COLLINEATIONS IN PATH-SPACE

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A geometry attached to systems of second order differential equations of the generic type

$$(1) \quad \ddot{x}^i + a^i(x, \dot{x}, t) = 0 \quad \dot{x}^i = \frac{dx^i}{dt} \text{ etc.} \quad (i = 1 \ldots n)$$

has been discussed elsewhere[1]. Curves representing solutions of (1) can be regarded as the generalized autoparallel lines or *paths* of a space, and the intrinsic differential geometry thereof is developed from two main assumptions: (*a*) the tensor invariance of all fundamental equations, including (1), and (*b*) the existence of a vectorial operator, the vanishing of which defines a parallelism making solutions of (1) autoparallel lines.

I here attempt to investigate a special type of path-space which allows continuous groups of deformations carrying paths into paths.

Let $u^i(x)$ be a vector field representing an infinitesimal transformation of such a group by means of the "small displacement"

$$x^i = x + u^i \, \delta\xi$$

Then the functions u^i must satisfy the equations of variation of (1):

$$(2) \quad \ddot{u}^i + a^i_{;r} \, u^r + a^i_{,r} \, u^r = 0$$

As usual, a repeated index denotes summation; moreover, $f_{;k} = \frac{df}{d\dot{x}^k}$ and $f_{,k} = \frac{df}{dx^k}$. Inasmuch as the operator for total differentiation with respect to t is

$$\frac{d}{dt} = -a^r \frac{d}{d\dot{x}^r} + \dot{x}^r \frac{d}{dx^r} + \frac{d}{dt}$$

we find that (2) reduce to

$$(2') \quad u^i_{,m,r} \dot{x}^m \dot{x}^r - a^r \, u^i_{,r} + a^i_{;j} \, u^j_{,r} \, \dot{x}^r + a^i_{,j} \, u^j = 0.$$

Let it be further assumed that a^i has the form of a polynomial in $\dot{x}$:

$$(3) \quad a^i = A^i + A^i_h \dot{x}^h + \Gamma^i_{h_r} \dot{x}^h \dot{x}^r + \ldots + A^i_{h_1 \ldots h_m} \dot{x}^{h_1} \ldots \dot{x}^{h_m} + \ldots$$

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(1) D. D. Kosambi *Rendiconti della Reale Accademia dei Lincei* pp.

D. D. Kosambi *Math. Zeitschrift*, *Bd.* 37, (1933), pp. 608, 618.

The coefficients A, Γ are functions of x alone, symmetric in all subscripts; the letter Γ has been used for the quadratic terms only, for reasons that will be apparent later.

In the previous papers referred to, as well as in a remarkable exposition by M. Cartan[2], it was shown that

$$a^i - \tfrac{1}{2}\dot{x}^r a^i{}_{;r} \quad \text{as also} \quad a^i{}_{;l;m;n}$$

and their further partial derivatives with respect to $\dot{x}$ are tensors of the rank expressed by the indices. It follows, since (1) are tensor invariant, that:

In a polynomial a^i of the form (3), *the terms of any degree except two have tensor co-efficients* ($A^i \ldots$). *The coefficients of the second degree terms* ($\Gamma^i{}_{jk}$) *have the same laws of transformation as those of a symmetric affine connection.*

We can, therefore, obtain a covariant differentiation with respect to the Γ's alone, by the usual rules:

$$(4) \qquad \lambda^i{}_{|h} = \lambda_{,h} + \lambda^r \Gamma^i{}_{hr}$$

and so on for tensors of any rank.

The equations (2′) also represent polynomials in $\dot{x}$, which must vanish identically, as our infinitesimal transformations form vector fields independent of the particular paths chosen. We thus obtain, from terms not of the second degree in $\dot{x}$,

$$(5) \quad u^m{}_{|r}[A^i{}_{mh_2..h_j}\delta^r{}_{h_1} + A^i{}_{h_1m..h_j}\delta^r{}_{h_2} + \cdot + A^i{}_{h_1h_2..m}\delta^r{}_{h_j} - \delta_{im} A^r{}_{h_1h_2..h_j}]$$
$$+ u^m A^i{}_{h_1h_2..h_j/m} = 0.$$

where the vertical bar before a subscript denotes covariant differentiation with respect to $\Gamma^i{}_{jk}$ as defined in (4); $\delta^i{}_j$ are the usual Kronecker symbols, zero or unity in value as the two indices are different or coincident. The second degree terms, however, give:

$$(6') \quad u^i{}_{|j|n} + u^i{}_{|k|j} = u^l[R^i{}_{jkl} + R^i{}_{kjl}].$$

$R^i{}_{jkl}$ being the curvature tensor for the Γ's. But with the following identities:—

$$R^i{}_{jkl} + R^i{}_{jlk} = 0$$
$$(7) \qquad R^i{}_{jkl} + R^i{}_{klj} + R^i{}_{ljk} = 0$$
$$u^i{}_{|j|k} - u^i{}_{|k|j} = -R^i{}_{hjk}u^h$$

we can reduce this to the normal form

$$(6) \qquad u^i{}_{|j|k} = R^i{}_{jkl}u^l.$$

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(2) E. Cartan *Math. Zeitschrift, Ibid.*, pp. 619, 622.

We have thus broken up the equations of variation into one system of partial differential equations of the second order, and several of the first order, all being tensorial in form.

The problem of determining whether any solutions of (5) and (6) exist is reducible to one of algebra[3], though not explicitly soluble as a rule. The general solution, if any exist, can be expressed in terms of p independent fundamental solutions ($p \leqq n^2+n$) as a linear combination of these with constant coefficients. But (6) has a further very important property, easily proved by means of its compatibility conditions and the identities (7). That is, if u^i, v^i be any two distinct solutions, the alternant or Poisson bracket

$$(u, v)^i \equiv u^r v^i_{,r} - v^r u^i_{,r} \equiv u^r v^i_{\ |r} - v^r u^i_{\ |r}$$

is also a solution of (6). Thus our independent infinitesimal transformations generate a group. It does not by any means follow that the common solutions of [5] and [6] generate a Lie group. This is the case, however, when none or only one such common solution exists, apart from this trivial case, the most general conditions can again be reduced to a problem of algebra, and in fact to the discussion of the independence of a series of linear or bilinear forms. One might consider the possibility of [5] being a consequence of the compatibility conditions of [6], or, of the equations [5] themselves possessing the group property. In general, it would not seem that such multiparameter groups exist when the a^i contain terms of degree higher than two in $\dot{x}$. The main point is that there exists a covariant derivation as for the affine onnections, and that the general operation of the *derivate* or *biderivate* which I have elsewhere defined, can be replaced by a known and familiar type. The analytic connections are not more general than those of the form $a^i = \Gamma^i_{jk} \dot{x}^j \dot{x}^k + E^i_j \dot{x}^j + \omega^i$ which, by the way, are the only ones that are analytic in the space of $n+1$ dimensions wherein t is taken as one of the x's.

The same discussion for the most general form of a^i has no meaning, but is easily extensible to a^i that are analytic in $\dot{x}$ and sufficiently differentiable in x to allow a discussion of compatibility conditions. Even more, convergence of the infinite series can be ignored if merely an expansion of the prescribed form exists. Formally, each power of x in the expansion yields just one equation, independent of all other terms except those of the second degree. Apart from the question of solving an infinite set of differential equations (present also in the analytic case) the only difficulty possible would be that of the absence of uniqueness of

(3) L. P. Eisenhart *Non-Riemannian Geometry* (1927), pp. 126, 132.

expansion. But in this last case, if it can occur at all, we may regard the various forms as given by the use of different ways of describing the same space; or as different spaces that are feasible for the same paths. Similarly, asymmetric components, corresponding to the torsion tensor and the like can be introduced in the various coefficients, though they will not appear in the actual equations (1) or (3).

The question of collineations (path-preserving continuous groups of transformations) in path-spaces for which the a^i *possess a formal expansion by polynomials homogeneous in* $\dot{x}$, *can be dealt with by methods similar to those used for manifolds with asymmetric affine connection. The particular connection, moreover, is represented by the coefficients of the quadratic terms in the expansion.*

References.

(1) D. D. Kosambi. Math Zeitschrift Bd. 37 (1933), pp. 608. 618.
(2) E. Cartan. Ibid, pp. 619, 622.
(3) L. P. Eisenhart. Non-Riemannian Geometry (1927), pp. 126, 132.