The homogeneous shifts

Bhaskar Bagchi

and

Gadadhar Misra

Theoretical Statistics and Mathematics Division
Indian Statistical Institute
Bangalore 560059
India

Abstract

A bounded linear operator T on a complex Hilbert space is called *homogeneous* if the spectrum of T is contained in the closed unit disc and all bi-holomorphic automorphisms of this disc lift to automorphisms of the operator modulo unitary equivalence. We prove that all the irreducible homogeneous operators are block shifts. Therefore, as a first step in classifying all of them, it is natural to begin with the homogeneous scalar shifts.

In this paper we determine all the homogeneous (scalar) weighted shifts. They consist of the un-weighted bilateral shift, two one-parameter families of unilateral shifts (adjoints of each other), an one-parameter family of bilateral shifts and a two-parameter family of bilateral shifts. This classification is obtained by a careful analysis of the possibilities for the projective representation of the Möbius group associated with an irreducible homogeneous shift.

1 Introduction

All Hilbert spaces in this paper are separable Hilbert spaces over the field of complex numbers. The set of all unitary operators on a Hilbert space $\mathcal{H}$ will be denoted by $\mathcal{U}(\mathcal{H})$. When equipped with any of the usual operator topology $\mathcal{U}(\mathcal{H})$ becomes a topological group. All these topologies induce the same Borel structure on $\mathcal{U}(\mathcal{H})$. We shall view $\mathcal{U}(\mathcal{H})$ as a Borel group with this structure.

$\mathbb{Z}$, $\mathbb{Z}^+$, $\mathbb{Z}^-$ will denote the set of all integers, non-negative integers and non-positive integers respectively. $\mathbb{R}$ and $\mathbb{C}$ will denote the Real and Complex numbers. $\mathbb{D}$ and $\mathbb{T}$ will denote the open unit disc and the unit circle in $\mathbb{C}$, and $\bar{\mathbb{D}}$ will denote the closure of $\mathbb{D}$ in $\mathbb{C}$. Möb will denote the Möbius group of all biholomorphic automorphisms of $\mathbb{D}$. Recall that Möb $= \{\varphi_{\alpha,\beta} : \alpha \in \mathbb{T}, \beta \in \mathbb{D}\}$, where

$$\varphi_{\alpha,\beta}(z) = \alpha \frac{z-\beta}{1-\bar{\beta}z}, \quad z \in \mathbb{D}. \tag{1.1}$$

Möb is topologised via the obvious identification with $\mathbb{T}\times\mathbb{D}$. With this topology, Möb becomes a topological group. Abstractly, it is isomorphic to $PSL(2,\mathbb{R})$ and to $PSU(1,1)$.

Recall from [1] that an operator T is called *homogeneous* if $\varphi(T)$ is unitarily equivalent to T for all φ in Möb for which $\varphi(T)$ makes sense. It was shown in Lemma 2.2 of [1] that the spectrum of such an operator is either $\mathbb{T}$ or $\bar{\mathbb{D}}$, so that $\varphi(T)$ actually makes sense (and is unitarily equivalent to T) for all elements φ of Möb.

In Section 2, we review the projective representations of Möb İt is well known that the universal cover of Möb (being a semi-simple Lie group) is of Type 1 (see Theorem 7 in **??** and the author's remark following it). As a simple consequence of this fact, as pointed out in Lemma **??**, all the projective (unitary) representations of Möb are direct integrals of its irreducible projective representations. The irreducible projective representations of Möbare clearly obtainable as push-downs of the ordinery irreducible representations of its universal covering group under the covering map. Therefore a complete list of these irreducible projective representations may easily be manufactured out of the known list (as obtainable from [10], for instance. We end Section 2 by making this list explicit. However, we have reprametrised the list in a convenient fashion in order to get a uniform description. Such a uniform description will greatly simplify the proof of the main theorem presented in the final section.

In Theorem 2.2 of Section 3, we show that to any irreducible homogeneous operator is associated an essentially (i.e., upto equivalence) unique projective unitary representation of Möb(The meaning of 'associated' is made precise in Definition 2.1.) The 'existence' part of this Theorem is already there in [6]. However, for the sake of completeness, we have presented proofs of both parts in this paper. We end this section by presenting a general non-sense construction of homogeneous operators.

In section 4, we present a list of all the homogeneous scalar shifts known to us. Excepting the un-weighted bilateral shift, all these examples are irreducible. Though many of these examples were previously known, the two-parameter family of bilateral homogeneous shifts (dubbed the complementary series examples) appears to be new.

In Section 5 we show that, as a consequence of Theorem 2.2, all irreducible homogeneous operators are block shifts (Theorem 5.1). Indeed, if T is an irreducible homogeneous operator with associated representation π (say) then the blocks of T are precisely the non-trivial $\mathbb{K}$- isotypic subspaces of the representation space of π. (Here $\mathbb{K}$ is the maximal compact subgroup of Möb) This theorem acquires substance from the fact (Lemma 2.2) that the blocks of an irreducible block shift are uniquely determined by the operator. As a consequence of Theorem 5.1 and Lemma **??** it follows that (Lemma 5.1) the projective representation associated with an irreducible scalar weighted shift must be one of the representations listed in Section 2. (With the exception of the sporadic principal series representation $P_{1,0}$, all the representations in this list are irreducible.) Finally, we find out all the homogeneous operators associated with the representations in this list. This proves (Theorem 5.2) that the irreducible homogeneous scalar shifts are precisely the ones listed in Section 4.

One surprising find of this proof technique is that each Principal series representation $P_{\lambda,s}$ is associated with two (generally distinct) homogeneous operators - both unitarily equivalent to the unweighted bilateral shift. The occurence of only one of these two operators (namely multiplication by the co-ordinate function on the representation space) is a priori evident. These two operators coallesce into one precisely when $s = 0$. We do not have any convincing explanation for the occurence of the second copy.

2 Homogeneous operators and weighted shifts

2.1 Generalities

Let $*$ denote the involution (i.e. automorphism of order two) of Möb defined by

$$\varphi^*(z) = \overline{\varphi(\bar{z})}, \quad z \in \mathbb{D}, \ \varphi \in \text{Möb}. \tag{2.1}$$

Thus $\varphi^*_{\alpha,\beta} = \varphi_{\bar{\alpha},\bar{\beta}}$ for $(\alpha, \beta) \in \mathbb{T} \times \mathbb{D}$. It is known that essentially (i.e. upto multiplication by arbitrary inner automorphisms), $*$ is the only outer automorphism of Möb. It also satisfies $\varphi^*(z) = \varphi(z^{-1})^{-1}$ for $z \in \mathbb{T}$. It follows that for any operator T whose spectrum is contained in $\bar{\mathbb{D}}$, we have

$$\varphi(T^*) = \varphi^*(T)^*, \quad \varphi(T^{-1}) = \varphi^*(T)^{-1} \tag{2.2}$$

the latter in case T is invertible, of course. It follows immediately from (2.2) that the adjoint T^* - as well as the inverse T^{-1} in case T is invertible - of a homogeneous operator $\mathbb{T}$ is again homogeneous.

Clearly a direct sum (more generally, direct integral) of homogeneous operators is again homogeneous.

Let I stand for either $\mathbb{Z}$, $\mathbb{Z}^+$ or $\mathbb{Z}^-$. Recall that an operator T on the Hilbert space $\mathcal{H}$ is called a weighted shift with weight sequence $w_n, \ n \in I$ if there is a distinguished orthonormal basis $x_n, \ \ n \in \mathbb{N}$ such that $Tx_n = w_n x_{n+1}$ for all $n \in I$. T is called a bilateral shift, forward unilateral shift or backward unilateral shift according as $I = \mathbb{Z}, \mathbb{Z}^+$ or $\mathbb{Z}^-$. To avoid trivialities, we shall assume throughout that all the weights w_n are non-zero. Every weighted shift (with non-zero weights) is unitarily equivalent to a unique weighted shift whose weights are strictly positive. The unweighted unilateral (respectively bilateral) shift is the unilateral (respectively bilateral) weighted shift all whose weights are equal to 1.

2.2 The reducible case

As already stated, the object of this paper is to classify the homogeneous shifts upto unitary equivalence. We first dispose off the case of reducible homogeneous shifts. To do so, we need :

Lemma 2.1 *If T is a homogeneous operator such that T^k is unitary for some positive integer k then T is unitary.*

Proof : Let $\varphi \in$ Möb . Since $\varphi(T)$ is unitarily equivalent to T, it follows that $(\varphi(T))^k$ is unitarily equivalent to T^k and hence is unitary. In particular, taking $\varphi = \varphi_\beta$ (for a fixed but arbitrary $\beta \in \mathbb{D}$), we find that the inverse and the adjoint of $(T - \beta I)^k (I - \bar{\beta} T)^{-k}$ are equal. That is,

$$(T - \beta I)^{-k} (I - \bar{\beta} T)^k = (T^* - \bar{\beta} I)^k (I - \beta T^*)^{-k}$$

and hence

$$(I - \bar{\beta} T)^k (I - \beta T^*)^k = (T - \beta I)^k (T^* - \bar{\beta} I)^k$$

for all $\beta \in \mathbb{D}$. (Note that the two factors on each side of this equation commute.) Expanding binomially, we get

$$\sum_{m,n=0}^{k} (-1)^{m+n} \binom{k}{m} \binom{k}{n} \bar{\beta}^m \beta^n T^m T^{*n} = \sum_{m,n=0}^{k} (-1)^{m+n} \binom{k}{m} \binom{k}{n} \bar{\beta}^m \beta^n T^{k-n} T^{*k-m}.$$

Equating coefficients of like powers, we get $T^m T^{*n} = T^{k-n} T^{*k-m}$ for $0 \leq m, n \leq k$. Noting that our hypothesis on T implies that T is invertible, we find $T^{m+n-k} = T^{*k-m-n}$ for all m, n in this range. In particular, taking $m + n = k - 1$, we have $T^{-1} = T^*$. Thus T is unitary. □

THEOREM 2.1 *Upto unitary equivalence, the only reducible homogeneous weighted shift (with non-zero weights) is the unweighted bilateral shift B.*

Proof : We shall see in Section ** that B is homogeneous. Being a non-trivial unitary, it is of course reducible. For the converse, let T be a reducible weighted shift with non-zero weights. Recall that by a Theorem of R. L. Kelly and N. K. Nikolskii, any such operator T is a bilateral shift, and its weight sequence $w_n,\ n \in \mathbb{Z}$ is periodic, say with period $k \geq 1$. That is, $w_{n+k} = w_n$ for all n. (See Problem 129 in [3] as well as [8].) Without loss of generality (replacing T by a unitarily equivalent copy if necessary), we may assume $w_n > 0$ for all n in $\mathbb{Z}$. The spectral radius $r(T)$ of T is given by the following formula (see Theorem 7 and its Corollary in [11]) : $r(T) = \max(r^-, r^+)$ where

$$r^+ = \lim_{n\to\infty} [\sup_{j\geq 0}(w_j w_{j+1} \cdots w_{n+j-1})]^{1/n}$$

and

$$r^- = \lim_{n\to\infty} [\sup_{j<0}(w_{j-1} w_{j-2} \cdots w_{j-n})]^{1/n}.$$

In our case, since the weight sequence w_n is periodic with period k, this formula for the spectral radius reduces to

$$r(T) = (w_0 w_1 \cdots w_{k-1})^{1/k}.$$

Now assume that T is also homogeneous. Then, by Lemma 2.2 of [1], $r(T) = 1$. Thus, $w_0 w_1 \cdots w_{k-1} = 1$. By the periodicity of the weight sequence, it then follows that

$$w_n w_{n+1} \cdots w_{n+k-1} = 1 \quad \forall n \in \mathbb{Z}.$$

Therefore, if $x_n,\ n \in \mathbb{Z}$ is the orthonormal basis such that $Tx_n = w_n x_{n+1}$ for all n, then we get $T^k x_n = x_{n+k} = B^k x_n$ for all n and hence $T^k = B^k$. Since B is unitary, this shows that T^k is unitary. Therefore, by Lemma 2.1, T is unitary. Hence $w_n = \|Tx_n\| = \|x_n\| = 1$ for all n. Thus $T = B$. □

2.3 Associated representations

In this section we make use of the standard notions of projective representations and their equivalence. However, for the sake of completeness we shall reproduce some of these definitions (along with some relevant results on these topics) in the following section.

DEFINITION 2.1 *If T is an operator on a Hilbert space $\mathcal{H}$ then a projective representation π of Möb on $\mathcal{H}$ is said to be associated with T if the spectrum of T is contained in $\bar{\mathbb{D}}$ and*

$$\varphi(T) = \pi(\varphi)^* T \pi(\varphi) \tag{2.3}$$

for all elements φ of Möb.

Clearly, if T has an associated representation then T is homogeneous. In the converse direction, we have the following theorem. It will be necessary in order to take care of the irreducible homogeneous shifts. The 'existence' part of this theorem is one of the main results in [6]. We include proofs of both parts for the sake of completeness and because the original existence proof in [6] uses a powerful selection theorem which is avoided here.

THEOREM 2.2 *If T is an* irreducible *homogeneous operator then T has a projective representation of Möb associated with it. Further, this representation is uniquely determined (upto equivalence) by T.*

Proof: For any $\varphi \in$ Möb , let $\mathcal{E}_\varphi$ be the set of all unitary operators U such that $\varphi(T) = U^*TU$. Since T is homogeneous, $\mathcal{E}_\varphi$ is non-empty for each φ. Also, for $U_1, U_2 \in \mathcal{E}_\varphi$, $U_1^1 U_2$ commutes with T and hence (by the irreducibility of T) is a scalar unitary. Thus each of the sets $\mathcal{E}_\varphi$ is a coset of the circle group $\mathbb{T}$ in $\mathcal{U}(\mathcal{H})$. Choose a Borel map $\pi :$ Möb $\to \mathcal{U}(\mathcal{H})$ such that $\pi(\varphi) \in \mathcal{E}_\varphi$ for all φ. (For instance, this may be done as follows. Fix a countable dense subset $\{f_n : n = 1, 2, \cdots\}$ of the Hilbert space $\mathcal{H}$. Let E denote the subset of $\mathcal{U}(\mathcal{H})$ consisting of all unitaries U such that $\langle Uf_1, f_1\rangle = \cdots \langle Uf_{n-1}, f_{n-1}\rangle = 0$ and $\langle Uf_n, f_n\rangle > 0$ for some $n = 1, 2, \cdots$. Clearly E is a Borel subset of $\mathcal{U}(\mathcal{H})$ which meets every coset of $\mathbb{T}$ in a singleton. Therefore we may choose $\pi(\varphi)$ to be the unique element of $E \cap E_\varphi$, for $\varphi \in$ Möb . It can be shown that if the graph of a map between standard Borel spaces is Borel in the product space, then the map is Borel. Since π defined here satisfies this requirement, it is a Borel map.) For $\varphi_1, \varphi_2 \in$ Möb , $\pi(\varphi_1\varphi_2)\pi(\varphi_2)^{-1}\pi(\varphi_1)^{-1}$ commutes with T and hence is a scalar. Thus π is a projective representation of Möb . By construction, it is associated with T.

Note that if $\pi_1,\ \pi_2$ are two projective representations associated with the same operator T then for each φ in Möb, $\pi_1(\varphi)^{-1}\pi_2(\varphi)$ commutes with T. If T is irreducible then this implies that $\pi_1(\varphi)^{-1}\pi_2(\varphi)$ is a scalar (necessarily of modulus 1) for all φ. Thus if T is an irreducible homogeneous operator, then the associated projective representation is unique upto equivalence. □

For any projective representation π of Möb , let $\pi^\#$ denote the projective representation of Möb obtained by composing π with the automorphism $*$ of Möb (cf. (2.1). That is,

$$\pi^\#(\varphi) := \pi(\varphi^*), \quad \varphi \in \text{Möb}. \tag{2.4}$$

For future use, we note :

PROPOSITION 2.1 *If the projective representation π is associated with a homogeneous operator T then $\pi^\#$ is associated with the adjoint T^* of T. If, further, T is invertible, then $\pi^\#$ is associated with T^{-1} also. It follws that T and T^{*-1} have the same associated representation.*

Proof: This is more or less obvious from (2.2). □

2.4 A construction

Let's say that a projective representation π of Möb is a *multiplier representation* if it is concretely realised as follows. π acts on a Hilbert space $\mathcal{H}$ of E - valued functions on Ω, where Ω is either $\mathbb{D}$ or $\mathbb{T}$ and E is a Hilbert space. The action of π on $\mathcal{H}$ is given by $\Big(\pi(\varphi)f\Big)(z) = c(\varphi, z)f(\varphi^{-1}z)$ for $z \in \Omega,\ f \in \mathcal{H},\ \varphi \in$ Möb. Here c is a suitable Borel function from Möb $\times\, \Omega$ into the Borel group of invertible operators on E.

Theorem 2.3 *Let $\mathcal{H}$ be a Hilbert space of functions on Ω such that the operator T on $\mathcal{H}$ given by*

$$(Tf)(x) = xf(x),\ x \in \Omega,\ f \in \mathcal{H}$$

is bounded. Suppose there is a multiplier representation π of Möb on $\mathcal{H}$. Then T is homogeneous and π is associated with T.

Proof : Let U be a sufficiently small neighbourhood of the identity in Möb so that $\varphi(T)$ makes sense for all $\varphi \in U$. According to [1, Lemma 2.2], it suffices to verify that $T\pi(\varphi) = \pi(\varphi)\varphi(T)$ for all $\varphi \in U$. Notice that for $\varphi \in U$, $\varphi(T)$ is just multiplication by $\varphi : (\varphi(T)f)(x) = \varphi(x)f(x)$. So, we need to verify that for any $x \in \Omega,\ f \in \mathcal{H}$,

$$xc(\varphi, x)f(\varphi^{-1}(x)) = c(\varphi, x)(y \mapsto \varphi(y)f(y))(\varphi^{-1}(x)).$$

But this is trivial. □

This easy but basic construction is from Proposition 2.3 of [1]. To apply this theorem, we only need a good supply of what we have called multiplier representations of Möb. Notice that most of the irreducible projective representations of Möb (as concretely presented in the following section) are multiplier representations.

2.5 Block shifts

Although this paper is essentially about ordinary weighted shifts, along the way we shall need the following more general notion.

Definition 2.2 *Let T be a bounded operator on a Hilbert space $\mathcal{H}$. Then T is called a block shift if there is an orthogonal decomposition $\mathcal{H} = \oplus_{n \in I} W_n$ of $\mathcal{H}$ into non-trivial subspaces W_n, $n \in I$ such that $T(W_n) \subseteq W_{n+1}$. Here $I = \mathbb{Z}, \mathbb{Z}^+$ or $\mathbb{Z}^-$. We say that T is a bi-lateral, forward unilateral or backward unilateral block shift according as $I = \mathbb{Z}, \mathbb{Z}^+$ or $\mathbb{Z}^-$. The subspaces $W_n,\ n \in I$ are called the blocks of T. In the case of backward block shift T, it is understood that $T(V_0) = \{0\}$. Notice that adjoint of a forward block shift is a backward block shift and vice-versa.*

Note that the weighted shifts are simply the block shifts all whose blocks are one-dimensional. To distinguish them from more general block shifts, they are sometimes called the scalar shifts.

One might imagine that the block shifts are too general a class to be of much significance. Indeed, one might think that most (if not all) operators can be realized as block shifts. Therefore the following result, showing that block shifts (at least the irreducible ones) has a very rigid structure, comes as a surprise. In the concluding section we shall see that all irreducible homogeneous operators are block shifts.

Lemma 2.2 *If T is an irreducible block shift then the blocks of T are uniquely determined by T.*

Proof (due to Marc Ordower): Fix an element $\alpha \in \mathbb{T}$ of infinite order (i.e., α is not a root of unity) Let $V_n,\ n \in I$ be blocks of T. Define a unitary operator S by $Sx = \alpha^n x$ for $x \in V_n,\ n \in I$. Notice that by our assumption on α the eigenvalues $\alpha^n,\ n \in I$ of S are distinct and the blocks V_n of T are precisely the eigenspaces of S. If $W_n,\ n \in J$ are also bloocks of T then define another unitary S_1 replacing the blocks V_n by the blocks W_n in the definition of S. A simple computation shows that we have $STS^* = \alpha T = S_1 T S_!^*$ and hence $S_1^* S$ commutes with T. Since $S_1^* S$ is unitary and T is irreducible, it follows from Schur's Lemma that $S_1^* S$ is a scalar. That is, $S_1 = \beta S$ for some $\beta \in \mathbb{T}$. Therefore, S has same eigenspaces as S. thus the blocks of T are uniquely determined as eigenspaces of of S. □

Remark 2.1 *After we conjectured (and were unable to prove) the validity of Lemma 2.2, our colleague V. Pati found a proof in the case of unilateral shifts. Finally, Marc Ordower found the beautiful proof (presented above) which works in all cases. In fact, thhis proof works equally well for commuting tuples of operators.*

Though limited in scope, Pati's proof has the advantage of being 'constructive': it gives an explicit description of the blocks of an irreducible unilateral block shift T in terms of the operator (in comparison, Ordower's proof is 'existential'). We present a brief sketch of this proof.

Let T be an irreducible forward block shift. (To get the proof for the backward case, just apply the following to the adjoint.) Let $\mathcal{S}$ be the multiplicative semigroup generated by T and T^*. Any element S of this semigroup can be written as a word in the letters T and T^*. Define the weight $w(S)$ of S to be the number of T^*s minus the number of Ts in such a word (although the expression of S as a word need not be unique, looking at the action of S on the blocks, it is clear that the weight is well defined). Then it can be shown that the initial block V_0 of T is the intersection of the kernels of the elements of $\mathcal{S}$ of weight 1. Also, for $n > 0$, the nth block V_n is the closed span of the images of the elements of weight $-n$.

3 Projective representations and Multipliers

3.1 Generalities

Through out this section, G is a locally compact second countable topological group. (However, in this paper, our interest is in the case of the Möbius group and its universal cover.) Then a measurable function $\pi : G \to \mathcal{U}(\mathcal{H})$ is called a *projective representation* of G on the Hilbert space $\mathcal{H}$ if there is a function (necessarily Borel) $m : G \times G \to \mathbb{T}$ such that

$$\pi(1) = I, \ \pi(g_1 g_2) = m(g_1, g_2)\pi(g_1)\pi(g_2) \tag{3.1}$$

for all g_1, g_2 in G. (More precisely, such a function π is called a projective unitary representation of G; however, we shall drop the adjective unitary since all representations considered in this paper are unitary.)

Two projective representations $\pi_1, \ \pi_2$ of G on the Hilbert spaces $\mathcal{H}_1$, $\mathcal{H}_2$ (respectively) will be called equivalent if there exists a unitary operator $U : \mathcal{H}_1 \to \mathcal{H}_2$ and a function (necessarily Borel) $\gamma : G \to \mathbb{T}$ such that $\pi_2(g)U = \gamma(g)U\pi_1(g)$ for all $g \in G$. We shall identify two projective representations if they are equivalent. Recall that a projective representation π of G is called *irreducible* if the unitary operators $\pi(g), \ g \in G$ have no common non-trivial reducing subspace. Clearly equivalence respects this property.

The function m associated with the projective representation π via (3.1) is called the multiplier of π. Clearly $m : G \times G \to \mathbb{T}$ is a Borel map. In view of Equation (3.1), m satisfies

$$m(g,1) = 1 = m(1,g), \quad m(g_1,g_2)m(g_1g_2,g_3) = m(g_1,g_2g_3)m(g_2,g_3). \tag{3.2}$$

for all group elements g, g_1, g_2, g_3. Any Borel function m into $\mathbb{T}$ satisfying Equation 3.2 is called a multiplier on the group. The multipliers form an abelian group under pointwise multiplication. This is called the multiplier group.

Recall that π is called an ordinary representation (and we drop the adjective "projective") if its multiplier is the constant function 1. The ordinary representation π which sends every group element to the identity operator on a one dimensional Hilbert space is called the identity (or trivial) representation. The following definition of equivalence of multipliers is standard (see [12] for instance):

Definition 3.1 *Two multipliers m and $\tilde{m}$ on the group G are called equivalent if there is a Borel function $\gamma : G \to \mathbb{T}$ such that $\gamma(g_1g_2)\tilde{m}(g_1,g_2) = \gamma(g_1)\gamma(g_2)m(g_1,g_2)$ for all g_1, g_2 in G.*

Clearly equivalent projective representations have equivalent multipliers. The multipliers equivalent to the trivial multiplier (viz. the constant function 1) are called exact. The exact multipliers form a subgroup of the multiplier group. The quotient is called the second cohomology group $H^2(G,\mathbb{T})$ (with respect to the trivial action of G on $\mathbb{T}$).

We shall need :

Theorem 3.1 *Let G be a connected semi-simple Lie group. Then every projective representation of G (say with multiplier m) is a direct integral of irreducible projective representations (all with the same multiplier m) of G.*

Proof: Let π be a projective representation of G. Let $\tilde{G}$ be the universal cover of G and let $p : \tilde{G} \to G$ be the covering homomorphism. Define a projective representation π_0 of $\tilde{G}$ by $\pi_0(\tilde{x}) = \pi(x)$ where $x = p(\tilde{x})$. A trivial computation shows that π_0 is indeed a projective representation of $\tilde{G}$ and its multiplier m_0 is given by $m_0(\tilde{x},\tilde{y}) = m(x,y)$ where $x = p(\tilde{x}),\ y = p(\tilde{y})$.

However, since $\tilde{G}$ is a connected and simply connected Lie group, $H^2(\tilde{G},\mathbb{T})$ is trivial. (This is an easy and well known consequence of Theorem 7.37 in [12] in conjunction with the Levy-Malcev theorem). Therefore, m_0 is exact. That is, there is a Borel function $\gamma : \tilde{G} \to \mathbb{T}$ such that

$$m(x,y) = m_0(\tilde{x},\tilde{y}) = \gamma(\tilde{x})\gamma(\tilde{y})/\gamma(\tilde{x}\tilde{y}) \tag{3.3}$$

for all $\tilde{x}, \tilde{y}$ in $\tilde{G}$, and $x = p(\tilde{x}),\ y = p(\tilde{y})$. Now define the ordinary representation $\tilde{\pi}$ of $\tilde{G}$ (equivalent to π_0) by : $\tilde{\pi}(\tilde{x}) = \gamma(\tilde{x})\pi_0(\tilde{x})$, for $\tilde{x}$ in $\tilde{G}$. Now, since $\tilde{G}$ is a locally compact and second countable group, by Theorem 2.9 in [4], the ordinary representation $\tilde{\pi}$ of $\tilde{G}$ may be written as a direct integral of (ordinary) irreducible representations $\tilde{\pi}_t$ of $\tilde{G}$: $\tilde{\pi}(\tilde{x}) = \int^{\oplus} \tilde{pi}_t(\tilde{x})d\ P(t),\quad \tilde{x} \in \tilde{G}$. Replacing $\tilde{\pi}$ by its definition in terms of π, we get that for each $x \in G$, $\pi(x) = \int^{\oplus} \gamma(\tilde{x})^{-1}\tilde{\pi}_t d\ P(t)$ for any $\tilde{x}$ such that $x = p(\tilde{x})$. So we would like to define $\pi_t : G \to \mathcal{U}(\mathcal{H})$ by $\pi_t(x) = \gamma(\tilde{x})^{-1}\tilde{\pi}_t(\tilde{x})$ for any $\tilde{x}$ as above and verify that π_t, thus defined, is an irreducible projective representation of G with multiplier m. But first we must show that π_t is well defined. That is, if $\tilde{x}$ and $\tilde{y}$ are elements of $\tilde{G}$ mapping into the same element x of G under p then we need to show

$$\gamma(\tilde{x})^{-1}\tilde{\pi}_t(\tilde{x}) = \gamma(\tilde{y})^{-1}\tilde{\pi}_t(\tilde{y}) \tag{3.4}$$

Let $\tilde{Z}$ be the kernel of the covering map p. Since $\tilde{Z}$ is a discrete normal subgroup of the connected topological group $\tilde{G}$, $\tilde{Z}$ is a central subgroup of $\tilde{G}$. Since for each t, $\tilde{\pi}_t$ is irreducible, it follows by Schur's lemma that there is a Borel function (indeed a continuous character of $\tilde{Z}$) $\gamma_t : \tilde{Z} \to \mathbb{T}$ such that $\tilde{\pi}_t(\tilde{z}) = \gamma_t(\tilde{z})I$ for all $\tilde{z} \in \tilde{Z}$. Also, we have $\tilde{\pi}(\tilde{z}) = \gamma(\tilde{z})\pi_0(\tilde{z}) = \gamma(\tilde{z})\pi(1) = \gamma(\tilde{z})I$ for all $\tilde{z} \in \tilde{Z}$. Therefore, evaluating $\tilde{\pi}(\tilde{z})$ using its direct integral representation, we find $\gamma(\tilde{z})I = \int^{\oplus} \gamma_t(\tilde{z})Id\ P(t)$ and hence $\gamma_t(\tilde{z}) = \gamma(\tilde{z})$ for all t ina set of full P-measure and all $\tilde{z} \in \tilde{Z}$ (Note that, being a discrete subgroup of the separable group $\tilde{G}$, $\tilde{Z}$ is countable.) Replacing the domain of integration by this subset if need be, we may assume that $\gamma_t = \gamma$ for all t. Thus,

$$\tilde{\pi}_t(\tilde{z}) = \gamma(\tilde{z})I \tag{3.5}$$

for all $\tilde{z}$ in $\tilde{Z}$ and for all t. Also, for $\tilde{x} \in \tilde{G}$ and $\tilde{z} \in \tilde{Z}$, we have $\gamma(\tilde{x})\gamma(\tilde{z})/\gamma(\tilde{x}\tilde{z}) = m_0(\tilde{x}, \tilde{z}) = m(x, 1) = 1$ (where $x = p(\tilde{x})$) and hence

$$\gamma(\tilde{x}\tilde{z}) = \gamma(\tilde{x})\gamma(\tilde{z}). \tag{3.6}$$

Now we come back to the proof of Equation (3.4). Since $p(\tilde{x}) = p(\tilde{y})$ there is a $\tilde{z} \in \tilde{Z}$ such that $\tilde{y} = \tilde{x}\tilde{z}$. Therefore $\gamma(\tilde{y})^{-1}\tilde{\pi}_t(\tilde{y}) = \gamma(\tilde{x})^{-1}\gamma(\tilde{z})^{-1}\tilde{\pi}_t(\tilde{x})\tilde{\pi}_t(\tilde{z})$ (by Equation (3.6)) $= \gamma(\tilde{x})^{-1}\tilde{\pi}_t(\tilde{x})$ (by Equation 3.5). This proves Equation 3.4 and hence shows that π_t is well defined.

Now, for $x, y \in G$,

$$\begin{aligned}
\pi_t(xy) &= \gamma(\tilde{x}\tilde{y})\tilde{\pi}_t(\tilde{x}\tilde{y}) \\
&= \gamma(\tilde{x}\tilde{y})\tilde{\pi}_t(\tilde{x})\tilde{\pi}_t(\tilde{y}) \\
&= (\gamma(\tilde{x})\gamma(\tilde{y})/\gamma(\tilde{x}\tilde{y}))\pi_t(x)\pi_t(y) \\
&= m_0(\tilde{x}, \tilde{y})\pi_t(x)\pi_t(y) \\
&= m(x, y)\pi_t(x)\pi_t(y),
\end{aligned}$$

where $\tilde{x}, \tilde{y}$ in $\tilde{G}$ are such that $x = p(\tilde{x}),\ y = p(\tilde{y})$. This shows that π_t is indeed a projective representation of G with multiplier m. Since from the definition of π_t it is clear that π_t and $\tilde{\pi}_t$ have the same invariant subspaces, and since the latter is irreducible, it follows that each π_t is irreducible. Thus we have the required decomposition of π as a direct integral of irreducible projective representations π_t with the same multiplier as π : $\pi = \int^{\oplus} \pi_t d\ P(t)$. □

As a consequence of (the proof of) Theorem 3.1, we have the following corollary. Here, as above $\tilde{G}$ is the universal cover of G, $p : \tilde{G} \to G$ is the covering map. Fix a Borel section $s : G \to \tilde{G}$ for p (that is, s is a Borel function such that $p \circ s$ is the identity function on G) such that $s(1) = 1$. Note that the kernel $\tilde{Z}$ of p is naturally identified with the fundamental group $\pi^1(G)$ of G. Define the map $\alpha : G \times G \to \tilde{Z}$ by :

$$\alpha(x, y) = s(xy)s(y)^{-1}s(x)^{-1}, \quad x, y \in G. \tag{3.7}$$

For any character (i.e., continuous homomorphism into the circle group $\mathbb{T}$) of $\pi^1(G)$, define $m_\chi : G \times G \to \mathbb{T}$ by

$$m_\chi(x, y) = \chi(\alpha(x, y)), \quad x, y \in G.$$

Since $\tilde{Z}$ is a central subgroup of $\tilde{G}$, it is easy to verify that α satisfies the multiplier identity 3.2. Hence m_χ is a multiplier on G for each character χ of $\tilde{Z}$.

Let $H_1(G)$ denote the first homology (with integer coefficients) of G as a manifold. Since $H_1(G)$ is the abelianisation of $\pi^1(G)$, the group of characters χ of $\pi^1(G)$ may be identified with the Pontryagin dual $\widehat{H_1(G)}$.

Finally, for any multiplier m on G, let $[m]$ denote its image in $H^2(G, \mathbb{T})$ under the quotient map. In terms of these notations, we have :

COROLLARY 3.1 *Let G be a connected semi-simple Lie group. Then the multipliers m_χ are mutually inequivalent, and every multiplier on G is equivalent to m_χ for a unique character χ. In other words, $\chi \mapsto [m_\chi]$ defines a group isomorphism :*

$$H^2(G, \mathbb{T}) \equiv \mathrm{Hom(H_1(G), \mathbb{T})}.$$

Proof: Let m be any multiplier on G. Define a projective representation π of G on the Hilbert space $L^2(G)$ by :

$$(\pi(g)F)(x) = m(x, g)F(xg), \quad g, x \in G, \ F \in L^2(G).$$

Then, using the defining equation (3.2) for a multiplier, it is easy to verify that π is indeed a projective representation of G and the multiplier associated with π is m. Therefore, the calculations done in proving Theorem 3.1 apply to m. Let χ denote the restriction to $\tilde{Z}$ of the Borel map γ which occurs in this proof. Equation (3.6) implies, in particular, that χ is a character of $\tilde{Z}$. Define the Borel map $f : G \to \mathbb{T}$ by $f = \gamma \circ s$. Then, for $x, y \in G$ ($s(xy)s(y)^{-1}s(x)^{-1} \in \tilde{Z}$ and hence) Equation (3.6) gives $f(xy) = \gamma(s(x)s(y))m_\chi(x, y)$. Also, Equation (3.3) (with the choice $\tilde{x} = s(x)$, $\tilde{y} = s(y)$) gives $m(x, y) = f(x)f(y)/\gamma(s(x)s(y))$. Hence $m(x, y) = \frac{f(x)f(y)}{f(xy)} m_\chi(x, y)$. Thus the multiplier m is equivalent to m_χ.

Finally, since $\chi \mapsto m_\chi$ is a group homomorphism, to show that the multipliers m_χ are mutually inequivalent, it suffices to show that $m_\chi \equiv 1$ implies that χ is the trivial character. So let χ be a character of $\tilde{Z}$ such that m_χ is exact. Hence there is a Borel function $g : G \to \mathbb{T}$ such that $m_\chi(x, y) = g(x)g(y)/g(xy)$ for $x, y \in G$. Hence we have

$$m_\chi(p(\tilde{x}), p(\tilde{y})) = h(\tilde{x})h(\tilde{y})/h(\tilde{x}\tilde{y})$$

for $\tilde{x}, \tilde{y} \in \tilde{Z}$. Here the Borel function $h : \tilde{G} \to \mathbb{T}$ is given by $h = g \circ p$. But, by Equation (3.3) (with $m = m_\chi$, $x = p(\tilde{x})$, $y = p(\tilde{y})$) shows that

$$m_\chi(p(\tilde{x}), p(\tilde{y})) = \gamma(\tilde{x})\gamma(\tilde{y})/\gamma(\tilde{x}\tilde{y})$$

for $\tilde{x}, \tilde{y} \in \tilde{G}$. Comparing these two equations we see that γ/h is a character of *tildeG*. But there is no non-trivial character of $\tilde{G}$. (A semi-simple Lie group is its own commutator, so there is no non-trivial homomorphism from such a group into any abelian group.) Therefore $\gamma = h = g \circ p$. But $g \circ p$ is a constant function on the kernel $\tilde{Z}$ of p, while the restriction of γ to $\tilde{Z}$ is the character χ thus χ is trivial. □

REMARK 3.1 *(a) The isomorphism $\chi \mapsto [m_\chi]$ in Corollary 3.1 appears to depend on the choice of the section s. But it is quite easy to prove that actually there is no such dependence. Thus the isomorphism of this corollary is a natural one.*

(b) The beginning of the proof of Corollary 3.1 shows that any multiplier m on a locally compact second countable group G is actually associated with ('comes from') some projective

unitary representation of G. In conjunction with Theorem 3.1, it then follows that if G is a connected semi-simple Lie group then any multiplier of G comes from an irreducible projective unitary representation.

(c) Let G be a connected semi-simple Lie group and let $\tilde{G}$ be its universal cover. Also, let $\tilde{Z}$ be as above. Finally, let χ be a character of $\tilde{Z}$. Let us say that an ordinary projective representation $\tilde{\pi}$ of $\tilde{G}$ is of pure of type χ *if $\tilde{\pi}(\tilde{z}) = \chi(\tilde{z})I$ for all $\tilde{z} \in \tilde{Z}$. The proof of Theorem 3.1 shows that there is a natural bijection $\pi \mapsto \tilde{\pi}$ between the (equivalence classes of) projective representations of G and the (equivalence classes of) pure ordinary representations of $\tilde{G}$. Further, under this bijection, the projective representations with multiplier m_χ correspond to the representations of pure type χ. Finally (since in general π and $\tilde{\pi}$ have the same invariant subspaces, and since by Schur Lemma the irreducible representations of $\tilde{G}$ are pure) the irreducible projective representations of G are in bijection with the irreducible representations of $\tilde{G}$ under this map.*

3.2 The irreducible representations of the Möbius group

In view of Theorem 3.1, to understand all the projective representations of Möb it suffices to know its irreducible projective representations. Most of these representations happen to arise out of the following construction.

For φ in Möb, φ' is a non-vanishing analytic function on $\bar{\mathbb{D}}$. Hence there is an analytic branch of $\log \varphi'$ on $\bar{\mathbb{D}}$. For the rest of this paper, fix such a branch for each φ such that (a) for $\varphi = 1$, $\log \varphi' \equiv 0$ and (b) the map $(\varphi, z) \mapsto \log \varphi'(z)$ from Möb $\times\, \bar{\mathbb{D}}$ into $\mathbb{C}$ is a Borel function. With such a determination of the logarithm, we define the functions $(\varphi')^{\lambda/2}$ (for any fixed real number $\lambda > 0$) and $\arg\varphi'$ on $\bar{\mathbb{D}}$ by $\varphi'(z)^{\lambda/2} = \exp(\frac{\lambda}{2} \log \varphi'(z)$ and $\arg\varphi'(z) = \operatorname{Im} \log\varphi'(z)$. (If our determination of the logarithms are changed then - it is easy to see - the representations of Möb introduced below as well as the multipliers on Möb defined in the next subsection remain unchanged modulo equivalence.)

For $n \in \mathbb{Z}$, let $f_n : \mathbb{T} \to \mathbb{T}$ be defined by $f_n(z) = z^n$. In all of the following examples, the Hilbert space $\mathcal{H}$ is spanned by an orthogonal set $\{f_n : n \in I\}$ where I is some subset of $\mathbb{Z}$. Thus the Hilbert space of functions is specified by the set I and $\{\|f_n\|, n \in I\}$. (In each case, $\|f_n\|$ behaves at worst like a polynomial in $|n|$ as $n \to \infty$, so that this really defines a space of function on $\mathbb{T}$.) For $\varphi \in$ Möb and complex parameters λ and μ, define the operator $R_\lambda, \mu(\varphi^{-1})$ on $\mathcal{H}$ by

$$(R_{\lambda,\mu}(\varphi^{-1})f)(z) = \varphi'(z)^{\lambda/2}|\varphi'(z)|^{\mu}(f(\varphi(z)), \quad z \in \mathbb{T}, \ f \in \mathcal{H}, \ \varphi \in \text{Möb}.$$

Of course, there is no a priori gurantee that this is a unitary (or even bounded) operator. But, when it is, it is easy to see that $R_{\lambda,\mu}$ is then a projective representation of Möb. (See the proof of Theorem 3.2 below.) Thus the description of the representation is complete if we specify I, $\{\|f_n\|^2, n \in I\}$ and the two parameters λ, μ. It turns out that all the irreducible projective representation of Möb have this form (excepting the anti-holomorphic discrete series representations which are of the form $R^{\#}_{\lambda,\mu}$).

By Remark 3.1(c), there is a natural bijection between the irreducible projective representations of Möb and the irreducible (ordinary) representations of its universal cover. But a complete list of the irreducible representations (upto equivalence) of the universal cover of Möb was obtained by Bargmann (see [10], for instance). Hence one obtains a complete list of the irreducible projective representations of Möb. This is as follows. (However, see Remark 3.2 of this section.)

LIST 3.1 *Holomorphic discrete series representations* D_λ^+. Here $\lambda > 0,\ \mu = 0,\ I = \mathbb{Z}^+$ and $\|f_n\|^2 = \frac{\Gamma(n+1)\Gamma(\lambda)}{\Gamma(n+\lambda)}$ for $n \geq 0$. For each f in the representation space there is an $\tilde{f}$, analytic in $\mathbb{D}$, such that f is the non-tangential boundary value of $\tilde{f}$. By the identification $f \leftrightarrow \tilde{f}$, the representation space may be identified with the functional Hilbert space $\mathcal{H}^{(\lambda)}$ of analytic functions on $\mathbb{D}$ with reproducing kernel $(1 - z\bar{w})^{-\lambda},\ z, w \in \mathbb{D}$.

Anti-holomorphic discrete series representations D_λ^-, $\lambda > 0$. D_λ^- may be defined as the composition of D_λ^+ with the automorphism $*$ of Equation (2.1). Thus, $D_\lambda^- = (D_\lambda^+)^\#$ (recall Equation (2.4)). This may be realized on a functional Hilbert space of anti-holomorphic functions on $\mathbb{D}$, in a natural way.

Principal series representations $P_{\lambda,s}$, $-1 < \lambda \leq 1$, *s purely imaginary.* Here $\lambda = \lambda,\ \mu = \frac{1-\lambda}{2} + s,\ I = \mathbb{Z},\ \|f_n\|^2 = 1$ for all n. (so the space is $L^2(\mathbb{T})$).

Complementary series representation $C_{\lambda,\sigma}$, $-1 < \lambda < 1,\ 0 < \sigma < \frac{1}{2}(1 - |\lambda|)$. Here $\lambda = \lambda,\ \mu = \frac{1}{2}(1 - \lambda) + \sigma,\ I = \mathbb{Z}$, and

$$\|f_n\|^2 = \prod_{k=0}^{|n|-1} \frac{k \pm \frac{\lambda}{2} + \frac{1}{2} - \sigma}{k \pm \frac{\lambda}{2} + \frac{1}{2} + \sigma},\ n \in \mathbb{Z},$$

where one takes the upper or lower sign according as n is positive or negative.

REMARK 3.2 *All the projective representations in the List 3.1 are mutually inequivalent. Moreover, they are all irreducible with the sole exception of* $P_{1,0}$ *for which we have the decomposition* $P_{1,0} = D_1^+ \oplus D_1^-$.

3.3 THE MULTIPLIERS OF THE MÖBIUS GROUP

Next we describe the multipliers of Möb upto equivalence. Let's define the Borel function $\boldsymbol{n}$: Möb $\times$ Möb $\to \mathbb{Z}$ by

$$\boldsymbol{n}(\varphi_1^{-1}, \varphi_2^{-1}) = \frac{1}{2\pi}(\arg(\varphi_2\varphi_1)'(0) - \arg\varphi_1'(0) - \arg\varphi_2'(\varphi_1(0))). \tag{3.8}$$

The chain rule implies that this is indeed an integer valued function. For any $\omega \in \mathbb{T}$, define m_ω : Möb $\times$ Möb $\to \mathbb{T}$ by

$$m_w(\varphi_1, \varphi_2) = \omega^{\boldsymbol{n}(\varphi_1,\varphi_2)}. \tag{3.9}$$

Then we have :

THEOREM 3.2 *(a)* m_ω *is a multiplier of Möb for each* $\omega \in \mathbb{T}$. *Upto equivalence,* m_ω, $\omega \in \mathbb{T}$ *are all the multipliers on Möb; further, these are mutually inequivalent multipliers. In other words,* $H^2(\text{Möb}, \mathbb{T})$ *is naturally isomorphic to* $\mathbb{T}$ *via the map* $\omega \mapsto m_\omega$.

(b) For each of the representations of Möb in List 3.1, the associated multiplier is m_w, *where (in terms of the parameter* λ *of the representation)* $w = e^{i\pi\lambda}$ *in each case, except for the anti-holomorphic discrete series representation(s) for which* $w = e^{-i\pi\lambda}$

Proof : We first prove Part (b). Let $\pi = R_{\lambda,\mu}$ be a representation in List 3.1. Thus π is not in the anti-holomorphic discrete series. From the definition of $R_{\lambda,\mu}$, one calculates that the associated multiplier m is given by :

$$m(\varphi_1^{-1}, \varphi_2^{-1}) = \frac{\Big((\varphi_2\varphi_1)'(z)\Big)^{\lambda/2}}{(\varphi_1'(z))^{\lambda/2}(\varphi_2'(\varphi_1(z)))^{\lambda/2}},\quad z \in \mathbb{T},$$

for any two elements φ_1, φ_2 of Möb. Notice that the right hand side of this equation is an analytic function of z for z in $\bar{\mathbb{D}}$ and it is of constant modulus 1 in view of the chain rule for differentiation. Therefore, by the maximum modulus principle, this formula is independent of z for z in $\bar{\mathbb{D}}$. Hence we may take $z = 0$ in this formula. This yields $m = m_\omega$ with $\omega = e^{i\pi\lambda}$. Notice that if m is the multiplier associated with the representation π then the multiplier associated with $\pi^\#$ is $\bar{m}$. Since $D^-_\lambda = D^{+\#}_\lambda$, it follows that if $\pi = D^-_\lambda$ is in the anti-holomorphic discrete series, then its multiplier is m_ω where $\omega = e^{-i\pi\lambda}$.

This argument also shows that m_ω is indeed a multiplier of Möb for each $\omega \in$ Möb . Further, since these multipliers include all the multipliers of Möb associated with irreducible projective representations, Remark 3.1(b) shows that modulo equivalence these are all the multipliers on Möb. Unfortunately, it seems very hard to see directly that the multipliers m_ω, $\omega \in \mathbb{T}$ are mutually inequivalent. (Since $\omega \mapsto [m_\omega]$ is clearly a group homomorphism from $\mathbb{T}$ onto $H^2(\text{Möb}, \mathbb{T})$, this amounts to verifying that m_ω is never exact for $\omega \neq 1$.) This fact may be deduced from Corollary 3.1 as follows.

Identify Möb with $\mathbb{T} \times \mathbb{D}$ via $\varphi_{\alpha,\beta} \mapsto (\alpha, \beta)$. The group law on $\mathbb{T} \times \mathbb{D}$ is given by :

$$(\alpha_1, \beta_1)(\alpha_2, \beta_2) = \Big(\alpha_1\alpha_2 \cdot \frac{1 + \bar{\alpha}_2\beta_1\bar{\beta}_2}{1 + \alpha_2\bar{\beta}_1\beta_2}, \frac{\beta_1 + \alpha_2\beta_2}{\alpha_2 + \beta_1\bar{\beta}_2}\Big).$$

The identity in $\mathbb{T} \times \mathbb{D}$ is $(1, 0)$ and the inverse map is $(\alpha, \beta)^{-1} = (\bar{\alpha}, -\alpha\beta)$.

Then the universal cover $\widetilde{\text{Möb}}$ is naturally identified with $\mathbb{R} \times \mathbb{D}$. Taking the covering map $p : \mathbb{R} \times \mathbb{D} \to \mathbb{T} \times \mathbb{D}$ to be $p(t, \beta) = (e^{2\pi it}, \beta)$, the group law on $\mathbb{R} \times \mathbb{D}$ is determined, by (continuity and) the requirement that p be a group homomorphism, as follows :

$$(t_1, \beta_1)(t_2, \beta_2) = \Big(t_1 + t_2 + \frac{1}{\pi}\text{Im Log}(1 + e^{-2\pi it_2}\beta_1\bar{\beta}_2), \frac{\beta_1 + e^{2\pi it_2}\beta_2}{e^{2\pi it_2} + \beta_1\bar{\beta}_2}\Big),$$

where 'Log' denotes the principal branch of the logarithm on the right half plane. The identity in $\mathbb{R} \times \mathbb{D}$ is $(0, 0)$ and the inverse map is $(t, \beta)^{-1} = (-t, -e^{2\pi it}\beta)$. The kernel $\tilde{Z}$ of the covering map p is identified with the additive group $\mathbb{Z}$ via $n \mapsto (n, 0)$.

Let's choose a Borel branch $\arg : \mathbb{T} \to \mathbb{R}$ of the argument function satisfying $\arg(\bar{z}) = -\arg(z)$, $z \in \mathbb{T}$. Let's then make an explicit choice of the Borel function $(\varphi, z) \mapsto \arg(\varphi'(z)$ (which occurs in the definition of $\boldsymbol{n}$ in Equation 3.8) as follows :

$$\arg\varphi'_{\alpha,\beta}(z) = \arg(\alpha) - 2\text{Im Log}(1 - \beta z).$$

Let's also choose the section $s : \mathbb{T} \times \mathbb{D} \to \mathbb{R} \times \mathbb{D}$ as follows : $s(\alpha, \beta) = (\frac{1}{2\pi}\arg(\alpha), \beta)$. An easy computation shows that, for these choices, we have $s(\varphi_1\varphi_2)s(\varphi_2)^{-1}s(\varphi_1)^{-1} = -\boldsymbol{n}(\varphi_1, \varphi_2)$ for φ_1, φ_2 in Möb. Hence we get that, for $\omega \in \mathbb{T}$, $m_\omega = m_\chi$ where $\chi = \chi_\omega$ is the character $nmapsto\omega^{-n}$ of $\mathbb{Z}$. Thus the map $\omega \mapsto [m_\omega]$ is but a special case of the isomorphism $\chi \mapsto m_\chi$ of Corollary 3.1. □

3.4 The simple representations of the Möbius group

Let $\mathbb{K}$ be the maximal compact subgroup of Möb given by $\{\varphi_{\alpha,0} : \alpha \in \mathbb{T}\}$. Of course, $\mathbb{K}$ is isomorphic to the circle group $\mathbb{T}$ via $\alpha \mapsto \varphi_{\alpha,0}$.

Definition 3.2 *Let π be a projective representation of Möb. We shall say π is normalised if $\pi|_{\mathbb{K}}$ is an ordinary representation of $\mathbb{K}$.*

Lemma 3.1 *Any projective representation of Möb is equivalent to a normalised representation.*

Proof: Take any projective representation σ of Möb. Then $\sigma|_{\mathbb{K}}$ is a projective representation of $\mathbb{K}$, say, with multiplier m. But $H^2(\mathbb{K})$ is trivial [9,]. So, there exists Borel function $f : \mathbb{K} \to \mathbb{T}$ such that

$$m(x,y) = \frac{f(x)f(y)}{f(xy)}, \quad x, y \in \mathbb{K}.$$

Extend f to a Borel function g : Möb $\to \mathbb{T}$. Define π by $\pi(x) = g(x)\sigma(x)$, $x \in$ Möb. Then π is normalised and equivalent to σ. $\square$

Notation 3.1 *For $n \in \mathbb{Z}$, let χ_n be the character of $\mathbb{T}$ given by $\chi_n(x) = x^{-n}$, $x \in \mathbb{T}$. For any normalised projective representation π of Möb and $n \in \mathbb{Z}$, let*

$$V_n(\pi) = \{v \in \mathcal{H} : \pi(x)v = \chi_n(x)v, \quad \forall\ x \in \mathbb{T}\}.$$

Then $\mathcal{H} = \oplus_{n\in\mathbb{Z}} V_n(\pi)$. The subspaces $V_n(\pi)$ are usually called the $\mathbb{K}$-isotypic subspaces of $\mathcal{H}$. Put

$$d_n(\pi) = \dim V_n(\pi) \text{ and } \mathcal{T}(\pi) = \{n \in \mathbb{Z} : d_n(\pi) \neq 0\}.$$

Definition 3.3 *(a) A subset A of $\mathbb{Z}$is said to be connected if for any three elements $a < b < c$ in $\mathbb{Z}$, $a, c \in A$ implies $b \in A$. If B is any subset of $\mathbb{Z}$, a connected component of B is a maximal connected subset of B (with respect to set inclusion). Since the union of two intersecting connected sets is clearly connected, the connected components of a set partition the set.*

(b) Let π be a normalised projective representation of Möb. We shall say that π is connected if $\mathcal{T}(\pi)$ is connected. π will be called simple if if π is connected and, further, $d_n(\pi) \leq 1$ for all $n \in \mathbb{Z}$. More generally, a projective representation is connected/simple if it is equivalent to a connected/simple (normalised) representation.

Remark 3.3 *Notice that if π and σ are equivalent normalised representations then there is an integer h such that $V_n(\sigma) = V_{n+h}(\pi)$ for all n in $\mathbb{Z}$. Consequently, $\mathcal{T}(\sigma)$ is an additive translate of $\mathcal{T}(\pi)$. Hence σ is connected/simple if and only if π is. Thus the definitions given above are consistent.*

Lemma 3.2 *Let π be any normalised projective representation of Möb. Then each connected component of $\mathcal{T}(\pi)$ is unbounded.*

Proof: By Theorem 3.1, we may write

$$\pi = \int^{\oplus} \pi_t \, dP(t),$$

where P is a regular measure and π_t is an irreducible projective representation of Möb for all t. An inspection of the entries in List 3.1 shows that $\mathcal{T}(\pi_t)$ is connected and unbounded for each t. So it suffices to show that the same must be true of their direct integral π. To this end, we claim that, for each n in $\mathbb{Z}$,

$$V_n(\pi) = \int^{\oplus} V_n(\pi_t) \, dP(t). \tag{3.10}$$

Indeed, the inclusion $\supseteq$ is trivial. To prove the inclusion $\subseteq$, take $v \in V_n(\pi)$. Then $v = \int^{\oplus} v_t \, dP(t)$ for some $v_t \in \mathcal{H}_t$. Consequently,

$$\begin{aligned}\int^{\oplus} \chi_n(x) v_t \, dP(t) &= \chi_n(x) v \\ &= \pi(x) v \\ &= \int^{\oplus} \pi_t(x) v_t \, dP(t).\end{aligned}$$

This implies that $\chi_n(x)v_t = \pi_t(x)v_t$ for almost all t. Therefore, $v_t \in V_n(\pi_t)$ for almost all t. This proves Claim (3.10).

Therefore, $n \in \mathcal{T}(\pi)$ if and only if $V_n(\pi_t) \neq 0$ for t in a set of positive P measure. Now suppose some component of $\mathcal{T}(\pi)$ is bounded. Then there exists $a < b < c$ in $\mathbb{Z}$such that b is in $\mathcal{T}(\pi)$ but a and c are not in $\mathcal{T}(\pi)$. It follows that a and c are not in $\mathcal{T}(\pi_t)$ for almost all t but b is in $\mathcal{T}(\pi_t)$ for all t in a set of positive measure. Therefore, there is a t for which $b \in \mathcal{T}(\pi_t)$ but $a, c \notin \mathcal{T}(\pi_t)$. Then the component of $\mathcal{T}(\pi_t)$ containing b is bounded. Contradiction. $\square$

THEOREM 3.3 *Upto equivalence, the only simple projective representations of Möb are the irreducible projective representations of Möb and the representations* $D^+_\lambda \oplus D^-_{2-\lambda}$, $0 < \lambda < 2$.

Proof: If π is irreducible then we have nothing to prove. So, assume $\pi = \pi_1 \oplus \pi_2$. By Equation (3.10), we have $V_n(\pi) = V_n(\pi_1) \oplus V_n(\pi_2)$. We have $d_n(\pi_1) + d_n(\pi_2) = d_n(\pi) \leq 1$. Since π is simple, we have $\mathcal{T}(\pi) = \mathcal{T}(\pi_1) \cup \mathcal{T}(\pi_2)$, $\mathcal{T}(\pi_1) \cap \mathcal{T}(\pi_2) = \emptyset$. Therefore, by Lemma 3.2, the connected components of $\mathcal{T}(\pi)$, $i = 1, 2$ together form a collection of pairwise disjoint unbounded connected sets. Since three unbounded connected subsets of $\mathbb{Z}$cannot be pairwise disjoint, it follows that this collection contains atmost (and hence exactly) two sets. Thus both π_1 and π_2 are (connected and hence) simple. Since the connected set $\mathcal{T}(\pi)$ is the disjoint union of the two unbounded connected sets $\mathcal{T}(\pi_1)$ and $\mathcal{T}(\pi_2)$, it follows that $\mathcal{T}(\pi) = \mathbb{Z}$. In consequence, (upto interchanging of π_1 and π_2) the connected sets $\mathcal{T}(\pi_1)$ (respectively $\mathcal{T}(\pi_2)$) must be bounded below (respectively bounded above).

The argument so far shows, in particular, that whenever simple projective representation π is reducible, $\mathcal{T}(\pi) = \mathbb{Z}$ is forced. Since π_1 and π_2 are simple but $\mathcal{T}(\pi_i)$ is a proper subset of $\mathbb{Z}$, it follows that π_1 and π_2 are irreducible. From the complete list of irreducible projective representations of Möb in Section 2.1, one sees that, upto equivalence, the only irreducible projective representations π_1 (respectively π_2) for which $\mathcal{T}(\pi_1)$ (respectively $\mathcal{T}(\pi_2)$ is bounded below (respectively above) are the holomorphic (repectively anti-holomorphic) discrete series representations. Therefore, there are positive real numbers λ and μ such that π_1 and π_2 are equivalent to D^+_λ and D^-_μ, respectively. Since D^+_λ and D^-_μ occur as a subrepresentation of a common projective representation (viz. an equivalent copy of π), they must have a common multiplier. In view of Remark **??**, this means $e^{-\pi i \lambda} = e^{\pi i \mu}$. Thus $\lambda + \mu$ is an even integer. Now, a computation shows that (upto additive translation) for $\pi = D^+_\lambda \oplus D^-_\mu$, we have

$$\mathcal{T}(\pi) = \{n \in \mathbb{Z} : n \geq 0\} \cup \{n \in \mathbb{Z} : n \leq -(\lambda + \mu)/2\}.$$

Since $\mathcal{T}(\pi) = \mathbb{Z}$, we must have $\lambda + \mu = 2$. Thus upto equivalence $\pi = D^+_\lambda \oplus D^-_{2-\lambda}$, $0 < \lambda < 2$. $\square$

4 Examples of homogeneous weighted shifts

Now we present a list of homogeneous weighted shifts. Later in this paper we shall see that this list is exhaustive.

List 4.1 *Principal series example.* The unweighted bilateral shift B (i.e., the bilateral shift with weight sequence $w_n = 1, \quad n \in \mathbb{Z},$) is homogeneous. To see this, apply Theorem 2.3 to any of the principal series representations of Möb. Being normal (in fact unitary) this operator is far from irreducible. By construction, all the Principal series representations are associated with it.

The Holomorphic discrete series examples. For any real number $\lambda > 0$, the unilateral shift $M^{(\lambda)}$ with weight sequence $\sqrt{\frac{n+1}{n+\lambda}}, \; n \in \mathbb{Z}^+$ is homogeneous. To see, this, apply Theorem 2.3 to the Discrete series representation D^+_λ. These are irreducible, and, by construction, the representation associated with $M^{(\lambda)}$ is D^+_λ.

The Anti-holomorphic discrete series examples. Being adjoints of homogeneous operators, the operators ${M^{(\lambda)}}^*, \quad \lambda > 0$, are homogeneous. Since ${D^+_\lambda}^\# = D^-_\lambda$, the representation associated with ${M^{(\lambda)}}^*$ is D^-_λ. It was shown in [5] that these operators are the only homogeneous operators in the Cowen-Douglas class $B_1(\mathbb{D})$.

The Complementary series examples. For any two distinct real numbers a and b in the open unit interval $(0, 1)$, the bilateral shift $K_{a,b}$ with weight sequence $\sqrt{\frac{n+a}{n+b}}, n \in \mathbb{Z}$, is homogeneous. To see this in case $0 < a < b < 1$, apply Theorem 2.3 to the Complementary series representation $C_{\lambda,\sigma}$ with $\lambda = a + b - 1$ and $\sigma = (b - a)/2$. If $0 < b < a < 1$ then $K_{a,b}$ is the inverse of the homogeneous operator $C_{b,a}$, and hence is homogeneous. Since $C^\#_{\lambda,\sigma} = C_{\lambda,\sigma}$, we see that for any two numbers $0 < a \neq b < 1$, the representation associated with the irreducible operator $K_{a,b}$ is $C_{\lambda,\sigma}$ with $\lambda = a + b - 1, \quad \sigma = |a - b|/2$.

The constant characteristic examples. For any strictly positive real number $\lambda \neq 1$, the bilateral shift B_r with weight sequence $\cdots, 1, 1, 1, r, 1, 1, 1, \cdots,$ (r in the zeroth slot, 1 elsewhere) is homogeneous. Indeed, if $0 < r < 1$ then B_r is a completely non unitary contraction with constant characteristic function $-r$; hence it is homogeneous because of Theorem 2.10 in [1]. (In [1] we show that apart from the unweighted unilateral shift and its adjoint, these are the only irreducible contractions with a constant characteristic function.) If $r > 1$, B_r is the inverse of the homogeneous operator B_s with $s = r^{-1}$, hence it is homogeneous. (In [1] we presented an unnecessarily convoluted argument to show that B_r is homogeneous for $r > 1$ as well.) It was shown in [1] that the representation $D^+_1 \oplus D^-_1$ is associated with each of the operators $B_r, \quad r > 0$. (Recall that this is the only reducible representation in the Principal series!)

5 Classification

Theorem 5.1 *If T is an irreducible homogeneous operator then T is a block shift. If π is a normalised representation associated with T then the blocks of T are precisely the $\mathbb{K}$-isotypic subspaces $V_n(\pi), \; n \in \mathcal{T}(\pi)$.*

Proof: Because of Lemma 2.2, it suffices to show that

$$T(V_n(\pi)) \subseteq V_{n+1}(\pi) \;\text{ for all } n \in \mathcal{T}(\pi). \tag{5.1}$$

Indeed, since T is irreducible, (5.1), shows that π is connected (if there were $a < b < c$ in $\mathbb{Z}$with $a, c \in \mathcal{T}(\pi)$ and $b \notin \mathcal{T}(\pi)$ then (5.1) would imply that $\oplus_{n<b} V_n(\pi)$ is a non-trivial reducing subspace of T). Since $\mathcal{T}(\pi)$ is also unbounded by Theorem 3.3, it follows that by re-indexing, the index can be taken to be either all integers or the non-negative integers or the non-positive integers. Therefore, T is a block shift.

So, it only remains to prove (5.1). To do this, fix $n \in \mathcal{T}(\pi)$ and $v \in V_n(\pi)$. For $x \in \mathbb{K}$, we have $\pi(x)v = \chi_n(x)v$. Consequently,

$$\begin{aligned}
\pi(x)Tv &= \pi(x^{-1})^* Tv \\
&= \pi(x^{-1})^* T\pi(x^{-1})(\pi(x)v) \\
&= (x^{-1}T)(x^{-n}v) \\
&= x^{-(n+1)}Tv.
\end{aligned}$$

So, $Tv \in V_{n+1}(\pi)$. This proves (5.1). □

LEMMA 5.1 *Let T be any homogeneous (scalar) weighted shift. Let π be the projective representation of Möb associated with T. Then upto equivalence, π is one of the representations in List 3.1. Further,*

(a) If T is a forward shift then the associated representation is holomorphic discrete series.

(b) If T is a backward shift then the associated representation is anti-holomorphic discrete series, and

(b) If T is bilateral shift then the associated representation is either principal series or complementary series.

Proof: Let T be a homogeneous shift. If T is reducible, then by Theorem 2.1, $T = B$ and hence the associated representations are Principal series. So assume T is irreducible. Notice that a scalar shift is by definition a block shift with one dimensional blocks. But by Lemma 5.1, the subspaces $V_n(\pi)$, $n \in \mathcal{T}(\pi)$ are blocks of T. Therefore, by Lemma 2.2, we have $d_n(\pi) \leq 1$ for all n. Also, by Lemma 5.1 π is connected. Thus π is simple. Thus by Lemma 3.3, either π is irreducible or $\pi = D_\lambda^+ \oplus D_{2-\lambda}^-$ for some λ in the range $0 < \lambda < 2$. In the first case we are done since the list in Section 2.1 includes alll irreducible projective representations. In the latter case, $\mathcal{T}(\pi) = \mathbb{Z}$ and hence T is a bi-lateral shift. Therefore, T^* is unitarily equivalent to T. Since π is associated with T, $\pi^\#$ is aqssociated with T^*. Therefore, by the uniqueness statement in Theorem 2.2, $\pi^\#$ is equivalent to π. That is,

$$D_{2-\lambda}^+ \oplus D_\lambda^- \equiv (D_{2-\lambda}^-)^\# \oplus (D_\lambda^+)^\# \equiv (D_{2-\lambda}^- \oplus D_\lambda^+)^\# \equiv D_\lambda^+ \oplus D_{2-\lambda}^-.$$

Hence we have $D_{2-\lambda}^+ \equiv D_\lambda^+$ and hence $2 - \lambda = \lambda$, i.e., $\lambda = 1$. Thus in this case, π is equivalent to $D_1^+ \oplus D_1^- = P_{1,0}$ which is a principal series representation belonging to our List 3.1.

Now a simple calculation shows that if π is a normalised representation equivalent to one of the representations in Section 2.1, then (upto additive translations) $\mathcal{T}(\pi) = \mathbb{Z}^+$ (respectively $\mathbb{Z}^-$) if π is holomorphic (respectively anti-holomorphic) discrete series and $\mathcal{T}(\pi) = \mathbb{Z}$ if π is principal or complementary series. Therefore, the staements (a), (b), (c) in the Lemma follow. □

THEOREM 5.2 *Upto unitary equivalence, the only homogeneous (scalar) weighted shifts (with non-zero weights) are the ones in List 4.1.*

Proof: Let T be a homogeneous weighted shift. If T is reducible, we are done by Theorem 2.1. So assume T is irreducible. Then by Theorem 2.2 there is a projective representation π of Möb associated with T. By Lemma 5.1, π is one of the representations in List 3.1. Further, replacing T by T^* if necessary, we may assume that T is either a forward or bi-lateral shift. Accordingly, π is either a holomorphic discrete series representation or a principal/complemantary series representation. Hence $\pi = R_{\lambda,\mu}$ for some parameters λ, μ. Recall that the representation space $\mathcal{H}_\pi$ is the closed linear span of the functions $f_n,\ n \in I$, where $f_n(z) = z^n,\ n \in I$ and $I = \mathbb{Z}^+$ in the former case and $I = \mathbb{Z}$ in the latter case. The elements $f_n,\ n \in I$, form a complete orthogonal set of vectors in $\mathcal{H}_\pi$ but these vectors are (in general) not unit vectors. Their norms are as given in List 3.1 (depending on π). Since T is a weighted shift with respect to the orthonormal basis of $\mathcal{H}_\pi$ obtained by normalising f_ns, there are scalars $a_n > 0,\ n \in I$ such that

$$Tf_n = a_n f_{n+1}, \quad n \in I.$$

Notice that since the f_ns are (in general) not normalised, the numbers a_n are not the weights of the weighted shift T. These weights are given by $w_n := a_n \|f_{n+1}/\|f_n\|,\ n \in I$. It follows that the adjoint T^* acts by :

$$T^* f_n = \frac{\|f_n\|^2}{\|f_{n-1}\|^2} a_{n-1} f_{n-1}, \quad n \in I,$$

where one puts $a_{-1} = 0$ in case $I = \mathbb{Z}^+$.

Let M be the multiplication operator on $\mathcal{H}_\pi$ defined by $Mf_n = f_{n+1},\quad n \in I$. Notice that for each representation in List 3.1, the corresponding operator M is in List 4.1. Also, in case M is invertible, M^{*-1} is also in the latter list.

Let β be a fixed but arbitrary element of $\mathbb{D}$, and let $\varphi_\beta := \varphi_{-1,\beta} \in$ Möb (recall the notation in Equation 1.1).Note that φ_β is an involution, and this simplifies the following computation of $\pi(\varphi_\beta)$ a little bit. Indeed, a straightforward calculation shows that, for $\pi = R_{\lambda,\mu}$, we have

$$\langle \pi(\varphi_\beta) f_m, f_n \rangle = c(-1)^n \bar{\beta}^{n-m} \|f_n\|^2 \sum_{k \geq (m-n)^+} C_k(m,n)\ r^k, \quad 0 \leq r \leq 1, \tag{5.2}$$

where we have put $r = |\beta|^2$, $c = \varphi_\beta'(0)^{\lambda/2+\mu}$ and

$$C_k(m,n) = \binom{-\lambda - \mu - m}{k + n - m} \binom{-\mu + m}{k}.$$

Since π is associated with T, from the defining equation (2.3) we have $T\pi(\varphi_\beta)(I - \bar{\beta}T) = \pi(\varphi_\beta)(\beta I - T)$. That is,

$$\bar{\beta} T \pi(\varphi_\beta) T + \beta \pi(\varphi_\beta) = T\pi(\varphi_\beta) + \pi(\varphi_\beta) T.$$

Fix m, n in I. Evaluate each side of the above equation at f_m and take the inner product of the resulting vectors with f_n. We have, for instance, $\langle T\pi(\varphi_\beta) T f_m, f_n \rangle = \langle \pi(\varphi_\beta) T f_m, T^* f_n \rangle =$

$a_m\bar{a}_{n-1}\|f_n\|^2/\|f_{n-1}\|^2\langle\pi(\varphi_\beta)f_{m+1},f_{n-1}\rangle$, and similarly for the other three terms. Now substituting from Equation 5.2 and cancelling the common factor $c(-1)^{n-1}\|f_n\|^2\bar{\beta}^{n-m}$, we arrive at the following identity in the indeterminate r :

$$a_m\bar{a}_{n-1}\sum_{k\geq(m-n+2)^+}C_k(m+1,n-1)r^k-\sum_{k\geq(m-n)^+}C_k(m,n)r^{k+1}$$
$$=\ \bar{a}_{n-1}\sum_{k\geq(m-n+1)^+}C_k(m,n-1)r^k-a_m\sum_{k\geq(m-n+1)^+}C_k(m+1,n)r^k. \tag{5.3}$$

Taking $m=n$ in Equation 5.3 and equating the co-efficients of r^1, we obtain :

$$(n+1-\mu)a_n=(n-\mu)\bar{a}_{n-1}+1,\quad n\in I. \tag{5.4}$$

References

[1] B. Bagchi and G. Misra, *Constant characteristic functions and Homogeneous operators*, J. Operator Theory, 37 (1997), 51-65.

[2] D. N. Clark and G. Misra, *On some Homogeneous contractions and Unitary representations of* $SU(1,1)$, J. Operator Theory, 30 (1993), 109-122.

[3] P. R. Halmos, A Hilbert space problem book, Springer Verlag, New York, 1974.

[4] G. W. Mackey, *The theory of unitary group representations*, The University of Chicago Press, 1976.

[5] G. Misra, *Curvature and Discrete Series Representation of* $SL_2(\mathbb{R})$, J. Int. Eqns and Operator Theory, 9 (1986), 452-459.

[6] G. Misra and N. S. N. Sastry, *Homogeneous tuples of Operators and Holomorphic Discrete series Representation of some Classical Groups*, J. Operator Theory, 24 (1990), 23-32.

[7] C. C. Moore, *Extensions and low dimensional cohomology theory of locally compact groups, I*, Trans. Amer. Math. Soc., 113 (1964), 40-63.

[8] N. K. Nikolskii, *On invariant subspaces of weighted shift operators*, Mat. Sbornik, 74 (116) (1967), 171-190.

[9] K. R. Parthasarathy, *Multipliers on locally compact groups*, Lecture Notes in Math. 93, Springer Verlag, New York, 1969.

[10] P. J. Sally, *Analytic continuation of the irreducible unitary representations of the universal covering group of SL(2,ℝ)*, Mem. Amer. Math. Soc. 69, Providence, 1967.

[11] A. L. Shields, *Weighted shift operators and analytic function theory*, Math. Surveys (AMS), 13 (1974), 49-128.

[12] V. S. Varadarajan, *Geometry of Quantum Theory*, Springer Verlag, New York, 1985.