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Measurement problem in quantum mechanics: Characteristics of an apparatus

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Abstract. The problem of measurement in Quantum Mechanics will be briefly reviewed. Since the measurement process involves a macroscopic apparatus, the attention is focussed on the dynamics of a pointer-like variable of the apparatus when it interacts with a quantum system. It is argued that since the measurement process requires an apparent collapse of the wave function in a certain basis, and collapse is an irreversible process, understanding of irreversibility in a quantum macroscopic system is crucial. The chief characteristics of an apparatus that are important in understanding measurement process are (a) its closely spaced energy levels and (b) its interaction with environment. The coupling with the environment drives the density matrix of the apparatus to diagonal form, but to have persistent correlations between system and apparatus states, it seems necessary to have a pointer variable that has a classical limit.

Keywords. Measurement; wavefunction collapse; irreversibility; density matrix; macroscopic apparatus.

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1. Quantum state and observables

In Quantum Mechanics (QM), the dynamical state of the system is described by a vector $|\psi\rangle$ in an appropriate linear vector space. But unlike classical description, the vector $|\psi\rangle$ does not contain an objective description of the dynamical variables of the system i.e. the quantum state does not, in general, attribute precise values to dynamical variables like position, velocity, angular momentum etc. Einstein, Podolsky and Rosenberg (EPR) [1,2] defined the so-called 'elements of reality' as those dynamical variables that can be measured precisely without disturbing the system, and required that an objective and complete theory should be able to specify all elements of reality at a given time. The quantum-mechanical description fails to do this (EPR). For example, position and momentum or three components of angular momentum cannot be specified simultaneously in a quantum state.

Furthermore, the measurable properties are not, in general, predicted deterministically by QM. Every dynamical variable, B, of the system is assigned an Hermitean operator B acting in the vector space of the state vector $|\psi\rangle$. B has a complete set of eigenfunctions with real eigenvalues, i.e.

$$B|\phi_n\rangle = b_n|\phi_n\rangle \tag{1}$$

A measurement of B yields only one of the eigenvalues b_n. The result of a single measurement is not predictable. QM predicts only the probabilities of all the possible outcomes b_n, of the measurement. These probabilities are obtained from the state vector $|\psi\rangle$ using the following prescription. Since the set of eigenvectors $|\phi_n\rangle$ form a complete set, one can write $|\psi\rangle$ as

$$|\psi\rangle = \sum_n C_n|\phi_n\rangle \tag{2}$$

Now the probability of obtaining the outcome b_n is given by $|C_n|^2$. Note that this uncertainty in knowing the value of the observable B is quite different from the way uncertainty occurs in classical terms. Classically, if we cannot predict the value of B precisely, it is because of (a) our ignorance of all the factors that may influence the dynamics of the system (b) finite resolution of the measurement procedure (c) imprecise knowledge of initial conditions etc. These uncertainties can in principle be reduced. The nature of quantum uncertainty is deeper, as it is there even when the wave function is completely known, and no values to the dynamical variables are assignable before the measurement is made. For a given eigenstate $|\phi_n\rangle$, the value of the variable B is known, but all other variables are uncertain. Thus the complete description in the sense of EPR is denied in QM.

Since the description of the quantum state and its time evolution are set in a linear vector space, it allows for situations that are completely paradoxical in the classical realm. The most discussed example is that of a wave-vector of a single particle that contains superpositions of states which correspond to well-separated positions in space. This is clearly incompatible with the classical notion of a particle, which is an entity having a unique position at a given time. Since the understanding and predictions that QM has provided have been enormously successful in all spheres it has been applied, such paradoxes only reflect the inadequacy of the classical concepts which have been derived from our experience at the macroscopic level. We have not satisfactorily understood the emergence of classical concepts from the underlying quantum dynamics.

2. Quantum measurement problem

For getting information on a system, we have to perform measurements, but in QM the measurement itself plays a role in giving attributes to the system that we understand classically. The measurement requires interaction of the system with an apparatus purporting to measure a variable B. The traditional way of describing the measurement is as follows. The measurement process disturbs the system and makes $|\psi\rangle$ collapse to one of the eigenfunctions $|\phi_n\rangle$, and the apparatus attributes the value b_n to B.

For the later sections of this article it is appropriate to describe this process in the language of density matrices. For an arbitrary state the density matrix ρ is:

$$\rho = |\psi\rangle\langle\psi| = \sum_{n,m} C_n^* C_m|\phi_m\rangle\langle\phi_n|$$

$$= \sum_n |C_n|^2 |\phi_n\rangle\langle\phi_n| + \sum_{n\neq m} C_n^* C_m |\phi_m\rangle\langle\phi_n| \tag{3}$$

The diagonal part of ρ corresponds to the system being in eigenstates $|\phi_n\rangle$, corresponding to values b_n for the variable B with corresponding probabilities $|C_n|^2$. The off-diagonal part has no classical interpretation. The collapse of the wave function can now be thought of as occurring in two steps. In the first stage

$$\rho \rightarrow \sum_n |C_n|^2 |\phi_n\rangle\langle\phi_n| \tag{4}$$

i.e. the off-diagonal elements of ρ vanish. At this stage ρ has the form of a classical probability distribution. Note that such a reduction cannot result from unitary evolution, as under unitary evolution the property $\rho^2 = \rho$ is preserved, which is not the case for (4). The second step of the measurement process is the selection of one outcome from all the possible outcomes, which is much like the realisation of one event out of the total space of possible events in a classical probabilistic process. But whereas in the classical situation, the factors neglected in dynamics and precise knowledge of initial conditions (which the observer is ignorant of) determine the outcome, no such factor that determines the outcome of measurement is known to us.

The problem of measurement lies essentially in understanding these two steps. If one describes the dynamics of system and apparatus coupled to each other quantum mechanically, one just has the unitary evolution governed by the total hamiltonian in the following way.

$$|\psi_{SA}(t)\rangle = \exp[-it(H_S + H_A + H_{SA})/\hbar]|\psi_{SA}(0)\rangle \tag{5}$$

The combined state vector $|\psi_{SA}(0)\rangle$ can be written in terms of separate basis vectors of the system and the apparatus. If the initial state has the form,

$$|\psi_{SA}(0)\rangle = \left(\sum_n a_n|\phi_n{}^S\rangle\right)\left(\sum_m b_m|\phi_m{}^A\rangle\right) \tag{6}$$

One requires that due to interaction, a correlation develops between the system states and the apparatus states to lead in time to a state vector in which each state of the system is coupled to one particular partner of the apparatus state, as in the following equation.

$$|\psi_{SA}(0)\rangle = f_1|\phi_1{}^S\rangle)|\phi_1{}^A\rangle) + f_2|\phi_2{}^S\rangle)|\phi_2{}^A\rangle) + \cdots \tag{7}$$

Barring exceptional conditions, the wave function would evolve into a superposed state i.e. a finite number of f_i's would be non-zero. But such a state would defy classical interpretation.

The founding fathers of QM recognised this difficulty. In fact, Von Neumann [3] invoked two dynamical processes for quantum evolution: U-process and R-process. U-process corresponds to standard unitary evolution and R-process describes the reduction of the wave vector to an eigenstate of an observable when an apparatus interacts with the system. Clearly, the R-process lies outside the realm of standard QM. Niels Bohr [4] further specified the conditions under which R-process occurs. He speculated that the R-Process occurs when the system interacts with a classical apparatus. While this proposal is in accord with our experience, it has serious conceptual difficulties. (a) R-process requires classical apparatus which cannot be described quantum mechanically. (b) This implies a Quantum–Classical dichotomy. Are there two kinds of dynamics? And if so, what is the dividing line between the two. (c) There doesn't seem to be a way to tackle the coupled quantum and classical dynamics. Till date, we do not have a universally acceptable understanding of the R-process.

3. Interaction with environment : Dephasing and classical behaviour

Though several kinds of solutions have been suggested, we adopt here a point of view that involves interaction of the apparatus with environment in an essential way. Such a point of view has been advocated by a number of authors and a considerable body of work in this direction exists [5-8]. The basic premise is that, the R-process as is apparent in our present understanding, involves a macroscopic apparatus and has an irreversibility associated with it. Now the dynamics of a macroscopic system occurs in a very high-dimensional and large phase space. The classical irreversibility is an apparent phenomenon at our time scales, which are too small compared to time scales of the true reversible motion of such large system. Furthermore, of all the degrees of freedom of the apparatus only one or few degrees of freedom are monitored. The key question is how to incorporate these ideas in the framework of QM.

Zeh [5] has particularly emphasized two particular characteristics of a macroscopic apparatus that are of great relevance in understanding their role in measurement. Energy levels of such a system are closely spaced. Due to small energy spacings the system is susceptible to very small disturbances. Thus such an object is not isolated and invariably interacts with its environment. Let us first qualitatively see how the environment influences the behaviour of a quantum system.

The time-dependent density matrix of a system can be written as

$$\rho_S(t) = \sum_{n,m} C_n C_m^* \exp[-it(\omega_n - \omega_m)] |\phi_n\rangle\langle\phi_m| = \sum_n |C_n|^2 |\phi_n\rangle\langle\phi_n| + \sum_{n\neq m} C_n C_m^* \exp[-it(\omega_n - \omega_m)] |\phi_n\rangle\langle\phi_m| \tag{8}$$

where $|\phi_n\rangle$ and ω_n are respectively the energy eigenstates and the energy eigenvalues. Now let the system interact with an environment with a large number of degrees of freedom. We can write the total density matrix $\rho_{SE}(t)$ of the system plus environment in a similar fashion.

$$\rho_{SE}(t) = \sum_{nE} |C_{nE}|^2 |\phi_{nE}\rangle\langle\phi_{nE}| + \sum_{nE_1\neq mE_2} C_{nE_1} C_{mE_2}^* |\phi_{nE_1}\rangle\langle\phi_{mE_2}| \exp[-it(\omega_{nE_1} - \omega_{mE_2})] \tag{9}$$

The information about any attributes of the system only is contained in the density matrix ${\rho_S}^{\rm red}$, from which environmental degrees of freedom have been traced over. Denoting by

$|\psi_E\rangle$, the energy eigenstates of the environment, $\rho_S{}^{\rm red}$ can be written as

$$\begin{aligned}\rho_S{}^{\rm red} &= \sum_{nE,E'} |C_{nE}|^2 |\langle\psi_{E'}||\phi_{nE}\rangle|^2 \\ &+ \sum_{nE_1\neq mE_2}\sum_{E'} C_{nE_1}C_{mE_2^*}\exp[-it(\omega_{nE_1}-\omega_{mE_2})] \\ &\langle\psi_{E'}||\phi_{nE_1}\rangle\langle\phi_{mE_2}||\psi_{E'}\rangle \end{aligned} \tag{10}$$

It is plausible that this expression can be written as

$$\begin{aligned}\rho^{\rm red}{}_S &= \sum_n |D_n|^2|\chi_n\rangle\langle\chi_n| \\ &+\sum_{n,m}|\chi_n\rangle\langle\chi_m| \sum_{E_1,E_2,E'} f_m(E_1,E_2,E') \\ &\exp[-it\omega_{nm}(E_1,E_2,E')]\end{aligned} \tag{11}$$

where $|\chi_n\rangle$ are related to products $\langle\psi_{E'}||\phi_{nE}\rangle$ in which environment coordinates have been integrated out, and may be quite different from the energy eigenfunctions $|\phi_n\rangle$. $\omega_{nm}(E_1, E_2, E')$ would clearly involve differences of frequencies ω_{mE} which are very large in number and closely spaced. The time-dependent sum in the second term of (11) has a typical wave-packet (in time) like behaviour. Such a sum decays rapidly from its large value at $t = 0$ and remains nearly zero for time $t << T$, where T is a recurrence time which is a multiple of all the time-periods occuring in the sum. T can be astronomically large even for say 1000 terms with arbitrary frequencies. This implies that for time scales of interest, $\rho_S{}^{\rm red}$ is diagonal in some system basis, as the non-diagonal term drops out. We thus argue that the reduced density matrix of a system coupled to an environment exhibits the features of a classical stochastic distribution. In the next section, we exploit this behaviour in the context of the measurement situation for two cases, which throw some light on the circumstances under which measurement is accomplished. The general scheme of measurement is as follows. We couple a quantum system to an apparatus which in turn is coupled to a system of large degrees of freedom, termed environment. One calculates the total density matrix and traces over the environmental degrees of freedom to obtain a reduced density matrix which contains only the system and the apparatus degrees of freedom. Alternatively one can think of it as the system being coupled to a macroscopic apparatus with many degrees of freedom. Of these only a few degrees of freedom are being monitored to measure the attributes of the system. The total Hamiltonian is :

$$H = H_S + H_A + H_{SA} + H_E + H_{AE} \tag{12}$$

where H_s, H_A and H_E are respective Hamiltonians for the system, the apparatus and the environment, H_{SA} is the interaction between the system and the apparatus and H_{AE} the interaction between the apparatus and the environment.

4. Models of measurements

A. *Zurek's model*

In this model [8] the system is a spin-$\frac{1}{2}$ particle represented by spin operator $\vec{\sigma}$. The apparatus is also a spin-$\frac{1}{2}$ particle represented by operator $\vec{L}$. The environment is a large collection of spin-$\frac{1}{2}$ particles represented by operators $\vec{J}_k$. The Hamiltonian of the system is taken to be:

$$H = g\sigma_z L_y + \sum_k g_k L_z J_{ky} \tag{13}$$

The initial state of the system and the apparatus can be generally written as,

$$|\psi(0)\rangle = (a|\uparrow\rangle + b|\downarrow\rangle)(c|+\rangle + d|-\rangle) \tag{14}$$

where $|\uparrow\rangle, |\downarrow\rangle$ are eigenstates of σ_z and $|+\rangle, |-\rangle$ are eigenstates of L_z. Zurek considered the following special situation. At $t = 0$, the apparatus is pointing in positive x-direction ($c = d = \frac{1}{\sqrt{2}}$) and H_{SA} is switched on. The interaction with the system causes this spin to rotate about the y-axis and at time, $t_o = \hbar/4g$ the wave function becomes

$$|\psi(0)\rangle = a|\uparrow\rangle|+\rangle + b|\downarrow\rangle|-\rangle \tag{15}$$

At this point a perfect correlation exists between the system states and the apparatus states. Now H_{SA} is switched off and H_{AE} is switched on. The reduced density matrix of the system and apparatus can be easily calculated to be of the form:

$$\begin{aligned}\rho^{\text{red}}(t) &= |a|^2(|\uparrow\rangle\langle\uparrow|)(|+\rangle\langle+|) + |b|^2(|\downarrow\rangle\langle\downarrow|)(|-\rangle\langle-|) \\ &\quad + ab^* Z(t)(|\uparrow\rangle\langle\downarrow|)(|+\rangle\langle-|) + \text{c.c}\end{aligned} \tag{16}$$

$Z(t)$ is a product of harmonically varying terms with frequencies given by $g_k/\hbar$, and becomes characteristically small with time, and though reversible, stays small for rather long times. Thus the model can serve as an example of the scheme of measurement outlined above. However, the scheme relies on two factors which are impractical, namely, very precise initial state and a rather precise duration of the system-apparatus interaction. Both these are required to achieve the perfect correlation between the system and apparatus states as in (15).

To examine the role of these precise requirements, we studied [9] the evolution of an arbitrary state as in (14) with the full Hamiltonian of (13). At long times, one finds that the reduced density matrix assumes the form:

$$\rho^{\text{red}}(t) = [|a|^2|\uparrow\rangle\langle\uparrow|) + |b|^2|\downarrow\rangle\langle\downarrow|][|+\rangle\langle+| + |-\rangle\langle-|] \tag{17}$$

Though, $\rho^{\text{red}}(t)$ does become diagonal, one notes that there is no correlation between system states and apparatus states. This is clearly due to the fact that in a low dimensional vector space, the quantum correlations are transitory, being superpositions of oscillations with a small number of frequencies.

B. *Stern–Gerlach measurement of spin*

This suggests that one should examine correlations of the system with apparatus which are associated with continuous or larger vector spaces. Accordingly, we next consider a model [10] for Stern–Gerlach measurement of spin. Here the spin of a particle is measured by monitoring the momentum of the particle which is a continuous degree of freedom. The

Hamiltonian of the system is taken to be:

$$H_S + H_A + H_{SA} = \lambda\sigma_x + \frac{p^2}{2m} + \epsilon x\sigma_x \tag{18}$$

$$H_{AE} + H_E = x\sum_k g_k q_k + \sum_k [p_k{}^2 + (m_k\omega_k q_k)^2] \tag{19}$$

where the notation is as follows. σ_x, p and x, denote respectively the x-components of spin, momentum and position of the particle. The first term in (18) gives the coupling of spin to uniform field, the second gives the kinetic energy of the particle, and the third gives the coupling of the particle to an inhomogenous magnetic field.The last term gives rise to a force on the particle which is along positive or negative direction depending on whether σ_x is -1 or $+1$. The two terms in (19) give the coupling of the particle position to environmental oscillators and the Hamiltonian of these oscillators. The idea is to calculate the full density matrix $\rho(s, x, q_n, s', x', q_n'; t)$ of the system and then trace over the environmental degrees of freedom q_k. While carrying out the trace, we assume that the environment oscillators are in thermal equilibrium, so the occupation of various energy states obeys the canonical distribution. The task of tracing has been exactly done using path-integral method by Feynman and Vernon [11]. It can be shown that in the weak coupling or high-temperature limit, the reduced density matrix $\rho_{s,s'}(x, x')$ obeys the following equation [12],

$$\partial\rho_{s,s'}(x, x')/\partial t = [L_{QM} + L_E]\rho_{s,s'}(x, x') \tag{20}$$

where L_{QM} is the Liouvillian operator for quantum evolution, and is given by $[\rho, H]/i\hbar$. L_E is present due to the evolution caused by interaction with the environment, and it leads to a non-unitary evolution. It should be noted that under non-unitary evolution, the density matrix loses the property: $\rho^2 = \rho$, which is the hallmark of unitary evolution. The explicit form of (20) is:

$$\begin{aligned}\frac{\partial\rho_{s,s'}(x, x')}{\partial t} = &\Big[\frac{\hbar}{2im}\Big(\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial x'^2}\Big) + \frac{i\lambda(s - s')}{\hbar}\Big]\rho \\ &+\Big[\frac{i\epsilon(sx - s'x')}{\hbar} - \gamma(x - x')\Big(\frac{\partial}{\partial x} - \frac{\partial}{\partial x'}\Big) \\ &-\frac{D}{4\hbar^2}(x - x')^2\Big]\rho\end{aligned} \tag{21}$$

where γ and D are analogs of frictional and diffusion coefficients and are related for a thermal environment. We now study the evolution for a particle in the initial state,

$$|\psi(0)\rangle = (a|\uparrow\rangle + b|\downarrow\rangle)\exp[i\tilde{p}x - \frac{x^2}{2\sigma^2}] \tag{22}$$

The general form of the density matrix is given by,

$$\begin{aligned}\rho(\mathrm{x}, \mathrm{x}', \mathrm{t}) = &|a|^2|\uparrow\rangle\langle\uparrow|\rho_{1,1}(x, x', t) + |b|^2|\downarrow\rangle\langle\downarrow|\rho_{-1,-1}(x, x', t) \\ &+ab^*|\uparrow\rangle\langle\downarrow|\rho_{1,-1}(x, x', t) + a^*b|\downarrow\rangle\langle\uparrow|\rho_{-1,1}(x, x', t)\end{aligned} \tag{23}$$

with $\rho_{s,s'}$ being given as solutions of (21). Equation 21 can be solved exactly. The spin-off diagonal parts decay in time irrespective of the spatial arguments like,

$$\rho_{-1,1} \sim \rho_{1,-1} \sim \exp(-At^3) \tag{24}$$

This makes the density matrix spin-diagonal at large times. But to achieve measurement one has to examine the spatial nonlocality of the spin-diagonal components. It turns out that these objects become diagonal with time in momentum space. To see this, we define

$$\rho_{s,s'}(Q,q,t) = \int\int dx dx' \exp[iQ(x+x')/2 + iq(x-x')]\rho_{s,s'}(x,x',t) \quad (25)$$

Now it is seen, that $\rho_{s,s'}(Q,q,t) \to 0$ with t, when $Q \neq o$, and for large t,

$$\rho_{s,s}(0,q,t) \sim 2\sqrt{\frac{\pi}{N(t)}} \exp\Big[-\frac{1}{N(t)}\big(q - \overline{p}\exp(-\gamma t) - \frac{\epsilon s}{\hbar\gamma}(1-\exp(-\gamma t))\big)^2\Big] \quad (26)$$

with

$$N(t) = D\left(1 - \frac{e^{-2\gamma t}}{\gamma} + \frac{e^{-2\gamma t}}{\sigma^2}\right) \quad (27)$$

The momentum-diagonal solution is similar to Ornstein–Zernike solution i.e. momentum distribution is gaussian with peaks centered around $\epsilon s/\hbar\gamma$. This means that the density matrix is completely diagonal with perfect correlation between spin component value and the average momentum of the particle.

5. Summary of results

We have examined here the conditions under which quantum measurement occurs. Following Bohr and others, we argue that a measurement occurs when a quantum system interacts with a macroscopic apparatus. It is associated with an irreversible collapse of the wave function. Regarding irreversibility as an apparent phenomenon occuring in systems with large number of degrees of freedom on a certain time scale, we consider a scheme of measurement in which the quantum system interacts with an apparatus, which in turn interacts with an environment having a large number of degrees of freedom. By considering two specific examples, we find that the interaction with environment always drives the density matrix of the apparatus to the diagonal form, which allows a classical stochastic interpretation, but the correlation between system and apparatus states persists only when the relevant apparatus degree of freedom has a 'classical limit'. In this sense, we have been able to provide a scheme for incorporating a concept like "classical apparatus" in a purely quantum formalism.

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