Uniqueness of Walkup's 9-vertex 3-dimensional Klein bottle

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Abstract

Via a computer search, Altshuler and Steinberg found that there are $1296 + 1$ combinatorial 3-manifolds on nine vertices, of which only one is non-sphere. This exceptional 3-manifold K^3_9 triangulates the twisted S^2-bundle over S^1. It was first constructed by Walkup. In this paper, we present a computer-free proof of the uniqueness of this non-sphere combinatorial 3-manifold. As opposed to the computer-generated proof, ours does not require wading through all the 9-vertex 3-spheres. As a preliminary result, we also show that any 9-vertex combinatorial 3-manifold is equivalent by proper bistellar moves to a 9-vertex neighbourly 3-manifold.

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1 Introduction and results

Recall that a *simplicial complex* is a collection of non-empty finite sets (sets of *vertices*) such that every non-empty subset of an element is also an element. For $i \geq 0$, the elements of size $i+1$ are called the *i-simplices* (or *i-faces*) of the complex. For a simplicial complex K, the maximum of k such that K has a k-simplex is called the *dimension* of K and is denoted by $\dim(K)$. If any set of $\lfloor \frac{d}{2} \rfloor + 1$ vertices form a face of a d-dimensional simplicial complex K, then one says that K is *neighbourly.*

All the simplicial complexes considered here are finite. The vertex-set of a simplicial complex K is denoted by $V(K)$. If K, L are two simplicial complexes, then a *simplicial isomorphism* from K to L is a bijection $\pi : V(K) \to V(L)$ such that for $\sigma \subseteq V(K)$, σ is a face of K if and only if $\pi(\sigma)$ is a face of L. Two complexes K, L are called *isomorphic* when such an isomorphism exists. We identify two simplicial complexes if they are isomorphic.

A simplicial complex is usually thought of as a prescription for construction of a topological space by pasting geometric simplices. The space thus obtained from a simplicial complex K is called the *geometric carrier* of K and is denoted by $|K|$. If a topological space X is homeomorphic to $|K|$ then we say that K is a *triangulation* of X. A *combinatorial d-manifold* is a triangulation of a closed pl d-manifold (see Section 2 for more).

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For a set V with $d+2$ elements, let S be the simplicial complex whose faces are all the non-empty proper subsets of V. Then S triangulates the d-sphere. This complex is called the *standard d-sphere* and is denoted by $S^d_{d+2}(V)$ or simply by S^d_{d+2}.

If σ is a face of a simplicial complex K then the *link* of σ in K, denoted by $\mathrm{lk}_K(\sigma)$ (or simply by $\mathrm{Lk}(\sigma)$), is by definition the simplicial complex whose faces are the faces τ of K such that τ is disjoint from σ and $\sigma \cup \tau$ is a face of K.

Let M be a d-dimensional simplicial complex. If α is a $(d-i)$-face of M, $0 < i \leq d$, such that $\mathrm{lk}_M(\alpha) = S^{i-1}_{i+1}(\beta)$ and β is not a face of M (such a face α is said to be a *removable* face of M) then consider the complex (denoted by $\kappa_\alpha(M)$) whose set of maximal faces is $\{\sigma : \sigma \text{ a maximal face of } M, \alpha \not\subseteq \sigma\} \cup \{\beta \cup \alpha \setminus \{v\} : v \in \alpha\}$. The operation $\kappa_\alpha : M \mapsto \kappa_\alpha(M)$ is called a *bistellar i-move*. For $0 < i < d$, a bistellar i-move is called a *proper bistellar move*. In [2], Altshuler and Steinberg found from their computer search that all the 9-vertex 3-spheres are equivalent via a finite sequence of proper bistellar moves. Here we prove:

Theorem 1. *Every 9-vertex combinatorial 3-manifold is obtained from a neighbourly 9-vertex combinatorial 3-manifold by a sequence of (at most 10) bistellar 2-moves.*

In [1], Altshuler has shown that every combinatorial 3-manifold with at most 8 vertices is a combinatorial 3-sphere. (This is also a special case of a more general result of Brehm and Kühnel [6].) Via a computer search, Altshuler and Steinberg found (in [2]) that there are 1297 combinatorial 3-manifolds on nine vertices, of which only one (namely, K^3_9 of Example 1 below) is non-sphere. Here, we present a computer-free proof of this fact. More explicitly, we prove:

Theorem 2. *Up to simplicial isomorphism, there is a unique 9-vertex non-sphere combinatorial 3-manifold, namely K^3_9.*

Note that Theorem 2 was a key ingredient in the proof of the main result (viz, non-existence of complementary pseudomanifolds of dimension 6) in [4]. The proof of Theorem 2 presented here makes the result of [4] totally independent of machine computations. This was one of the prime motivations for the present paper.

2 Preliminaries

For $i = 1, 2, 3$, the i-faces of a simplicial complex K are also called the *edges*, *triangles* and *tetrahedra* of K, respectively. A simplicial complex K is called *connected* if $|K|$ is connected. For a simplicial complex K, if $U \subseteq V(K)$ then $K[U]$ denotes the induced subcomplex of K on the vertex-set U. If the number of i-simplices of a d-dimensional simplicial complex K is $f_i(K)$, then the vector $f = (f_0, \dots, f_d)$ is called the *f-vector* of K and the number $\chi(K) := \sum_{i=0}^{d} (-1)^i f_i(K)$ is called the *Euler characteristic* of K.

For a face σ in a simplicial complex K, the number of vertices in $\mathrm{lk}_K(\sigma)$ is called the *degree* of σ in K and is denoted by $\deg_K(\sigma)$. The induced subcomplex $C(\sigma, K)$ on the vertex-set $V(K) \setminus \sigma$ is called the *simplicial complement* of σ in K.

By a *subdivision* of a simplicial complex K we mean a simplicial complex K' together with a homeomorphism from $|K'|$ onto $|K|$ which is facewise linear. Two complexes K and L are called *combinatorially equivalent* (denoted by $K \approx L$) if they have isomorphic subdivisions. So, $K \approx L$ if and only if $|K|$ and $|L|$ are pl homeomorphic. If a simplicial complex X is combinatorially equivalent to S^d_{d+2} then it is called a *combinatorial d-sphere*.

A simplicial complex K is called a *combinatorial d-manifold* if the link of each vertex is a combinatorial $(d-1)$-sphere. Thus, a simplicial complex K is a combinatorial d-manifold if and only if $|K|$ is a closed pl d-manifold with the pl structure induced from K (see [9]).

A *graph* is an 1-dimensional simplicial complex. For $n \geq 3$, the n-vertex combinatorial 1-sphere (*n-cycle*) is the unique n-vertex 1-pseudomanifold and is denoted by S^1_n. A *coclique* in a graph is a set of pairwise non-adjacent vertices.

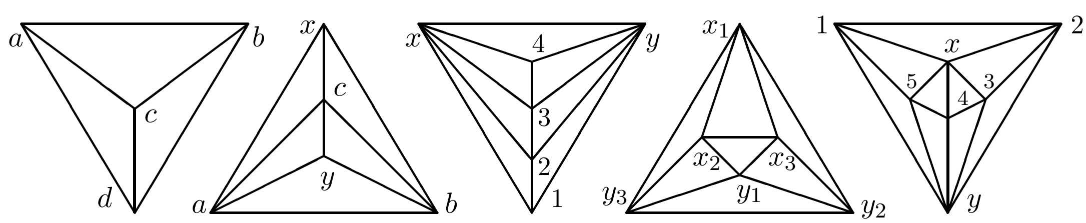
A simplicial complex K is called *pure* if all the maximal faces of K have the same dimension. A maximal face in a pure simplicial complex is also called a *facet*. For a pure d-dimensional simplicial complex K, let $\Lambda(K)$ be the graph whose vertices are the facets of K, two such vertices being adjacent in $\Lambda(K)$ if and only if the corresponding facets intersect in a $(d-1)$-simplex. A d-dimensional pure simplicial complex K is called a *d-pseudomanifold* if each $(d-1)$-face is contained in exactly two facets of K and $\Lambda(K)$ is connected. If the link of a vertex in a pseudomanifold is not a combinatorial sphere then it is called a *singular vertex*. Clearly, any connected combinatorial d-manifold is a d-pseudomanifold without singular vertices. Since a connected $(d+1)$-regular graph has no $(d+1)$-regular proper subgraph, a d-pseudomanifold has no proper d-dimensional sub-pseudomanifold.

For two simplicial complexes K, L with disjoint vertex-sets, the *join* $K * L$ is the simplicial complex $K \cup L \cup \{\sigma \cup \tau \, : \, \sigma \in K, \tau \in L\}$. Clearly, if both K and L are pseudomanifolds then $K * L$ is a pseudomanifold of dimension $\dim(K) + \dim(L) + 1$.

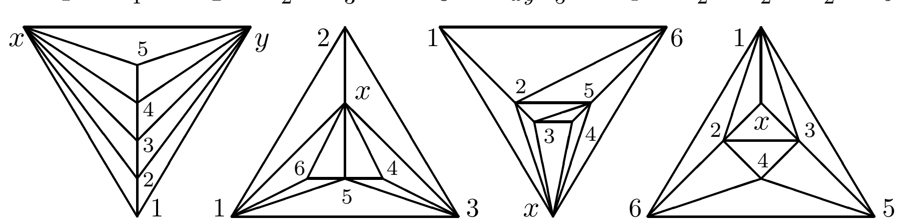
Let K be an n-vertex d-pseudomanifold. If u is a vertex of K and v is not a vertex of K then consider the pure simplicial complex $\Sigma_{uv}K$ on the vertex set $V(K) \cup \{v\}$ whose set of facets is $\{\sigma \cup \{u\} : \sigma$ is a facet of K and $u \notin \sigma\} \cup \{\tau \cup \{v\} : \tau$ is a facet of $K\}$. Then $\Sigma_{uv}K$ is a $(d+1)$-pseudomanifold and is called the *one-point suspension* of K (see [3]). It is easy to see that the links of u and v in $\Sigma_{uv}K$ are isomorphic to K.

Example 1. For $d \geq 2$, let K^d_{2d+3} be the d-dimensional pure simplicial complex whose vertices are the vertices of the $(2d+3)$-cycle S^1_{2d+3} and the facets are the sets of $d+1$ vertices obtained by deleting an interior vertex from the $(d+2)$-paths in the cycle. The simplicial complex K^d_{2d+3} is a combinatorial d-manifold. Indeed, it was shown in [7] that K^d_{2d+3} triangulates $S^{d-1} \times S^1$ for d even, and it triangulates the twisted product of S^{d-1} and S^1 for d odd. In particular, K^3_9 triangulates the twisted product of S^2 and S^1 (often called the 3-*dimensional Klein bottle*). It was first constructed by Walkup in [10].

Example 2. Some combinatorial 2-spheres on 5, 6 and 7 vertices.



$\mathcal{S}_1 = S^2_4$ $\quad$ $\mathcal{S}_2 = S^0_2 * S^1_3$ $\quad$ $\mathcal{S}_3 = \Sigma_{xy}S^1_5$ $\quad$ $\mathcal{S}_4 = S^0_2 * S^0_2 * S^0_2$ $\quad$ $\mathcal{S}_5 = S^0_2 * S^1_5$

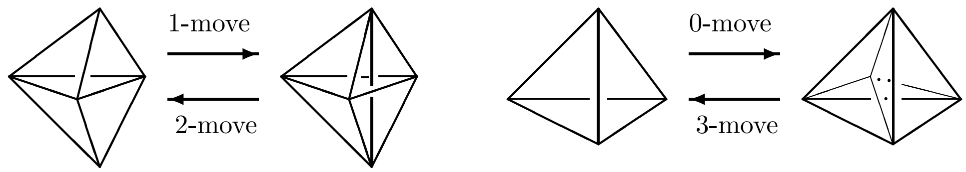


$\mathcal{S}_6 = \Sigma_{xy}S^1_6$ $\quad$ $\mathcal{S}_7$ $\quad$ $\mathcal{S}_8$ $\quad$ $\mathcal{S}_9$

The following result (which we need later) follows from the classification of combinatorial 2-spheres on ≤ 7 vertices (e.g., see [1, 3]).

Proposition 2.1. *Let K be an n-vertex combinatorial 2-sphere. If $n \leq 7$ then K is isomorphic to $\mathcal{S}_1 \dots, \mathcal{S}_8$ or $\mathcal{S}_9$ above.*

If κ_α is a proper bistellar i-move on a pure simplicial complex M and $\mathrm{lk}_M(\alpha) = S^{i-1}_{i+1}(\beta)$ then β is a removable i-face of $\kappa_\alpha(M)$ (with $\mathrm{lk}_{\kappa_\alpha(M)}(\beta) = S^{d-i-1}_{d-i+1}(\alpha)$) and $\kappa_\beta : \kappa_\alpha(M) \mapsto M$ is a bistellar $(d-i)$-move. For a vertex u, if $\mathrm{lk}_M(u) = S^{d-1}_{d+1}(\beta)$ then the bistellar d-move $\kappa_{\{u\}} : M \mapsto \kappa_{\{u\}}(M) = N$ deletes the vertex u (we also say that N is obtained from M by *collapsing* the vertex u). The operation $\kappa_\beta : N \mapsto M$ is called a bistellar 0*-move*. We also say that M is obtained from N by *starring* the vertex u in the facet β of N.



Bistellar moves in dimension 3

If M is a 3-pseudomanifold and $\kappa_\alpha : M \mapsto N$ is a bistellar 1-move then, from the definition, $(f_0(N),\, f_1(N), f_2(N), f_3(N)) = (f_0(M), f_1(M)+1, f_2(M)+2, f_3(M)+1)$ and $\deg_N(v) \geq \deg_M(v)$ for any vertex v.

Two simplicial complexes K and L are called *bistellar equivalent* (denoted by $K \sim L$) if there exists a finite sequence of bistellar moves leading from K to L. Let κ_α be a proper bistellar i-move and $\mathrm{lk}_M(\alpha) = S^{i-1}_{i+1}(\beta)$. If K_1 is obtained from K by starring ([3]) a new vertex in α and K_2 is obtained from $\kappa_\alpha(K)$ by starring a new vertex in β then K_1 and K_2 are isomorphic. Thus, if $K \sim L$ then $K \approx L$. Conversely, it was shown in [8], that if two combinatorial manifolds are combinatorially equivalent then they are bistellar equivalent.

Let $\tau \subset \sigma$ be two faces of a simplicial complex K. We say that τ is a *free face* of σ if σ is the only face of K which properly contains τ. (It follows that $\dim(\sigma) - \dim(\tau) = 1$ and σ is a maximal simplex in K.) If τ is a free face of σ then $K' := K \setminus \{\tau, \sigma\}$ is a simplicial complex. We say that there is an *elementary collapse* of K to K'. We say K *collapses* to L and write $K \overset{s}{\searrow} L$ if there exists a sequence $K = K_0, K_1, \dots, K_n = L$ of simplicial complexes such that there is an elementary collapse of K_{i-1} to K_i for $1 \leq i \leq n$. If L consists of a 0-simplex (a vertex) we say that K is *collapsible* and write $K \overset{s}{\searrow} 0$.

Suppose $P' \subseteq P$ are polyhedra and $P = P' \cup B$, where B is a pl k-ball (for some $k \geq 1$). If $P' \cap B$ is a pl $(k-1)$-ball then we say that there is an *elementary collapse* of P to P'. We say that P collapses to Q and write $P \searrow Q$ if there exists a sequence $P = P_0, P_1, \dots, P_n = Q$ of polyhedra such that there is an elementary collapse of P_{i-1} to P_i for $1 \leq i \leq n$. For two simplicial complexes K and L, if $K \overset{s}{\searrow} L$ then clearly $|K| \searrow |L|$. The following is a consequence of the Simplicial Neighbourhood Theorem (see [5]).

Proposition 2.2. *Let σ be a facet of a connected combinatorial d-manifold X. Put $L = C(\sigma, X)$, the simplicial complement of σ in X. Also, let $Y = X \setminus \{\sigma\}$. Then*

(*a*) $|Y| \searrow |L|$.

(*b*) *If, further, L is collapsible then X is a combinatorial sphere.*

3 Proof of Theorem 1

For $n \geq 4$, by an S^2_n we mean a combinatorial 2-sphere on n vertices. If a combinatorial 3-manifold has at most 9 vertices then it is connected and hence is a 3-pseudomanifold.

Lemma 3.1 . *Let N be an n-vertex combinatorial 3-manifold with minimum vertex-degree $k \leq n-2$ and $n \leq 9$. Let u be a vertex of degree k in N. Then there exists a bistellar 1-move $\kappa_\beta : N \mapsto \widetilde{N}$ such that $\deg_{\widetilde{N}}(u) = k+1$.*

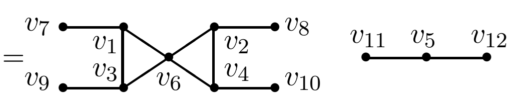
Proof. Let $X = \mathrm{lk}_N(u) = S^2_k$. If $k = 4$ then $X = S^2_4(\{a,b,c,d\})$ for some $a,b,c,d \in V(N)$. Let $\beta = abc$. Suppose $\mathrm{lk}_N(\beta) = \{u,x\}$. If $x = d$ then the induced subcomplex $K = N[\{u,a,b,c,d\}]$ is a 3-pseudomanifold. Since $n \geq 6$, K is a proper subcomplex of N. This is not possible. Thus, $x \neq d$ and hence ux is a non-edge in N. So, κ_β is a bistellar 1-move, as required. So, let $k \geq 5$.

Suppose the result is false. Let $\mathcal{B}$ denote the collection of all facets $B \in N$ such that $u \notin B$ and B contains a triangle of X. Then $\mathcal{B}$ is a set of 4-sets satisfying (a) each element of $\mathcal{B}$ is contained in $V(X)$, (b) each triangle of X is contained in a unique member of $\mathcal{B}$, and (c) each member of $\mathcal{B}$ contains one or two triangles of X. (Indeed, if $B \in \mathcal{B}$ is not contained in $V(X)$ and $\beta \subseteq B$ is a triangle of X, then κ_β is a 1-move on N which increases $\deg(u)$, contrary to our assumption that N does not admit such a move. This proves (a). Now, (b) is immediate since N is a pseudomanifold. If $B \in \mathcal{B}$ contains three triangles of X then these three triangles have a common vertex x, and $\deg_N(x) = 4$, contrary to our assumption that the minimum vertex-degree of N is ≥ 5. This proves (c)).

Let G denote the graph whose vertices are the 4-subsets of $V(X)$ containing 1 or 2 triangles of X. Two vertices are adjacent if the corresponding 4-subsets have a triangle of X in common. It follows that $\mathcal{B}$ is a maximal coclique in G. So, we look for the maximal cocliques of G for each admissible choice of X.

In case $k = 5$, X is of the form $S^0_2(xy) * S^1_3(abc)$. In this case, G has a unique maximal coclique $C = \{xyab, xyac, xybc\}$. If $\mathcal{B} = C$, then N contains a proper 3-dimensional sub-pseudomanifold $S^1_3(uxy) * S^1_3(abc)$, a contradiction. So, $k \geq 6$.

Let $k = 6$. Then X is isomorphic to $\mathcal{S}_3$ or $\mathcal{S}_4$ (defined in Example 2). Consider the case $X = \mathcal{S}_3$. Then the vertices of G are $v_1 = x123$, $v_2 = y123$, $v_3 = x234$, $v_4 = y234$, $v_5 = xy14$, $v_6 = xy23$, $v_7 = x124$, $v_8 = y124$, $v_9 = x134$, $v_{10} = y134$, $v_{11} = xy13$, $v_{12} = xy24$ and

$G =$ 

Note that $\mathrm{Aut}(X) \cong \mathbb{Z}_2 \times \mathbb{Z}_2$, generated by (x,y) and $(1,4)(2,3)$. Up to this automorphism group, the maximal cocliques of G are $C_1 = \{v_6, v_7, v_8, v_9, v_{10}, v_{11}, v_{12}\}$, $C_2 = \{v_5, v_6, v_7, v_8, v_9, v_{10}\}$, $C_3 = \{v_3, v_4, v_7, v_8, v_{11}, v_{12}\}$, $C_4 = \{v_2, v_3, v_7, v_{10}, v_{11}, v_{12}\}$, $C_5 = \{v_3, v_4, v_5, v_7, v_8\}$ and $C_6 = \{v_2, v_3, v_5, v_7, v_{10}\}$.

If $\mathcal{B} = C_i$, for $1 \leq i \leq 4$, then the available portion of $\mathrm{lk}_N(x)$ can not be completed to a 2-sphere, a contradiction. If $\mathcal{B} = C_5$ then $\mathrm{lk}_N(1) = S^0_2(u4) * S^1_3(xyz)$, so that $\deg_N(1) = 5 < k$, a contradiction. If $\mathcal{B} = C_6$ then $\mathrm{lk}(x)$ and $\mathrm{lk}(y)$ are S^2_6's with vertex-sets $\{u,1,2,3,4,y\}$ and $\{u,1,2,3,4,x\}$ respectively. It follows that the remaining (one or two) vertices can only be joined to each other and with $1,2,3,4$. Then these vertices have degree ≤ 5, a contradiction.

Next assume that $X = \mathcal{S}_4$. Up to automorphism of X, there are two maximal cocliques of G, namely, $C_1 = \{x_1y_1x_2x_3, x_1y_1x_2y_3, x_1y_1y_2x_3, x_1y_1y_2y_3\}$ and $C_2 = \{x_1y_1x_2x_3, x_1y_1x_2y_3,$

$x_1y_2x_3y_3, y_1y_2x_3y_3\}$. If $\mathcal{B} = C_1$ or C_2, then $\mathrm{lk}_N(x_2) = S^0_2(x_1y_1) * S^1_3(ux_3y_3)$ and hence $\deg_N(x_2) = 5 < k$, a contradiction.

Thus $k = 7$ and $n = 9$. Let uv be a non-edge. Then $f_1(N) \leq 35$ and hence $f_3(N) \leq 26$. Since there are 10 facets through u and 10 facets through v, it follows that $\#(\mathcal{B}) \leq 6$. Since there are 10 triangles in X and each member of $\mathcal{B}$ contains at most two triangles of X, $\#(\mathcal{B}) \geq 5$. Thus $\mathcal{B}$ is a maximal coclique of G of size 5 or 6.

Since X has 7 vertices, X is isomorphic to $\mathcal{S}_5, \dots, \mathcal{S}_8$ or $\mathcal{S}_9$ (of Example 2).

Consider the case, $X = \mathcal{S}_5$. Then the vertices of G are $v_1 = x124$, $v_2 = x235$, $v_3 = x134$, $v_4 = x245$, $v_5 = x135$, $v_6 = y124$, $v_7 = y235$, $v_8 = y134$, $v_9 = y245$, $v_{10} = y135$, $v_{11} = xy12$, $v_{12} = xy23$, $v_{13} = xy34$, $v_{14} = xy45$, $v_{15} = xy15$, $v_{16} = x123$, $v_{17} = x234$, $v_{18} = x345$, $v_{19} = x145$, $v_{20} = x125$, $v_{21} = y123$, $v_{22} = y234$, $v_{23} = y345$, $v_{24} = y145$ and $v_{25} = y125$. Note that $\mathrm{Aut}(X) \cong \mathbb{Z}_2 \times D_{10}$, generated by (x, y), $(1, 2, 3, 4, 5)$ and $(1, 2)(3, 5)$. Up to this automorphism group, there are only 11 maximal 6-cocliques and 3 maximal 5-cocliques. These are $C_1 = \{v_1, v_2, v_{13}, v_{14}, v_{15}, v_{21}\}$, $C_2 = \{v_1, v_2, v_{13}, v_{19}, v_{21}, v_{24}\}$, $C_3 = \{v_1, v_2, v_{15}, v_{18}, v_{21}, v_{23}\}$, $C_4 = \{v_1, v_3, v_{12}, v_{19}, v_{23}, v_{25}\}$, $C_5 = \{v_1, v_6, v_{12}, v_{13}, v_{14}, v_{15}\}$, $C_6 = \{v_1, v_6, v_{12}, v_{13}, v_{19}, v_{24}\}$, $C_7 = \{v_1, v_6, v_{12}, v_{15}, v_{18}, v_{23}\}$, $C_8 = \{v_1, v_6, v_{17}, v_{19}, v_{22}, v_{24}\}$, $C_9 = \{v_1, v_7, v_{17}, v_{19}, v_{23}, v_{25}\}$, $C_{10} = \{v_1, v_8, v_{14}, v_{15}, v_{17}, v_{21}\}$, $C_{11} = \{v_1, v_8, v_{17}, v_{19}, v_{21}, v_{24}\}$, $C_{12} = \{v_{11}, v_{12}, v_{13}, v_{14}, v_{15}\}$, $C_{13} = \{v_{11}, v_{12}, v_{13}, v_{19}, v_{24}\}$ and $C_{14} = \{v_{11}, v_{17}, v_{19}, v_{22}, v_{24}\}$. If $\mathcal{B} = C_i$, for $i = 1, \dots, 5$ or 7, then x becomes a singular vertex, a contradiction. If $\mathcal{B} = C_9$, then x becomes a vertex of degree 6, a contradiction. For $i \in \{6, 8, 10, 11, 12, 13, 14\}$, if $\mathcal{B} = C_i$ then 5 becomes a vertex of degree 5, a contradiction.

Now, let $X = \mathcal{S}_6$. Then the vertices of G are $v_1 = xy13$, $v_2 = xy14$, $v_3 = xy25$, $v_4 = xy35$, $v_5 = x124$, $v_6 = y124$, $v_7 = x125$, $v_8 = y125$, $v_9 = x134$, $v_{10} = y134$, $v_{11} = x235$, $v_{12} = y235$, $v_{13} = x245$, $v_{14} = y245$, $v_{15} = x145$, $v_{16} = y145$, $v_{17} = xy15$, $v_{18} = xy23$, $v_{19} = xy34$, $v_{20} = x123$, $v_{21} = y123$, $v_{22} = x234$, $v_{23} = y234$, $v_{24} = x345$, $v_{25} = y345$. Note that $\mathrm{Aut}(X) \cong \mathbb{Z}_2 \times \mathbb{Z}_2$, generated by (x, y) and $(1, 5)(2, 4)$. Up to $\mathrm{Aut}(X)$, there are 11 maximal 6-cocliques and one maximal 5-coclique. These are $C_1 = \{v_1, v_3, v_{20}, v_{21}, v_{24}, v_{25}\}$, $C_2 = \{v_1, v_4, v_{20}, v_{21}, v_{24}, v_{25}\}$, $C_3 = \{v_2, v_3, v_{20}, v_{21}, v_{24}, v_{25}\}$, $C_4 = \{v_5, v_6, v_{17}, v_{18}, v_{24}, v_{25}\}$, $C_5 = \{v_5, v_8, v_{17}, v_{18}, v_{24}, v_{25}\}$, $C_6 = \{v_5, v_{11}, v_{17}, v_{21}, v_{24}, v_{25}\}$, $C_7 = \{v_5, v_{13}, v_{17}, v_{21}, v_{22}, v_{25}\}$, $C_8 = \{v_5, v_{15}, v_{17}, v_{21}, v_{22}, v_{25}\}$, $C_9 = \{v_7, v_8, v_{17}, v_{18}, v_{24}, v_{25}\}$, $C_{10} = \{v_7, v_{11}, v_{17}, v_{21}, v_{24}, v_{25}\}$, $C_{11} = \{v_7, v_{15}, v_{17}, v_{21}, v_{22}, v_{25}\}$, $C_{12} = \{v_{17}, v_{20}, v_{21}, v_{24}, v_{25}\}$. For $i \in \{1, 3, 4, 5, 6, 7, 9, 11\}$, if $\mathcal{B} = C_i$ then x becomes a singular vertex, a contradiction. If $\mathcal{B} = C_2$ or C_{12} then $\deg(2) = 5$, a contradiction. If $\mathcal{B} = C_8$ or C_{10} then $\mathrm{lk}(x)$ is an S^2_7 with vertex-set $\{u, y, 1, \dots, 5\}$. So, uv and xv are non-edges and hence $\deg(v) \leq 6$, a contradiction.

Consider the case, $X = \mathcal{S}_7$. Then the vertices of G are $v_1 = x135$, $v_2 = x124$, $v_3 = x125$, $v_4 = x235$, $v_5 = x236$, $v_6 = x346$, $v_7 = x134$, $v_8 = x145$, $v_9 = x245$, $v_{10} = x256$, $v_{11} = x356$, $v_{12} = x136$, $v_{13} = x146$, $v_{14} = 1234$, $v_{15} = 1236$, $v_{16} = 2345$, $v_{17} = 3456$, $v_{18} = 1256$, $v_{19} = 1456$, $v_{20} = x126$, $v_{21} = x234$, $v_{22} = x456$, $v_{23} = 1235$, $v_{24} = 1345$, $v_{25} = 1356$. In this case, $\mathrm{Aut}(X) \cong D_6$, generated by $(1, 3, 5)(2, 4, 6)$ and $(1, 3)(4, 6)$. There is no maximal 5-coclique and up to $\mathrm{Aut}(X)$, there are only 3 maximal 6-cocliques. These are $C_1 = \{v_{16}, v_{18}, v_{20}, v_{21}, v_{22}, v_{23}\}$, $C_2 = \{v_{16}, v_{19}, v_{20}, v_{21}, v_{22}, v_{23}\}$, $C_3 = \{v_{17}, v_{19}, v_{20}, v_{21}, v_{22}, v_{23}\}$. If $\mathcal{B} = C_1$ or C_2 then 3 becomes a vertex of degree 6, a contradiction. If $\mathcal{B} = C_3$ then 5 becomes a singular vertex, a contradiction.

Consider the case, $X = \mathcal{S}_8$. Then the vertices of G are $v_1 = x124$, $v_2 = x125$, $v_3 = x236$, $v_4 = x134$, $v_5 = x346$, $v_6 = x145$, $v_7 = x245$, $v_8 = x356$, $v_9 = x136$, $v_{10} = x146$, $v_{11} = 1236$, $v_{12} = 1246$, $v_{13} = 2456$, $v_{14} = 1235$, $v_{15} = 1345$, $v_{16} = 3456$, $v_{17} = x123$, $v_{18} = x234$, $v_{19} = x456$, $v_{20} = x156$, $v_{21} = x235$, $v_{22} = x256$, $v_{23} = 1256$, $v_{24} = 2345$, $v_{25} = 2356$. Here, $\mathrm{Aut}(X) \cong \mathbb{Z}_2$, generated by $(1, 4)(2, 5)(3, 6)$. There is no maximal 5-coclique and up to

Aut(X), there are 10 maximal 6-cocliques. These are $C_1 = \{v_4, v_{10}, v_{17}, v_{19}, v_{23}, v_{24}\}$, $C_2 = \{v_5, v_9, v_{17}, v_{19}, v_{23}, v_{24}\}$, $C_3 = \{v_4, v_9, v_{17}, v_{19}, v_{23}, v_{24}\}$, $C_4 = \{v_1, v_6, v_{18}, v_{20}, v_{23}, v_{24}\}$, $C_5 = \{v_2, v_7, v_{18}, v_{20}, v_{23}, v_{24}\}$, $C_6 = \{v_1, v_7, v_{18}, v_{20}, v_{23}, v_{24}\}$, $C_7 = \{v_4, v_6, v_{17}, v_{20}, v_{23}, v_{24}\}$, $C_8 = \{v_4, v_7, v_{17}, v_{20}, v_{23}, v_{24}\}$, $C_9 = \{v_5, v_6, v_{17}, v_{20}, v_{23}, v_{24}\}$, $C_{10} = \{v_5, v_7, v_{17}, v_{20}, v_{23}, v_{24}\}$. If $\mathcal{B} = C_i$, for $i = 1, 2, 4, 5$ or 7, then lk(x) is an S^2_7 with vertex-set $\{u, y, 1, \dots, 5\}$. So, uv and xv are non-edges and hence $\deg(v) \leq 6$, a contradiction. If $\mathcal{B} = C_i$, for $i = 3, 6, 8, 9$ or 10 then x becomes a singular vertex, a contradiction.

Consider the case, $X = \mathcal{S}_9$. Then the vertices of G are $v_1 = x124$, $v_2 = x125$, $v_3 = x235$, $v_4 = x236$, $v_5 = x134$, $v_6 = x136$, $v_7 = x156$, $v_8 = x246$, $v_9 = x345$, $v_{10} = x456$, $v_{11} = 1234$, $v_{12} = 1235$, $v_{13} = 1236$, $v_{14} = x126$, $v_{15} = x234$, $v_{16} = x135$, $v_{17} = 1246$, $v_{18} = 1256$, $v_{19} = 2345$, $v_{20} = 2346$, $v_{21} = 1345$, $v_{22} = 1356$, $v_{23} = 1456$, $v_{24} = 2456$, $v_{25} = 3456$. Here, Aut$(X) \cong D_6$, generated by $(1,2,3)(4,5,6)$ and $(1,2)(4,5)$. There is no maximal 5-coclique and up to Aut(X), there are 13 maximal 6-cocliques. These are $C_1 = \{v_1, v_5, v_{15}, v_{17}, v_{22}, v_{25}\}$, $C_2 = \{v_1, v_6, v_{15}, v_{17}, v_{22}, v_{25}\}$, $C_3 = \{v_2, v_5, v_{15}, v_{17}, v_{22}, v_{25}\}$, $C_4 = \{v_2, v_6, v_{15}, v_{17}, v_{22}, v_{25}\}$, $C_5 = \{v_1, v_5, v_{15}, v_{17}, v_{21}, v_{23}\}$, $C_6 = \{v_1, v_6, v_{15}, v_{17}, v_{21}, v_{23}\}$, $C_7 = \{v_2, v_5, v_{15}, v_{17}, v_{21}, v_{23}\}$, $C_8 = \{v_2, v_6, v_{15}, v_{17}, v_{21}, v_{23}\}$, $C_9 = \{v_8, v_9, v_{14}, v_{15}, v_{16}, v_{23}\}$, $C_{10} = \{v_1, v_9, v_{15}, v_{16}, v_{17}, v_{23}\}$, $C_{11} = \{v_2, v_9, v_{15}, v_{16}, v_{17}, v_{23}\}$, $C_{12} = \{v_1, v_7, v_{15}, v_{16}, v_{17}, v_{25}\}$, $C_{13} = \{v_2, v_7, v_{15}, v_{16}, v_{17}, v_{25}\}$. If $\mathcal{B} = C_3, C_4, C_8$ or C_{13}, then 5 becomes a singular vertex, a contradiction. If $\mathcal{B} = C_1$ or C_5 then $\deg(x) = 5$, a contradiction. If $\mathcal{B} = C_2, C_6, C_9$ or C_{12}, then $\deg(2) = 6$, a contradiction. If $\mathcal{B} = C_7, C_{10}$ or C_{11}, then $\deg(3) = 6$, a contradiction. This completes the proof. $\Box$

Proof of Theorem 1. Let M be a 9-vertex combinatorial 3-manifold. Then, by Lemma 3.1, there exists a sequence of bistellar 1-moves $\kappa_{\beta_1}, \dots, \kappa_{\beta_l}$ such that $N := \kappa_{\beta_l}(\cdots(\kappa_{\beta_1}(M)))$ is neighbourly. Since $f_1(M) \geq 4 \times 9 - 10 = 26$ (see [10]) and each bistellar 1-move produces an edge, $l \leq 10$. Let $\alpha_i = \mathrm{lk}_{\kappa_{\beta_{i-1}}(\cdots(\kappa_{\beta_1}(M)))}(\beta_i)$ for $2 \leq i \leq l$ and $\alpha_1 = \mathrm{lk}_M(\beta_1)$. Then $M = \kappa_{\alpha_1}(\cdots(\kappa_{\alpha_l}(N)))$. This completes the proof. $\Box$

Remark 1 . Let C^3_7 be the cyclic 3-sphere whose facets are those 4-subsets of the vertices of the 7-cycle $S^1_7(1 \cdots 7)$ on which the 7-cycle induces a subgraph with even-sized components. Let K be the simplicial complex $C^3_7 \setminus \{1234\}$. Let K' be the simplicial complex on the vertex-set $\{1', \dots, 7'\}$ isomorphic to K (by the map $i \mapsto i'$). Consider the simplicial complex M which is obtained from $K \sqcup K'$ by identifying i with i' for $1 \leq i \leq 4$. Then M is a combinatorial 3-sphere (connected sum of two copies of C^3_7). The minimum vertex-degree in M is 6. The degree of the vertex 6 in M is 6 with non-edges $65'$, $66'$ and $67'$. But, there is no bistellar 1-move $\kappa_\beta : M \mapsto \kappa_\beta(M)$ such that $\deg_{\kappa_\beta(M)}(6) = 7$. So, Lemma 3.1 is not true for $n = 10$.

Remark 2 . Let X be an S^2_k. Let $\alpha = \alpha(X)$ denote the number of 4-subsets of $V(X)$ which contain one or two triangles of X. While proving Lemma 3.1, we noticed that when $k = 6$, $\alpha(X) = 12$, and when $k = 7$, $\alpha(X) = 25$: independent of the choice of X! This is no accident. Indeed, we have $\alpha(S^2_k) = (k-2)(2k-9)$, regardless of the choice of S^2_k. This may be proved by noting that any two S^2_k are equivalent by a sequence of proper bistellar moves, and α is invariant under such moves. The explicit value of α may then be computed by making a judicious choice of S^2_k.

4 Proof of Theorem 2

Throughout this section M_9^3 will denote a fixed but arbitrary neighbourly non-sphere combinatorial 3-manifold. Thus, $\chi(M_9^3) = 0$, and $f(M_9^3) = (9, 36, 54, 27)$.

Lemma 4.1 . *The f-vector of the simplicial complement of any facet of M_9^3 is either $(5, 10, 7, 1)$ or $(5, 10, 6, 0)$.*

Proof. Let σ be a facet of M_9^3 and let (f_0, f_1, f_2, f_3) be the f-vector of the simplicial complement $C(\sigma, M_9^3)$. By Proposition 2.2, the geometric carrier of the simplicial complex $M_9^3 \setminus \{\sigma\}$ collapses to that of $C(\sigma, M_9^3)$. Since the Euler characteristic is invariant under collapsing, we get $\chi(C(\sigma, M_9^3)) = \chi(M_9^3 \setminus \{\sigma\}) = 1$. Thus, $f_0 - f_1 + f_2 - f_3 = 1$. Also, as M_9^3 is neighbourly, $f_0 = 5$ and $f_1 = \binom{5}{2} = 10$. Hence $f_2 = f_3 + 6$. So, to complete the proof, it is sufficient to show that $f_3 \leq 1$.

Since $f_2 \leq \binom{5}{3} = 10$, it follows that $f_3 \leq 4$. Clearly, there are unique simplicial complexes with f-vectors $(5, 10, 10, 4)$, $(5, 10, 9, 3)$ and $(5, 10, 8, 2)$, and all these are collapsible. But, if $C(\sigma, M_9^3)$ was collapsible then, by Proposition 2.2, M_9^3 would be a sphere. So, $f_3 \leq 1$. □

Lemma 4.2 . *Let σ_1, σ_2 be two disjoint facets of M_9^3 and let x be the unique vertex of M_9^3 outside $\sigma_1 \cup \sigma_2$. Then the induced subcomplex of $\mathrm{lk}_{M_9^3}(x)$ on σ_1 (as well as on σ_2) is an S_3^1 together with an isolated vertex.*

Proof. By Lemma 4.1, the simplicial complement $C(\sigma_2, M_9^3)$ of σ_2 has only one facet (viz. σ_1) and seven triangles, four of which are the triangles in σ_1. So, $C(\sigma_2, M_9^3)$ contains exactly three triangles through x. Up to isomorphism, there are two choices for these three triangles, one of which leads to a collapsible complex $C(\sigma_2, M_9^3)$, which is not possible by Proposition 2.2. In the remaining case, we get the situation as described in the lemma. □

Lemma 4.3 . *Suppose each vertex of M_9^3 is in exactly two edges of degree 3. Then M_9^3 has an edge of degree ≥ 6.*

Proof. Fix any facet σ of M_9^3. Since M_9^3 is neighbourly, the link of each vertex is an S_8^2 and hence has 12 triangles. Thus each vertex of M_9^3 is in 12 facets. Therefore, by the inclusion-exclusion principle, the number of facets meeting σ in at least one vertex is $\binom{4}{1} \times 12 - \sum\limits_{e \subset \sigma} \deg(e) + \binom{4}{3} \times 2 - \binom{4}{4} \times 1 = 55 - \sum\limits_{e \subset \sigma} \deg(e)$. Hence, by subtraction, the number of facets disjoint from σ is $\sum\limits_{e \subset \sigma} \deg(e) - 28$. But, by Lemma 4.1, at most one facet can be disjoint from σ. Hence

$$\sum_{e \subset \sigma} \deg(e) = 29 \text{ or } 28 \tag{1}$$

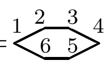
according as there is a (necessarily unique) facet of M_9^3 disjoint from σ, or not. Here the sum is over all the six edges of M_9^3 contained in the facet σ.

Now suppose, if possible, that all the edges of M_9^3 are of degree 3, 4 or 5. Then (1) implies that any facet of M_9^3 contains at most one edge of degree 3. Let G denote the graph with vertex-set $V(M_9^3)$ whose edges are precisely the edges of degree 3 in M_9^3. By, our assumption, G is a 9-vertex regular graph of degree 2, i.e., a disjoint union of cycles. If $e = xy$ is any edge of G then, putting $A = V(\mathrm{lk}(xy))$, we see that $A \cup \{x\}$ and $A \cup \{y\}$ are two cocliques of size 4 in G. This is because no facet of M_9^3 contains more than one edge of G. But, we see by inspection that the 9-cycle S^1_9 is the only 9-vertex union of cycles in which there is such a pair of 4-cocliques corresponding to every edge e. Thus, $G = S^1_9$. Also, for every edge $e = xy$ of S^1_9, there is a unique set A of vertices of S^1_9 such that $A \cup \{x\}$ and $A \cup \{y\}$ are 4-cocliques of S^1_9.

This observation uniquely determines the link in M^3_9 of all its degree 3 edges. Hence all the 27 distinct facets of M^3_9 are determined. But we now see that any two vertices at a distance 2 in S^1_9 form an edge of degree 6 in M^3_9, a contradiction. □

Lemma 4.4. *There is at least one pair of disjoint facets in M^3_9.*

Proof. Suppose not. Thus, any two facets of M^3_9 intersect. Let e be an edge of degree 7. Then $2 \times 12 - 7 = 17$ facets intersect e and hence $27 - 17 = 10$ facets are disjoint from e. These facets are 4-sets in the heptagon lk(e), each of which meets all the edges of the heptagon. But, one sees that the heptagon contains only seven such 4-sets, a contradiction. So, M^3_9 has no edge of degree 7.

Next, let e be an edge of degree 6. Let lk$(e) =$ 

and let x be the unique vertex outside $e \cup \{1, \dots, 6\}$. Each of the $27 - 2 \times 12 + 6 = 9$ facets disjoint from e is a 4-set meeting all the edges of the hexagon. There are only eleven such 4-sets, namely, $x135$, $x246$, 1235, 2346, 1345, 2456, 1356, 1246, 1245, 2356, 1346. Since at most two of the four sets $x135$, 1235, 1345, 1356 can be facets, and at most two of the four sets $x246$, 2346, 2456, 1246 can be facets, we have no way to choose nine of these eleven sets as facets of M^3_9. So, M^3_9 has no edge of degree 6.

Thus all the edges of M^3_9 have degree 3, 4 or 5. For $3 \leq i \leq 5$, let ε_i be the number of edges of degree i. Since the total number of edges is $\binom{9}{2} = 36$, we have

$$\varepsilon_3 + \varepsilon_4 + \varepsilon_5 = 36. \tag{2}$$

Also, counting in two ways the ordered pairs (e, σ), where e is an edge in a facet σ, we get

$$3\varepsilon_3 + 4\varepsilon_4 + 5\varepsilon_5 = 27 \times \binom{4}{2} = 162. \tag{3}$$

Since any two facets intersect, Equation (1) shows that $\sum\limits_{e \subset \sigma} \deg(e) = 28$ for each facet σ. Since the only permissible edge-degrees are 3, 4 and 5, it follows that there are only two types of facets. A facet of type 1 contains one edge of degree 3 (and five of degree 5) while a facet of type 2 contains two edges of degree 4 (and four of degree 5). Counting in two ways pairs (e, σ) with $e \subset \sigma$, where (i) e is an edge of degree 3 and σ is a facet (of type 1) and (ii) e is an edge of degree 4 and σ is a facet (of type 2), we see that there are $3\varepsilon_3$ facets of type 1 and $2\varepsilon_4$ facets of type 2. Since the total number of facets is 27, we get

$$3\varepsilon_3 + 4\varepsilon_4 = 27. \tag{4}$$

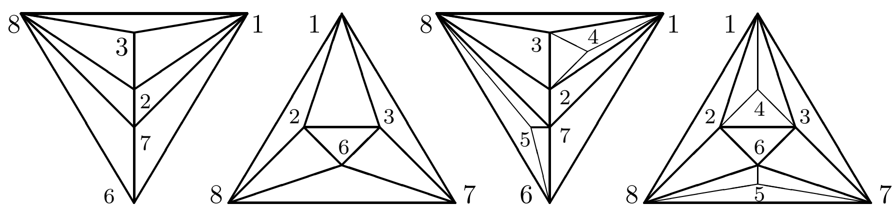
Solving Equations (2), (3) and (4), we obtain $\varepsilon_3 = 9$, $\varepsilon_4 = 0$, $\varepsilon_5 = 27$. Thus all the edges have degree 3 or 5. Therefore, for any vertex x, lk(x) is an S^2_8 all whose vertices have degree 3 or 5. Since an S^2_8 has 18 edges, its vertex degrees add up to 36. So, lk(x) has exactly two vertices of degree 3. That is, each vertex x of M^3_9 is in exactly two edges of degree 3. Hence, by Lemma 4.3, there is an edge of degree ≥ 6, a contradiction. This proves the lemma. □

Let's say that a vertex x of M^3_9 is *good* if there is a partition of $V(M^3_9) \setminus \{x\}$ into two facets. By, Lemma 4.4, there is at least one good vertex. Next we prove.

Lemma 4.5 . *The link of any good vertex in M_9^3 is isomorphic to the 2-sphere $\mathcal{S}$ given below.*

Proof. Let v be a good vertex. Let 1235 and 4678 be two disjoint facets not containing v. Let $L = \mathrm{lk}(v)$. In view of Lemma 4.2, we may assume that the induced subcomplex of L on 1235 and 4678 are $S_3^1(\{1,2,3\}) \cup \{5\}$ and $S_3^1(\{6,7,8\}) \cup \{4\}$ respectively. Hence $V(\mathrm{lk}_L(5)) \subseteq \{4,6,7,8\}$ and $V(\mathrm{lk}_L(4)) \subseteq \{1,2,3,5\}$. It follows that no triangle of L contains 45, so that 45 is not an edge of L. Therefore, $\mathrm{lk}_L(5) = S_3^1(\{6,7,8\})$ and $\mathrm{lk}_L(4) = S_3^1(\{1,2,3\})$. Since each vertex of L is adjacent in L with 4 or 5, and no two degree 3 vertices are adjacent in an S_8^2, it follows that 4 and 5 are the only two vertices of degree 3 in L. Note that the partition $\{1235, 4678\}$ of $V(M_9^3) \setminus \{v\}$ into two facets of M_9^3 is uniquely recovered from L as the pair of stars of the degree 3 vertices of L. (Star of a vertex u in a simplicial complex K is the join $\{u\} * \mathrm{lk}_K(u)$.) Thus, there is a natural bijection between good vertices and pairs of disjoint facets.

Collapsing the two degree 3 vertices 4 and 5 in L, we obtain an S_6^2 with two disjoint triangle 123 and 678. Therefore, L is obtained from an S_6^2 by starring a vertex in each of two disjoint triangles. We know (see Proposition 2.1) that there are exactly two different S_6^2, namely, $\mathcal{S}_3$ and $\mathcal{S}_4$. Observe that each of these two S_6^2 has a unique pair of disjoint triangles, up to automorphisms of the S_6^2. Thus L is isomorphic to $\mathcal{S}$ or $\mathcal{T}$ below.

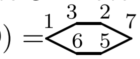


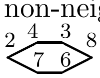
$\mathcal{S}_3 = \Sigma_{18} S_5^1$ $\mathcal{S}_4 = S_2^0 * S_2^0 * S_2^0$ $\mathcal{S}$ $\mathcal{T}$

If possible, let $L = \mathcal{T}$. We claim that the facet of M_9^3 (other than $v124$) containing 124 must be 1245. Indeed, it can not be 1234 since then there would be three facets of M_9^3 disjoint from it, contradicting Lemma 4.2. Also, it can not be $124i$ for $6 \leq i \leq 8$ (since the induced subcomplex on its complement contradicts Lemma 4.2). Thus, 1245 is a facet of M_9^3. Similarly, we get six facets $1245, 1345, 2345, 4567, 4568, 4578$ of M_9^3. Then $\mathrm{lk}_{M_9^3}(45)$ is the disjoint union of two circles. This is not possible since M_9^3 is a manifold. $\square$

Lemma 4.6 . *If M_9^3 is a 9-vertex non-sphere neighbourly combinatorial 3-manifold then $M_9^3 = K_9^3$.*

Proof. By Lemma 4.4, there is a good vertex, say 9 is a good vertex. By Lemma 4.5, the link of 9 is isomorphic to $\mathcal{S}$. Assume that the link of 9 is $\mathcal{S}$. The facet (other than 2349) containing 234 must be 2346 (since for every vertex $x \neq 6, 9$, there are two facets through 9 disjoint from $234x$). So, 2346 and 5789 are disjoint facets. Then 1 is a good vertex. Similarly, $3567 \in M_9^3$ and 8 is a good vertex. A similar argument shows that the facet (other than 1349) through 134 must be 1345 (since the induced subcomplex of lk(2) on 5689 is not an S_3^1 together with an isolated vertex, by Lemma 4.2, 1347 can not be a facet). Similarly, the facet (other than 5689) through 568 must be 4568.

Now, consider the links of 1 and 8. By Lemma 4.5, both are isomorphic to $\mathcal{S}$. Note that the only non-neighbour in $\mathcal{S}$ of a vertex of degree 6 has degree 3. Since lk(89) = 

and lk(19) = 

, it follows that deg(48) = 3 = deg(15). Since 3567 and 1249 are disjoint facets not containing 8 and since 189, 289 $\in M_9^3$, Lemma 4.2 implies that 128 $\in M_9^3$ and 148 $\notin M_9^3$. Since $\mathrm{lk}(8)$ is a copy of $\mathcal{S}$ and 12 is an edge and 14 is a non-edge in $\mathrm{lk}(8)$, it follows that 123 is a triangle of $\mathrm{lk}(8)$ and hence $\deg_{\mathrm{lk}(8)}(3) = 3$. Since the two degree 3 vertices are non-adjacent in the edge-graph of $\mathcal{S}$ and $\deg_{\mathrm{lk}(8)}(3) = 3 = \deg_{\mathrm{lk}(8)}(4)$, it follows that $\mathrm{lk}_{\mathrm{lk}(8)}(4) = S_3^1(5,6,7)$. Similarly, since 2346 and 5789 are disjoint facets not containing 1 and 189, 179 $\in M_9^3$, it follows that 178 $\in M_9^3$, 158 $\notin M_9^3$ and hence $\mathrm{lk}_{\mathrm{lk}(1)}(5) = S_3^1(2,3,4)$. Then, from the links of 1 and 8 we get facets 1235, 1245, 1238, 1278, 1678, 4578, 4678.

Now, trying to complete the links of 2 and 7, we get facets 2356, 2456, 3457, 3467. This implies that K_9^3 is a subcomplex of M_9^3. Since both are 3-pseudomanifolds, $M_9^3 = K_9^3$. $\square$

Proof of Theorem 2. Observe that the degree 3 edges in K_9^3 are 15, 59, 94, 48, 83, 37, 72, 26, 61 and the automorphism group D_{18} of K_9^3 acts transitively on these nine edges. But, none of these nine edges are removable. (Since $\mathrm{lk}_{K_9^3}(15) = S_3^1(2,3,4)$ and 234 is a face in K_9^3, 15 is not removable.) So, there is no bistellar 2-move on K_9^3.

Now, let N_9^3 be a 9-vertex non-sphere combinatorial 3-manifold. If N_9^3 is not neighbourly then, by Theorem 1, there is a 9-vertex neighbourly 3-manifold M_9^3 obtainable from N_9^3 by a sequence of bistellar 1-moves. Since N_9^3 is non-sphere, so is M_9^3. Therefore, by Lemma 4.6, $M_9^3 = K_9^3$. Thus, N_9^3 is obtainable from K_9^3 by a sequence of bistellar 2-moves. But, we just observe that K_9^3 does not admit any bistellar 2-move. A contradiction. So, N_9^3 is neighbourly. Now, by Lemma 4.6, $N_9^3 = K_9^3$. $\square$

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