J. Univ. Bombay 2, 18-20, 1933

THE CLASSIFICATION OF INTEGERS

By D. D. KOSAMBI

Fergusson College, Poona.

In dealing with number-theoretic operations, we find certain operators extensible to continuous variables, and some for which no immediate extension is to be seen from natural numbers to the continuum. Here, I consider an operator of the latter type, and show its uses in classifying the entire scheme of positive integers. This note also propounds a problem, or rather many related problems without giving explicit solutions. In some cases, a good deal has been discovered about the answers, whereas the expert technician will have no difficulty in recognizing that the rest are practically insoluble in our present state of knowledge regarding the theory of numbers.

1. For any positive integer n, we define the operator:

$K[n]$ = The sum off all factors of n including unity, but excluding n itself.

Thus: $K[1] = 0 \quad K[2] = K[3] = K[p] = 1$ $\quad p$ any prime.

$K[2^t] = 2^t - 1$ etc.

A first classification of the numbers can be effected into:

abundant numbers: $K[n] > n$

deficient numbers: $K[n] < n$

and the boundary class, usually included with the abundant, of the perfect numbers: $K[n] = n$. The problem of the distribution, or rather the proportion of these, has been attacked with some success, with the knowledge that the deficient numbers form slightly more than a half of the total. As the multiples of any abundant or perfect number are also abundant, a more essential problem is the number of *primitive* abundant numbers, one to which Dickson has made several most brilliant contributions.

It is seen that $K[n] = m$ regarded as an equation for n, given m, has not always a solution; as for example, $m = 2, 5, \ldots.$ Again, the solution whenever it exists, need not be unique.

2. The operator $K[n]$ may be repeated:

$K[K[n]] = K^2[n]$ and so on.

If $K^r[n] = 1$ then r is defined as the *class* of the integer n. The class of unity will be defined as zero from the outset, and it will then be the only number in its class. Whereas every integer has another derived from it by the application of the operator K, not every integer has a class. For instance:

The perfect numbers $K[n] = n$

The amicable numbers $K[n] = m \quad K[m] = n$

We shall call numbers of this type cyclic numbers, the cycle being the least value of s for which

$$K^s [n] = n$$

holds true. The members of the cycle are then given by

$$K[n] \qquad K^2[n] \qquad K^3[n] \ldots\ldots K^{s-1} [n]$$

I do not recall having seen any discussion of numbers of cycle greater than two.

There is still a further possibility of numbers without class: the numbers of class or cycle infinity. By this is meant numbers n, if any, such that for every a, another b exists to satisfy the inequality:

$$K^a [n] < K^b [n]$$

3. The problems arising out of this classificatiou may be viewed *en bloc*:

1. What is the nature and distribution of the integers m which cannot be derived by an operation, *i.e.*, for which there is no n satisfying

$$K[n] = m$$

2. When are the solutions of the above equation unique, and what is the maximum number of solutions possible for such an equation?

3. Are there numbers in every class, and how many? For instance, primes are integers of class one, and known to be infinite in number. Here, we query whether

$$K^r [n] = 1$$

has any solutions for a given r and the number of these.

4. Do there exist numbers of any preassigned cycle: are there solutions for every s of

$$K^s [n] = n$$

and how many sets?

The questions 3 and 4 concern the existence and number of solutions of simple operational equations; the general problem is too vague to be of interest at present.

5. Are there numbers without class or cycle?

6. Are there integers of mixed class and cycle? *i. e.*, integers which, after a certain number of operations, reduce to a cyclic number. [These can be shown to exist, as for example $K[25] = 6$, $K[6] = 6$. This also shows the non-uniqueness of the solutions of operational

equations]. Are there integers of the mixed type for every class and every possible cycle? And what is their distribution?

7. The integers being written down in their order, what is the distribution into classes and cycles? In the set

$$1, 2, 3, 4, 5, 6, \ldots\ldots\ldots, N.$$

how many classes and cycles are there represented, and what is the number of integers in each class and cycle represented, under the limit N? How many mixed integers are there under the limit?

It should be noted that this is a generalization of the classical problem regarding the distribution of primes. It may be hoped that the two supplement each other, and that some light will be thrown in the near future on the more general problem as related to the older one. As stated, this may be called the problem of the classification of integers.