# The Problem of Differential Invariants

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The classical problem of characterizing a surface regardless of the co-ordinate system used, was shown by Gauss to be that of determining its invariant curvature. For the general $n$ dimensional Riemannian space, the solution depends on the Riemann-Christoffel curvature tensor. In fact, we say after Lie and Ricci that all essential differential invariants of such a space are given by the curvature tensor and its successive covariant derivatives.

Further generalizations of the concept of space, such as those with an affine connection have an additional number of invariants, as the *torsion*, and other well-known tensors depending on the special type of connection or parallelism used. I have shown elsewhere * that a geometry can be associated with second order differential equations, the *paths* being integral curves of

$$\ddot{x}^i + \alpha^i(x, \dot{x}, t) = 0 \tag{1}$$

There are two procedures for obtaining differential invariants; the first (my own) is motivated, but incomplete; the second is more powerful, but has a slight disadvantage in needing a good many *a priori* assumptions.

With the tensor-invariance of (1) and their equations of variation

$$\dot{u}^i + \alpha^i_{\ ;r}\, \dot{u}^r + \alpha^i_{\ ,r}\, u^r = 0 \tag{2}$$

we deduce the existence of a vector differential operator, the *biderivate*;

$$\mathrm{D}\,(u)^i \equiv \dot{u}^i + \tfrac{1}{2}\, \alpha^i_{\ ;r}\, u^r$$

(1) can now be written as

$$\mathrm{D}\,(\dot{x})^i + (\alpha^i - \tfrac{1}{2}\, \alpha^i_{\ ;r}\, \dot{x}^r) = 0 \tag{3}$$

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* *Cf.* D. D. Kosambi: *Math. Zeitschrift* vol. 37 (1933) pp. 608-618.
E. Cartan: *Ibid.* pp. 619-622.

It is seen that

(4) $$\varepsilon^i = \alpha^i - \tfrac{1}{2}\,\alpha^i_{\ ;r}\,\dot{x}^r$$

is a vector, a first differential invariant of the system. The second can be obtained by reducing the equations of variation to the normal form:

(5) $$D^2(u)^i = \overset{i}{\underset{r}{P}}\,u^r$$

$$P^i_{\ j} = -\alpha^i_{\ ,j} + \tfrac{1}{2}\dot{x}^k\,\alpha^i_{\ ,k;j} + \tfrac{1}{2}\frac{\partial}{\partial t}\alpha^i_{\ ;j} - \tfrac{1}{2}\alpha^k\,\alpha^i_{\ ;j;k} + \tfrac{1}{4}\alpha^i_{\ ;k}\,\alpha^k_{\ ;j}$$

The mixed tensor $P^i$ corresponds to the Riemann-Christoffel tensor in this scheme. Our invariants are thus $\varepsilon^i$ and $P^i_{\ j}$ with the two differential processes

(6) $$\partial/\partial\dot{x}^r \qquad \text{and} \qquad D^i$$

For more general connections, we may use

(7) $$D(u)^i \equiv \dot{u}^i + \gamma^i_{\ r}\,u^r$$

where the $\gamma^i_{\ j}$ are only restricted to having the same law of transformation as $\tfrac{1}{2}\alpha^i_{\ ;j}$. The consequent scheme of invariants can be deduced from the preceding *intrinsic invariants*, with the one *invariant of the connection*:

(8) $$\sigma^i_{\ j} = \tfrac{1}{2}\,\alpha^i_{\ ;j} - \gamma^i$$

It is, however, not at all clear that all invariants have been so obtained. To settle this important point, we follow the procedure of Prof. Elie Cartan of Paris, the second procedure mentioned above. The space considered is now of $2n+1$ dimensions in $x$, $\dot{x}$, $t$, the last being an absolute time-like parameter. We take

(9) $$\omega^i_{\ d} \equiv dx^i - \dot{x}^i\,dt \neq 0$$

and ascribe to this Pfaffian, a vector character under all admissible transformations. A tensorial operator is then defined *a priori*,

(10) $$\mathfrak{D}\,(u)^i = du^i + \gamma^i_{\ r}\,u^r + \gamma^i_{\ kr}\,u^r\,\omega^k_{\ d}$$

The difference $\mathfrak{D}\omega^i_{\ \delta} - \Delta\,\omega^i_{\ d}$ can be expressed as

(11) $$\mathfrak{D}\,\omega^i_{\ \delta} - \Delta\,\omega^i_{\ d} = \theta^i_{\ \delta}\,dt - \theta^i_{\ d}\,\delta t + \left(\gamma^i_{\ kr} - \gamma^i_{\ rk}\right)\omega^k_{\ d}\,\omega^r_{\ \delta}$$

where

(12) $$\overset{i}{\theta}_{d} = d\dot{x}^i + \alpha^i\,dt + \gamma^i_{\ r}\,\omega^r_{\ d}$$

This is also tensor-invariant, $\omega^i_{\ d} = \theta^i_{\ d} = 0$ being equivalent to the original system (1). In all succeeding formulæ, we eliminate $dx$, $d\dot{x}$, by the use of the Pfaffian differential vectors in (9) and (12).

A first set of differential invariants now appears as the coefficients of the various linear and bilinear terms in $dt$, $\omega$, $\theta$, in the expression $D\,\theta^i_\delta - \Delta\,\theta^i_d$

An intrinsic choice of coefficients would be one that minimises the number of these invariants, and for that we must choose

$$\gamma^i_k = \tfrac{1}{2}\,\alpha^i_{;k} \qquad \gamma^i_{kr} = \tfrac{1}{2}\,\alpha^i_{;k;r}$$

Another differential invariant appears in computing $(\mathcal{D}\Delta - \Delta\mathcal{D})\,u^i$ and the full intrinsic set is then seen to be:

$$(13) \qquad \dot{x}^i \qquad \varepsilon^i \qquad P^i_j \qquad \alpha^i_{;j;k;l}$$

The invariantive vector differential processes are now *three* in number to be obtained by writing

$$\mathcal{D}(u)^i = D(u)^i\,dt + u^i_{/k}\,\omega^k_d + u^i_{;k}\,\theta^k_d$$

where

$$(14) \qquad D(u)^i = \frac{\partial u^i}{\partial t} + \dot{x}^r\frac{\partial u^i}{\partial x^r} - \alpha^h\frac{\partial u^i}{\partial \dot{x}^k} + \tfrac{1}{2}\,\alpha^i_{;r}\,u^r$$

$$u^i_{/k} = u^i_{,k} - \tfrac{1}{2}\,\alpha^r_{;k}\,u^i_{;r} + \tfrac{1}{2}\,\alpha^i_{;k;r}\,\dot{u}^r$$

For the most general connections of this type, we shall have to add the following invariants of the connection:

$$\sigma^i_j = \tfrac{1}{2}\,\alpha^i_{;j} - \gamma^i_j \qquad \sigma^i_{jk} = \tfrac{1}{2}\,\alpha^i_{;j;k} - \gamma^i_{jk}$$

It is seen that with proper restrictions on our absolute parameter and the transformation group, no further differential invariants are to be obtained, except by using the differential operators on these. There are differential relations between the invariants, but none that prevent the set from consisting of independent members. The various special types of spaces hitherto considered can be described by the vanishing of one or more of these invariants.

NOTE: I find that not even the procedure of Prof. Cartan includes all the differential invariants for the transformation group [A].:

$$\bar{x}^i = F^i\,[x^k \cdots] \qquad \bar{t} = t$$

These can be derived from the rather arbitrary but classical procedure of alternating all fundamental operations which are tensor-invariant. This, in essence, is the method of Christoffel for the derivation of the invariants from compatibility conditions, and is equivalent to calculating the Poisson brackets for a system of linear partial differential equations.

The fundamental differential operations which carry a vector $u^i$ into another are:

(15) $$Du^i \equiv \dot{u}^i + \tfrac{1}{2}\alpha^i_{;r} u^r \qquad \frac{\partial u^i}{\partial \dot{x}^j} = u^i_{;j} \qquad \frac{\partial u^i}{\partial t}$$

In the preceding, I only considered the first two. We find on alternating upon a vector $u^i$:

(16) $$u^i_{;j;k} - u^i_{;k;j} = 0 \qquad \frac{\partial}{\partial t} u^i_{;j} - \left(\frac{\partial u^i}{\partial t}\right)_{;j} = 0$$

The alternant of D and $\partial/\partial\dot{x}$ is another operator, precisely the covariant differentiator suggested by Cartan:

(17) $$(Du^i)_{;k} - D(u^i_{;k}) = u^i_{|k} = u^i_{,k} - \tfrac{1}{2}u^i_{;r}\,\alpha^r_{;k} + \tfrac{1}{2}u^r \alpha^i_{;r;k}$$

The rest give us immediately:

(18) $$u^i_{|j;k} - u^i_{;k|j} = \tfrac{1}{2}u^r \alpha^i_{;r;j;k}$$

(19) $$u^i_{|j|k} - u^i_{|k|j} = u^r R^i_{jk;r} - u^i_{;r} R^r_{jk}$$

where $$R^i_{jk} = \tfrac{1}{3}(P^i_{j;k} - P^i_{k;j})$$

(20) $$(Du^i)_{|j} - Du^i_{|j} = u^r(R^i_{jr} - P^i_{j;r}) + u^i_{;r} P^r_j$$

(21) $$\frac{\partial}{\partial t} D u^i - D\frac{\partial u^i}{\partial t} = u^r \frac{\partial}{\partial t}\tfrac{1}{2}\alpha^i_{;r} - u^i_{;r}\frac{\partial \alpha^r}{\partial t}$$

(22) $$\frac{\partial}{\partial t} u^i_{|j} - \left(\frac{\partial u^i}{\partial t}\right)_{|j} = u^r \frac{\partial}{\partial t}\tfrac{1}{2}\alpha^i_{;r;j} - \tfrac{1}{2}u^i_{;r}\frac{\partial \alpha^r_{;j}}{\partial t}$$

Thus, for intrinsic invariants, the fundamental list is:

(23) $$\dot{x}^i \qquad P^i_j \qquad \alpha^i_{;j;k;l}$$

The rest are derivable from these by means of the operations $\frac{\partial}{\partial t}$, D and $\partial/\partial\dot{x}$. The following relations are seen to hold for the invariants:

(24)
$$\begin{aligned}
\dot{x}^i_{;k} &= \delta^i_k \\
D\dot{x}^i &= -\varepsilon^i \\
\dot{x}^i_{|k} &= -\varepsilon^i_{;k} \\
-\varepsilon^i_{;k;l} &= \tfrac{1}{2}\alpha^i_{;k;l;m}\dot{x}^m \\
D\varepsilon^i &= -P^i_j\,\dot{x}^j + \frac{\partial \alpha^i}{\partial t} \\
\varepsilon^i_{|k} &= -P^i_k - \dot{x}^r R^i_{rk} + \frac{\partial}{\partial t}\alpha^i_{;k}
\end{aligned}$$

Thus, the term $\frac{\partial \alpha^i_{;j}}{\partial t}$ can be omitted from $P^i_j$ without destroying its tensor invariance.