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Photon states in anisotropic media

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Abstract. Quantum aspects of optical polarization are discussed for waves traveling in anisotropic dielectric media with a view to relate the dynamics of polarization with that of photon spin and its manipulation by classical polarizers.

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1. Motivation

The motivation for this study came from a suggestion that the photon polarization states form a two state quantum system and should be able to serve as qubits for quantum computation. This would require (i) ability to manipulate the polarization states of single photons, (ii) ability to create polarization entangled states of two and more photons. Concerning the first requirement, one notes that polarization of classical electromagnetic waves can be easily manipulated using devices like polarizers, quarter-wave plate etc., which are made of birefringent medium. So the question here is related to the understanding of the dynamical behaviour of photon spin states, as they propogate in an anisotropic medium.

Concerning the second requirement, one notes that in the past twenty years, the entangled photon polarization states have played a central role in experiments investigating quantum entanglement. For example, the non-local EPR (Einstein–Podolsky–Rosen) correlations as codified in Bell's inequalities have been tested for the polarization-entangled states of two photons. Similarly, quantum teleportation and quantum encryption have also been accomplished using photon polarization states.

In this paper, we shall concern ourselves with the first aspect, namely, the manipulation of photon spin states by classical elements. When light passes through an optically active medium or a quarter-wave plate, its polarization state changes continuously, which is understood by saying that the passage through the medium introduces a further phase difference between the oscillation of the two components of the electric field. Quantum mechanically this must be understood in terms of the evolution of the wave function for photon polarization. Here, we provide the field-theoretic underpinning for such a description, by discussing the quantization of the electromagnetic (EM) waves in an anisotropic medium. This quantization differs in some significant aspects from the quantization of EM field in vacuum or isotropic medium. In the latter situation, it is most convenient to work in the radiation gauge, where the EM energy can be written as a piece containing only transverse EM fields and a piece that contains external charges interacting via Coulomb potential. In the absence of charges one needs to quantize only the first piece to yield photons with two transverse polarization states, which can also be written as eigenstates of spin angular momentum.

For the anisotropic medium on the other hand, the electric field is not transverse even in the absence of external charges. So the separation of the EM energy into a radiation-related piece and a charge-related piece is not based on transverse condition. We present a gauge in which such a decomposition is made and proceed with the quantization. One is led to two photon polarization states which are dependent on the direction of the wave vector. These states are not transverse, not orthogonal to each other and are not eigenstates of spin angular momentum. All these results carry over from the classical analysis.

The paper is organized as follows. Starting from the classical description of polarization states as harmonic oscillations, we first review the quantum mechanical analog of these oscillations. We then make a correspondence with the classical description in terms of coherent states for EM fields in vacuum. This is then generalized for the anisotropic medium by incorporating the effect of the medium through the introduction of dielectric constant in the Maxwell equations.

2. Coherent state description of classical polarization

The classical view of the polarization of electromagnetic waves is related to the direction of oscillation of the electric field vector. Since the electric field vector can oscillate in a plane perpendicular to the propagation vector $\vec{k}$, the oscillations of the electric field are described in terms of oscillations of the two perpendicular components of the electric field in the plane. The different states of polarization then depend upon the relative magnitudes and phase difference of the oscillations of the two components. For a plane wave moving in the z-direction, we have

$$E_x(z,t) = \varepsilon_x \cos(kz - \omega t),$$
$$E_y(z,t) = \varepsilon_y \cos(kz - \omega t + \phi). \tag{1}$$

Then the electric field traverses the curve given by

$$\frac{E_x^2}{\varepsilon_x^2} + \frac{E_y^2}{\varepsilon_y^2} - \frac{2E_xE_y}{\varepsilon_x\varepsilon_y}\cos\phi = 1. \tag{2}$$

The quantum mechanical treatment of polarization requires us to treat the oscillations of the two electric field components by their wave functions. Representing the two components as X and Y, the wave function of oscillation in a typical eigenstate is

$$|n_x, n_y\rangle \propto \mathrm{e}^{-X^2/2} H_{n_x}(X) \mathrm{e}^{-Y^2/2} H_{n_y}(Y), \tag{3}$$

where H_n's are the Hermite polynomials.

But this state has no description of the amplitude or phase for either of the oscillators. Thus the notion of polarization does not exist for an energy eigenstate. One expects these notions to emerge when quantum numbers n_x and n_y become large. Since the phase is associated with the time-dependence of oscillation one needs to consider appropriate superpositions of the above energy eigenstates. For the oscillator problem these are the well known coherent states [1].

Coherent states are superposition of infinite number of energy eigenstates. So here the phase emerges as a variable conjugate to the number operater $\widehat{N}$ whose eigenvalues are n. A coherent state for a two-dimensional oscillator characterized by amplitudes ε_x and ε_y is given by

$$|\varepsilon_x,\varepsilon_y;0\rangle = \mathrm{e}^{-[|\varepsilon_x|^2+|\varepsilon_y|^2]/2} \sum_{n_x=0}^{\infty}\sum_{n_y=0}^{\infty} \frac{(\varepsilon_x)^{n_x}}{\sqrt{n_x!}} \frac{(\varepsilon_y)^{n_y}}{\sqrt{n_y!}} |n_x,n_y,0\rangle. \tag{4}$$

The time-dependence of these states is given by

$$|\varepsilon_x,\varepsilon_y,t\rangle = |\varepsilon_x \mathrm{e}^{-i\omega t},\varepsilon_y \mathrm{e}^{-i\omega t},0\rangle. \tag{5}$$

Coherent states are the right eigenstates of the 'annihilation operator'. This property allows us to write

$$\langle\varepsilon_x,\varepsilon_y,t|X|\varepsilon_x,\varepsilon_y,t\rangle = \sqrt{\frac{\hbar}{2\omega}}\,\mathrm{Re}(\varepsilon_x \mathrm{e}^{-i\omega t)},$$
$$\langle\varepsilon_x,\varepsilon_y,t|Y|\varepsilon_x,\varepsilon_y,t\rangle = \sqrt{\frac{\hbar}{2\omega}}\,\mathrm{Re}(\varepsilon_y \mathrm{e}^{-i\omega t}). \tag{6}$$

These expectation values have the behaviour identical to the classical two-dimensional oscillator. The above discussion is straightforwardly generalized to EM waves.

3. Field theoretic treatment of the electromagnetic waves

The quantum field theoretic description of the EM field is done in terms of the vector potential, $\vec{A}(\vec{r},t)$. The vector field is associated with an internal spin of value one. The polarization is regarded as the spin of the photon. Physically, the photon exists in only two transverse polarization states, which theoretically is a consequence of the additional gauge condition on the A-field. We now review the quantization of the EM field in vacuum, as a prelude to considering the same in a dielectric medium. The Hamiltonian of the field is given by

$$H = \frac{1}{8\pi}\int [\vec{E}^2(\vec{r},t) + \vec{B}^2(\vec{r},t)]\mathrm{d}^3 r. \tag{7}$$

One expresses the EM fields in terms of potentials ϕ and $\vec{A}$,

$$\vec{B} = \vec{\nabla}\times\vec{A},$$
$$\vec{E} = -\frac{1}{c}\frac{\partial\vec{A}}{\partial t} - \vec{\nabla}\phi. \tag{8}$$

In the radiation gauge, $\vec{\nabla}.\vec{A} = 0$, the potential ϕ obeys the Poisson equation, and can thus be eliminated in favour of external charge densities $\rho(\vec{r})$. This allows us to write the EM-Hamiltonian as [2]

$$H = \frac{1}{8\pi}\int\left[\left(\frac{1}{c}\frac{\partial\vec{A}}{\partial t}\right)^2 + (\vec{\nabla}\times\vec{A})^2\right]\mathrm{d}^3r + \int \mathrm{d}^3r\int \mathrm{d}^3r'\,\frac{\rho(\vec{r})\rho(\vec{r}')}{|\vec{r}-\vec{r}'|}. \tag{9}$$

The second term is the energy related to field generated by matter while the first term describes the source-free radiation field. Since the vector potential is transverse, this part involves only the transverse EM fields.

The field $\vec{A}$ can be expanded in terms of eigenmodes, which we take to be plane waves,

$$\vec{A}(\vec{r},t) = \left(\frac{4\pi c^2}{V}\right)^{1/2}\sum_{k,\alpha}\sqrt{\frac{\hbar}{2\omega_k}}\left[a_{k\alpha}\vec{e}_\alpha \mathrm{e}^{i\vec{k}.\vec{r}}\mathrm{e}^{-i\omega_k t} + a^*_{k\alpha}\vec{e}^{\,*}_\alpha \mathrm{e}^{-i\vec{k}.\vec{r}}\mathrm{e}^{i\omega_k t}\right], \tag{10}$$

where $\omega_k = ck$ denote the eigenfrequencies, V is the volume and $\vec{e}_\alpha$'s denote the two polarization modes ($\alpha = 1,2$), which are perpendicular to $\vec{k}$ due to the transverse condition on $\vec{A}$. We also record below the expansion of the electric field in terms of these eigenmodes,

$$\begin{aligned}\vec{E}(\vec{r},t) &= \frac{-i}{\sqrt{V}}\sum_{k,\alpha}\sqrt{\frac{\hbar\omega_k}{2}}\left[a_{k\alpha}\vec{e}_\alpha \mathrm{e}^{i\vec{k}.\vec{r}}\mathrm{e}^{-i\omega_k t} - a^*_{k\alpha}\vec{e}^{\,*}_\alpha \mathrm{e}^{-i\vec{k}.\vec{r}}\mathrm{e}^{i\omega_k t}\right]\\ &= \vec{E}^{+}(\vec{r},t) + \vec{E}^{-}(\vec{r},t).\end{aligned} \tag{11}$$

In the quantum treatment, one treats $a_{k\alpha}$ and $a^*_{k\alpha}$ as the creation and annihilation operators for the plane wave modes, which obey the usual bosonic commutation rule $[a_{k\alpha}, a^\dagger_{k'\beta}] = \delta_{k,k'}\delta_{\alpha\beta}$. In terms of these operators the Hamiltonian can be written as a sum of two-dimensional harmonic oscillators for each mode in the following way:

$$H = \frac{1}{2}\sum_{k\alpha}\hbar\omega_k\left[a^\dagger_{k\alpha}a_{k\alpha} + a_{k\alpha}a^\dagger_{k\alpha}\right]. \tag{12}$$

The operator $a^\dagger_{k,\alpha}$ creates a one-photon state in the plane wave mode with wave vector $\vec{k}$ and polarization given by $\vec{e}_\alpha$. For $\vec{k}$ along the z-axis one can generate one-photon states with arbitrary polarization wave functions using linear combinations of operators $a^\dagger_{kx}$ and $a^\dagger_{ky}$. For example, the operator

$$a^\dagger_{k\theta} = [\cos\theta\, a^\dagger_{kx} + \sin\theta\, a^\dagger_{ky}] \tag{13}$$

generates the analog of linearly polarized state, in the wave function sense. If we measured the polarization, we would find it fully polarized in the x-direction with probability $\cos^2\theta$, and fully polarized in the y-direction with probability $\sin^2\theta$. On the other hand, the coherent state constructed using this operator,

$$|e_{\vec{k},\theta}\rangle = \mathrm{e}^{-1/2|e^2_{\vec{k},\theta}}\sum_{n_{\vec{k},\theta}=0}^{\infty}\frac{(e_{\vec{k},\theta})^{n_{\vec{k},\theta}}}{n_{\vec{k},\theta}!}a^{\dagger\;n_{\vec{k},\theta}}_{k\theta}|0\rangle \tag{14}$$

is the eigenstate of the annihilation part of the field operator

$$\vec{E}^{+}(\vec{r},t)|e_{\vec{k},\theta}\rangle = -i\sqrt{\frac{\hbar\omega_k}{2}}\vec{e}_{\vec{k},\theta}\,\mathrm{e}^{i\vec{k}.\vec{r}}|e_{\vec{k},\theta}\rangle\mathrm{e}^{-i\omega_k t}. \tag{15}$$

This state has the well-defined polarization at an angle θ to the x-axis. Similarly coherent states of arbitrary polarization as in eq. (2) may be constructed by taking linear combinations of the operators $a^{\dagger}_{kx}$ and $a^{\dagger}_{ky}$ with complex coefficients.

It is of interest to consider the spin angular momentum associated with these states. The angular momentum of the EM field is defined by

$$\vec{G} = \frac{1}{4\pi c}\int \vec{r}\times(\vec{E}\times\vec{B})\mathrm{d}^3 r. \tag{16}$$

$\vec{G}$ can be separated into an orbital part which depends on the origin of coordinates, and a spin part $\vec{G}_{\mathrm{s}}$, which is independent of $\vec{r}$ [2]. The latter is given by

$$\vec{G}_{\mathrm{s}} = \frac{1}{4\pi c}\int (\vec{E}\times\vec{A})\mathrm{d}^3 r. \tag{17}$$

Writing it in terms of photon operators with linear polarizations in two perpendicular directions, one finds

$$\vec{G}_{\mathrm{s}} = i\hbar\sum_k \hat{k}[a^{\dagger}_{k2}a_{k1} - a^{\dagger}_{k1}a_{k2}]. \tag{18}$$

One can bring this expression to a diagonal form using the operators

$$a_{k\mathrm{l}} = \frac{a_{k1}+ia_{k2}}{\sqrt{2}};\quad a_{k\mathrm{r}} = \frac{a_{k1}-ia_{k2}}{\sqrt{2}}, \tag{19}$$

which correspond to left- and right-circularly polarized states. Then

$$\vec{G}_{\mathrm{s}} = \hbar\sum_k \hat{k}[a^{\dagger}_{k\mathrm{r}}a_{k\mathrm{r}} - a^{\dagger}_{k\mathrm{l}}a_{k\mathrm{l}}]. \tag{20}$$

Thus these states are eigenstates of the angular momentum in the direction $\hat{k}$ and of values $\pm\hbar$. Note these states are degenerate.

4. EM fields in a dielectric medium

We now consider the quantization of EM waves in a nondispersive dielectric medium. If one treats the medium at the atomic level, EM fields vary rapidly with position and time. For most physical purposes, such as propagation of optical waves etc., it is sufficient to work with space-time averaged fields, which are again smooth. This is done by introducing fields $\vec{D}$ and $\vec{B}$ which incorporate the induced electric dipole density $\vec{P}$ and magnetic dipole density $\vec{M}$ of the medium,

$$\begin{aligned} D_\alpha &= E_\alpha + 4\pi P_\alpha = \varepsilon_{\alpha\beta}E_\beta,\\ B_\alpha &= H_\alpha + 4\pi M_\alpha = \mu_{\alpha\beta}H_\beta, \end{aligned} \tag{21}$$

where $\varepsilon_{\alpha\beta}$ and $\mu_{\alpha\beta}$ are the dielectric permittivity and magnetic permeability tensors. We shall take them to be frequency independent and in the following treatment take the medium to be nonmagnetic, for which $\mu_{\alpha\beta} = \delta_{\alpha\beta}$. There is a straightforward generalization for the magnetic medium, which is mentioned towards the end of the section.

We first recall the Maxwell equations obeyed by the fields,

$$\vec{\nabla}.\vec{D} = 4\pi\rho_{\rm ext}, \tag{22}$$

$$\vec{\nabla}.\vec{B} = 0, \tag{23}$$

$$\vec{\nabla}\times\vec{E} = -\frac{1}{c}\frac{\partial\vec{B}}{\partial t}, \tag{24}$$

$$\vec{\nabla}\times\vec{H} = \frac{1}{C}\frac{\partial\vec{D}}{\partial t} + \frac{4\pi}{c}\vec{J}_{\rm ext}. \tag{25}$$

The EM field energy is now given by

$$H = \frac{1}{8\pi}\int\left[E_\alpha\varepsilon_{\alpha\beta}E_\beta + B^2\right]{\rm d}^3r. \tag{26}$$

Again the homogeneous equations are solved by potentials $\vec{A}$ and ϕ defined in eq. (8). Substituting them in the other two Maxwell equations yields

$$\frac{1}{c}\frac{\partial}{\partial t}(\varepsilon_{\alpha\beta}\partial_\alpha A_\beta) + \varepsilon_{\alpha\beta}\nabla_\alpha\nabla_\beta\phi = -4\pi\rho_{\rm ext}, \tag{27}$$

$$\nabla_\alpha\nabla_\beta A_\beta - \nabla^2 A_\alpha + \frac{1}{c^2}\varepsilon_{\alpha\beta}\frac{\partial^2 A_\beta}{\partial t^2} = \frac{4\pi}{c}\left(J_\alpha - \frac{1}{4\pi}\varepsilon_{\alpha\beta}\nabla_\beta\frac{\partial\phi}{\partial t}\right). \tag{28}$$

Now we make a choice of the gauge by the equation

$$\varepsilon_{\alpha\beta}\partial_\alpha A_\beta = 0. \tag{29}$$

In this gauge, the charge conservation equation allows us to identify the longitudinal component of current to be

$$J^{\rm l}_\alpha = \varepsilon_{\alpha\beta}\nabla_\beta\frac{\partial\phi}{\partial t}. \tag{30}$$

Thus the inhomogenous Maxwell equations take the form

$$\varepsilon_{\alpha\beta}\nabla_\alpha\nabla_\beta\phi = -4\pi\rho_{\rm ext}, \tag{31}$$

$$\nabla_\alpha\nabla_\beta A_\beta - \nabla^2 A_\alpha + \frac{1}{c^2}\varepsilon_{\alpha\beta}\frac{\partial^2 A_\beta}{\partial t^2} = \frac{4\pi}{C}J^{\rm t}_\alpha, \tag{32}$$

where $J^{\rm t}_\alpha$ refers to the transverse part of the current.

The field energy can now be expressed in terms of these potentials as

$$H = \frac{1}{8\pi}\int{\rm d}^3r\left[\frac{1}{c^2}\frac{\partial A_\alpha}{\partial t}\varepsilon_{\alpha\beta}\frac{\partial A_\beta}{\partial t} + (\vec{\nabla}\times\vec{A})^2\right] \tag{33}$$

$$+\frac{1}{2}\int\frac{\rho_{\rm ext}(\vec{r})\rho_{\rm ext}(\vec{r}')}{|\vec{\xi}(\vec{r}) - \vec{\xi}(\vec{r}')|}{\rm d}^3r{\rm d}^3r', \tag{34}$$

where the potential ϕ has been eliminated in favour of the charge density $\rho_{\rm ext}(\vec{r})$, using the solution of eq. (31). In the coordinate system coinciding with the principal axes of the dielectric tensor, $\vec{\xi}(\vec{r}) = (x/\varepsilon_1, y/\varepsilon_2, z/\varepsilon_3)$, where $\varepsilon_1, \varepsilon_2$ and ε_3 denote the diagonal values of the dielectric tensor along the three respective axes. Thus, in this gauge, one can again separate the energy in two parts, one associated with the radiation field, and the other with the charges. In the absence of external charges and currents, we need to consider the dynamics of only the $\vec{A}$ field. The corresponding Lagrangian is

$$L_r = \frac{1}{8\pi}\int \mathrm{d}^3r \left[\frac{1}{c^2}\frac{\partial A_\alpha}{\partial t}\varepsilon_{\alpha\beta}\frac{\partial A_\beta}{\partial t} - (\vec{\nabla}\times\vec{A})^2\right]. \tag{35}$$

As before, we expand $\vec{A}(\vec{r},t)$ in terms of plane waves,

$$A_\alpha(\vec{r},t) = \sqrt{\frac{4\pi c^2}{V}}\sum_{k,\alpha}\left[q_{k\alpha}\mathrm{e}^{i\vec{k}.\vec{r}} + q^*_{k\alpha}\mathrm{e}^{-i\vec{k}.\vec{r}}\right] \tag{36}$$

and obtain

$$L_r = \frac{1}{2}\sum_{k,\alpha}\left[\dot{q}_{k\alpha}\varepsilon_{\alpha\beta}\dot{q}_{k\beta} - q_{k\alpha}V_{\alpha\beta}(\hat{k})q_{k\beta}\right], \tag{37}$$

where

$$V_{\alpha\beta}(\hat{k}) = c^2k^2(\delta_{\alpha\beta} - \hat{k}_\alpha\hat{k}_\beta). \tag{38}$$

This leads to the standard generalized eigenvalue problem [3] of the form

$$V_{\alpha\beta}e_{\beta\nu}(\vec{k}) = \omega^2_\nu(\vec{k})e_{\alpha\nu}(\vec{k}). \tag{39}$$

The eigenvalue equation for each $\vec{k}$ is the well-known Fresnel equation [4]

$$\det|\omega^2\varepsilon_{\alpha\beta} - V_{\alpha\beta}(\vec{k})| = 0. \tag{40}$$

Of the three eigenvalues, one is eliminated by the gauge condition, as the corresponding eigenvector does not satisfy eq. (29). The eigenvectors, which give the polarization obey the orthogonality and normalization corresponding to the metric $\varepsilon_{\alpha\beta}$,

$$e^*_{\mu\alpha}\varepsilon_{\alpha\beta}e_{\beta\nu} = (\vec{e}_\mu, \vec{e}_\nu) = \delta_{\mu\nu}. \tag{41}$$

Now one can introduce the normal coordinates $\xi_{k\nu}$,

$$q_{k\alpha} = e_{\alpha\nu}\xi_{k\nu}, \tag{42}$$

and the Lagrangian can be written as

$$L_r = \frac{1}{2}\sum_{k\nu}\left[(\dot{\xi}_{k\nu})^2 - \omega^2_\nu(\vec{k})\xi^2_{k\nu}\right]. \tag{43}$$

It should be noted that the two polarization vectors and $\vec{k}$ are mutually orthogonal with respect to the metric $\varepsilon_{\alpha\gamma}$, i.e.,

$$(\vec{k},\vec{e}_\mu) = 0 \ , \qquad (\vec{e}_\mu,\vec{e}_\nu) = \delta_{\mu\nu}, \tag{44}$$

with the scalar product being defined in eq. (41).

Now the quantization proceeds in exactly the same manner as in the previous section. The expansions of the $\vec{A}$ field and the electric field are the same as in eqs (10) and (11) with polarization vectors obtained from eq. (39). The eigenstates in the medium are no longer eigenstates of the angular momentum. Thus if a photon of a given spin state enters the medium, its spin wave function continuously evolves in time, as the two polarization eigenfunctions have different frequencies.

One can similarly construct counterparts of coherent states to make correspondence with the classical polarization states. The generalization to magnetic media is also straight forward. It simply leads to the following change in the matrix $V_{\alpha\beta}(\vec{k})$,

$$V_{\alpha\beta}(\vec{k}) = K_{\alpha\eta}\mu_{\eta\nu}K_{\nu\beta} \quad ; \quad K_{\alpha\eta} = \varepsilon_{\alpha\gamma\eta}k_\gamma, \tag{45}$$

where $\varepsilon_{\alpha\beta\gamma}$ is the usual third-rank antisymmetric tensor.

To conclude, within the linear regime, the photon states in the medium have a direct correspondence with the classical states. The various polarization phenomena related to double refraction have thus analogs in terms of photon wave functions and the action of various polarization devices can be understood in terms of changes in the photon wave function.

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