ON DIFFERENTIAL EQUATIONS WITH THE GROUP PROPERTY

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The purpose of the present work is the discussion of the possible types of homogeneous linear differential equations of which the solutions generate continuous groups of transformations in a space of n dimensions. The geometrical interpretations of the results form perhaps the sole interesting element in a tedious formal presentation. A more elegant form of reasoning is greatly to be wished for, but seems to be impossible at present. The entire paper considers problems inverse to the usual discussions of continuous groups of deformation of a space into itself.

1. A given set of r vectors of n components each is said to generate a continuous group if the alternant of any two λ, μ,

$$(\lambda^r \mu^i{}_{,r} - \mu^r \lambda^i{}_{,r}) = [\lambda, \mu] \qquad \dots \quad (1.1)$$

is also a vector of the set, or more generally, a vector of the r dimensional manifold defined by the vector fields in a space of n dimensions. The repeated index denotes summation over all n values and the subscript with a comma is used for ordinary partial differentiation. The actual transformations are derived from the infinitesimal generators

$$\lambda^i \frac{\partial}{\partial x^i}, \ \mu^i \frac{\partial}{\partial x^i} \dots,$$

and it is seen that our definition of the group condition is essentially the same as that usually given.

Consider first the case where the set satisfies a system of the first order of partial differential equations :

$$\text{(A)} \qquad u^i{}_{,j} = \phi^i_j (x, u)$$

there being r and only r solutions $u = \lambda, \mu, \dots \dots$, etc.

The group condition can be worked out at once from

$$[\lambda, \mu]^i{}_{,j} = \phi^i_j(x, [\lambda, \mu]) \quad \ldots \quad (1.2)$$

which reduces to

$$\begin{aligned} &\phi^r{}_j(x, \lambda)\, \phi_r{}^i(x, \mu) - \phi_j{}^r(x, \mu)\, \phi^i{}_r(x, \lambda) \\ &\quad + \lambda^r \phi^i{}_{r,j}(x, \mu) - \mu^r \phi^i{}_{r,j}(x, \lambda) \\ &\quad + \frac{\lambda^r \partial \phi^i{}_r}{\partial \mu^k} \phi^k{}_j(x, \mu) - \mu^r \frac{\partial \phi^i{}_r}{\partial \lambda^k} \phi_j{}^k(x, \lambda) \\ &\quad - \phi^i{}_j(x, \lambda^r \phi_r{}^i(x, \mu) - \mu^r \phi^i{}_r(x, \lambda)) = 0. \quad \ldots \quad (1.3) \end{aligned}$$

With these, however, there must be taken the compatibility conditions of (A), of the form

$$u^i{}_{j,k} - u^i{}_{k,j} \equiv \phi^i{}_{j,k}(x,u) - \phi^i{}_{k,j}(x, u) + \frac{\partial \phi^i{}_j}{\partial u^r} \phi^r{}_k - \frac{\partial \phi^i{}_k}{\partial u^r} \phi_j{}^r = 0$$

$$\text{i.e.} \qquad \psi^i{}_{jk}(x, u) = 0. \quad \ldots \quad (1.4)$$

These must hold identically, or in virtue of (A), or must themselves be taken as the equations of an extended system to be adjoined to A.

If we form the antisymmetric product

$$\lambda^k \psi^i{}_{jk}(\mu) - \mu^k \psi^i{}_{jk}(\lambda)$$

we have

$$\begin{aligned} &\lambda^k \phi^i{}_{j,k}(x, \mu) - \mu^k \phi_{j,k}(x, \lambda) - \lambda^k \phi^i{}_{k,j}(x, \mu) + \mu^k \phi^i{}_{kj}(x, \lambda) \\ &\qquad + \lambda^k \phi^r{}_k(x, \mu) \frac{\partial \phi^i{}_j}{\partial \mu^r} = 0. \quad \ldots \quad (1.5) \end{aligned}$$

Of this, four terms coincide with those in (1.3), but for sign. Adding the two, we obtain a further system which, when taken with the equations of compatibility, is to replace the group condition:

$$\begin{aligned} &\phi^r{}_j(x, \lambda)\, \phi^i{}_r(x, \mu) - \phi^r{}_j(x, \mu)\, \phi^i{}_r(x, \lambda) \\ &\quad - \phi^i{}_j(x, \lambda^r \phi^i{}_r(x, \mu) - \mu^r \phi^i{}_r(x\lambda)) \\ &\quad + \lambda^r \phi^i{}_{j.r}(x, \mu) - \mu^r \phi_{j.r}(x, \lambda) \\ &\quad + \lambda^r \phi_r{}^h(x, \mu) \frac{\partial \phi^i{}_j}{\partial \mu^h} - \mu^h \phi^r{}_h(x, \lambda) \frac{\partial \phi^i}{\partial x^r} = 0 \\ &\qquad\qquad \ldots \quad (1.6) \end{aligned}$$

2. Since the formulae of the last section are utterly unwieldy, and incapable of being interpreted to any particular extent, I shall consider a special case. We recall the condition that $a\lambda + b\mu$ is a generator with λ and μ, a and b being arbitrary constants not both zero. To this end, it is sufficient that the system (A) be homogeneous. We take then, the system

$$\text{(B)} \qquad u^i_{\ |j} \equiv u^i_{\ ,j} + \Gamma^i_{jk}\, u^k = 0.$$

This is to be regarded as tensor invariant and it then follows at once that the Γ^i_{jk} can be taken to form the co-efficients of affine connections for an n dimensional manifold. The group and compatibility conditions are respectively:

$$\Omega^h_{\ ij|k}\, \lambda^i \mu^j = 0\,; \qquad \Omega^h_{\ ij} = \Gamma^h_{ij} - \Gamma^h_{ji} \qquad \dots \quad (2.1)$$

$$u^l\, R^h_{\ ljk} = 0. \qquad \dots \quad (2.2)$$

Here, the tensor R is the Riemann-Christoffel tensor of the connection, and Ω is twice the torsion-tensor. The vertical bar is used for covariant differentiation and the equations (B) are simply those of a parallel vector. The case where (2.2) are identically satisfied is trivial, since the space then becomes Euclidean, the R's vanishing identically and with proper choice of co-ordinates the Γ's also. In case the equations (2.2) are not identically satisfied, further equations may be derived by covariant differentiation. Of all these, the distinct ones must be finite in number, say q, that have precisely the r independent solutions that form our generators. From each of these, a certain number of bilinear forms in λ, μ can be built up. Now if the group condition is not to place extra restrictions on the solutions, then the form (2.1) must belong to the field of the forms derivable from our compatibility conditions. We may sum up the results by:

The homogeneous system** (B) **possesses the group property if and only if** (a) **the torsion-tensor has a vanishing covariant derivative, or** (b) **the covariant derivative of the torsion-tensor can be linearly expressed in terms of the curvature tensor** $R^i_{\ jkl}$ **and its first** $q-1$ **covariant derivatives.

As a corollary, it is seen that the symmetric affine connections, possess the group property for parallel fields. In all cases, it must be remembered that the connection is to admit r and only r independent parallel fields.

3. The case is different for equations of the higher order. We consider homogeneous equations of the second order, and again postulate a first order operator of covariant differentiation. In particular, take the system

(C) $$u^i_{\ ;jk} = A^i_{jkl}\, u^l$$

This has the compatibility conditions:

(3.1) $$[A^h_{\ ijk} - A^h_{\ jik} R^h_{\ kij}]\, u^k = 0.$$

(3.2) $$u_{hr} R^r_{\ ijk} - u^l_{\ i} R^h_{\ ljk} - u^l_{\ k} A^h_{\ ijl} + u^l_{\ j} A^h_{\ ikl} + u^l [A^h_{\ ikl\|j} - A^h_{\ ijl\|k}] = 0$$

Using now the fact that λ, μ are both solutions of (C), we proceed to compute the second covariant derivative of the alternant, and find the group property to be

$$\begin{aligned}(\lambda^l_{\ i}\,\mu^k - \mu^l_{\ i}\,\lambda^k)\, A^h_{\ ljk} &+ (\lambda^r\,\mu^h_{\ l} - \mu^r\,\lambda^h_{\ l})\, A^l_{\ ijr} \\ &-(\lambda^r\,\mu^l_{\ r} - \mu^r\,\lambda^l_{\ r})\, A^h_{\ ijl} \\ &+ (\lambda^l_{\ j}\,\mu^r - \mu^l_{\ j}\,\lambda^r)\,[A^h_{\ lir} - A^h_{\ ril}] \\ &+ (\lambda^l\,\mu^r\,[A^h_{\ lir;j} - A^h_{\ ril\|j}] = 0 \qquad \ldots \quad (3.3)\end{aligned}$$

From (3.2), we can derive as before a binary expression in λ, μ. This, however, is no longer a form. Eliminating the curvature tensor by means of (3.1) we find an equation of the same type as (3.3). The two coincide when and only when the tensor A^i_{jkl} obeys the same subscript identities as the curvature tensor, and actually coincides with it, or when the tensor A^i_{jkl} vanishes identically. In the former case we find the equations to be those of "*affine collineation.*"

A similar result seems to hold for those homogeneous equations wherein first covariant derivatives are assumed to enter : for any symmetric connection, the first covariant derivative cannot enter without extra conditions on the solutions, and a consequent loss of the group property. As the computations are too long to trust without verification, I omit them, and state the result in the form of a conjecture.

For symmetric affine connections, the only homogeneous partial differential equations of the second order possessing the group property are

$$u^i_{\ |jk} = 0 \qquad \text{and} \qquad u^i_{\ |jk} = R^i_{\ jkl}\, u^l$$

4. There are still some considerations which remain. For instance, it may be possible for a certain subset of the set of solutions to possess the group property. This would mean essentially that the group conditions do introduce new restrictions on the solutions, in the form of further partial differential equations which form a completely integrable system with the original one, but naturally with fewer common solutions. There is again the line of approach indicated by the classical theorems of Lie on the constants of compositions of a group; geometrically, this leads to the investigation of the Ricci "coefficients of rotation," and is of special interest for transitive groups, since here it is possible to associate certain covariant vectors with the contravariant vectors that form the solutions under consideration. A case of some interest is also furnished by the introduction of a metric in the space, which allows among other things, the equations of Killing.

The second order case is of interest because of the highly specialized type of possible systems. The principal system corresponds to the "equations of variation" for the paths of the space, as follows from my work on parallelism. The same paper (under publication) shows that the general covariant derivative is non-distributive, and may lead to non-homogeneous linear equations, a case not considered here since the methods outlined give a successful treatment of that type as well.