on the boundary, but actually greater than any interior value.

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The same reasoning shows that u^2+v^2 cannot have a non-zero minimum in the interior of R. Furthermore, neither u nor v can have a maximum or a minimum within R. A general theorem covers all of these cases :

THEOREM 2.

Under the assumptions of theorem 1, no function $\varphi(u,v)$ with continuous first partial derivatives can have a maximum or a minimum at an interior point of R unless $\partial\varphi/\partial u = \partial\varphi/\partial v = 0$ at that point.

The proof is as indicated before.

Whereas the transformations considered are not conformal, and hence, do not correspond to the more restricted class of functions satisfying the Cauchy-Riemann differential equations, it is clear nevertheless that they are not so general as might be wished, being in fact *schlicht* in R, due to the non-vanishing Jacobian. For the general case, exception would have to be made of points where the inverse transformation failed because of the vanishing of J.