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Recurrences of strange attractors

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Abstract. The transitions from or to strange nonchaotic attractors are investigated by recurrence plot-based methods. The techniques used here take into account the recurrence times and the fact that trajectories on strange nonchaotic attractors (SNAs) synchronize. The performance of these techniques is shown for the Heagy–Hammel transition to SNAs and for the fractalization transition to SNAs for which other usual nonlinear analysis tools are not successful.

Keywords. Recurrence plots, recurrence times, strange chaotic and nonchaotic attractors.

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1. Introduction

A strange attractor is an object in the phase space generated by a nonlinear dynamical system, which usually corresponds to chaotic behaviour of the system. However, another type of attractor was found about two decades ago, i.e., strange nonchaotic attractors (SNAs) [1]: here we have a strange attractor but the corresponding dynamical system is not chaotic. SNAs mainly appear in quasiperiodically forced nonlinear systems. These objects lie in between quasiperiodicity and chaos. They are strange because the relations showing the dependence of the dynamical variables on the phases are not smooth. Their nonchaotic behaviour, which is due to the fact that they are not sensitive to changes of initial conditions, can be characterized by the largest Lyapunov exponent which is negative. SNAs have been reported in many situations using usual methods of nonlinear analysis and a quite complete description of these exotic attractors has been recently given [2,3].

In 1987, a new method of data analysis, called recurrence plots (RPs), was introduced [4]. Many contributions have been reported since then, showing the relevance of this method for short, nonstationary data and its applicability to data from different fields of research, such as physiological and climate data [5]. Since RPs have been shown to be efficient in detecting various qualitative transitions in the dynamics of different systems, we have intended to propose a recurrence approach in order to detect the transitions from or to SNAs too.

The outline of this paper is as follows: in §2, we present the recurrence approach to detect the different transitions. This approach is applied in §3 to detect transitions to SNAs in the quasiperiodically forced logistic map. Section 4 examines the transition from SNAs to chaos. Section 5 summarizes the results.

2. Recurrence approach

RPs allow visualizing the recurrence of states in phase space. A particularity of RPs is that they even enable the visualization of higher-order dimensional phase space trajectories through a two-dimensional representation of their recurrences. Considering an m-dimensional phase space trajectory $\vec{\mathbf{x}}_i \in R^m$ $(i = 1, \ldots, N)$, the RP is obtained by calculating a $N \times N$ matrix:

$$R_{i,j} = \Theta(\delta - \parallel \vec{\mathbf{x}}_i - \vec{\mathbf{x}}_j \parallel), \quad i, j = 1, \ldots, N, \tag{1}$$

where δ is a predefined threshold, $\Theta(\cdot)$ the Heaviside function and $\parallel \cdot \parallel$ denotes a norm, here the maximum norm. Points that are closer (respectively further) than δ yield an entry '1' (respectively '0') in the matrix $R_{i,j}$. Then, the values '1' and '0' are plotted in the two-dimensional (i–j) plot, providing a visual representation of the system's dynamics.

A closer inspection of the RP reveals small structures which are single dots and lines which can be diagonal, vertical as well as horizontal lines. A recurrence quantification analysis (RQA) based on diagonal lines in the RP has been introduced in [6,7]. Marwan *et al* [8] have proposed further measures of complexity to quantify vertical structures in RPs. Recently, we have introduced statistical measures based on recurrence times and on the main diagonal line of the cross-recurrence plot (CRP) in order to detect the transitions from or to SNAs [9]. In the following, we recall briefly these statistics.

Transition from a quasiperiodic to a SNA: To detect the transition from quasiperiodic dynamics to SNAs, we are interested in the vertical distances between recurrence points in the RP. These vertical time distances between recurrence points have been first analyzed by Gao who has called them *recurrence times* [10]. We evaluate the frequency distribution $P(w)$ of the lengths w of these time distances. We then compute the mean value of this distribution which is the mean recurrence time (T_{MRT}):

$$T_{\mathrm{MRT}} = \sum_{w=1}^{N} wP(w) \Bigg/ \sum_{w=1}^{N} P(w). \tag{2}$$

Furthermore, we extract the maximum value of the distribution and call it number of recurrence of the most probable recurrence time (N_{MPRT}):

$$N_{\mathrm{MPRT}} = \max(\{P(w)\};\ w = 1, \ldots, N). \tag{3}$$

Finally, we compute the variances σ_{MRT} of T_{MRT} and σ_{MPRT} of N_{MPRT}. For a given sufficiently long trajectory, the variances are evaluated by dividing the trajectory into k segments and computing T_{MRT} and N_{MPRT} for each segment separately. These are

$$\sigma_{\mathrm{MRT}} = \frac{1}{k-1}\sum_{l=1}^{k}(T_{\mathrm{MRT}}(l) - \bar{T}_{\mathrm{MRT}})^2 \tag{4}$$

and

$$\sigma_{\mathrm{MPRT}} = \frac{1}{k-1}\sum_{l=1}^{k}(N_{\mathrm{MPRT}}(l) - \bar{N}_{\mathrm{MPRT}})^2, \tag{5}$$

where the overbar indicates the mean value.

Transition from SNAs to chaos: To detect the transition from SNAs to chaotic dynamics, we mainly explore the property that two trajectories on a SNA, starting at different initial conditions but driven by the same quasiperiodic force with identical phase, completely synchronize [11]. Note that, this is not the case for two chaotic trajectories. In order to identify the complete synchronization, we compute the cross-recurrence plot (CRP) of the two trajectories. A CRP is an extension of the RP which enables a nonlinear analysis of bivariate data [12]. A CRP consists of the visualization of the cross-recurrence matrix

$$\mathrm{CR}_{i,j} = \Theta(\delta - \parallel \vec{\mathbf{x}}_i - \vec{\mathbf{y}}_j \parallel) \tag{6}$$

of the two separate trajectories $\vec{\mathbf{x}}_i$ and $\vec{\mathbf{y}}_j$ with $i, j = 1, \ldots, N$; $\vec{\mathbf{x}}_i \in R^m$ and $\vec{\mathbf{y}}_j \in R^m$. We will reconstruct the trajectories by delay coordinates [13], with embedding dimension m and delay τ. In this case, the value for the threshold δ is of the order of the average standard deviation σ of the two time series. Although it is possible to use real coordinates here, we have found it useful to study a case of data generated by embedding which is of relevance in experiments where usually only one observable of the system is available. Our indicator of synchronization is based on the main diagonal line or line of identity of the CRP, more precisely, on the measure of complexity determinism (DET) [5] computed on the main diagonal line. If both trajectories synchronize, the main diagonal line of the CRP is continuous, otherwise it is interrupted. The determinism (DET) is given by

$$\mathrm{DET} = \sum_{l=l_{\min}}^{N} lD(l) \bigg/ \sum_{l=1}^{N} lD(l), \tag{7}$$

where $D(l)$ denotes the frequency distribution of the length l of diagonal lines. Computing DET only for coordinates $i = j$, we find $\mathrm{DET} = 1$ when both trajectories $\vec{\mathbf{x}}_i$ and $\vec{\mathbf{y}}_j$ synchronize and $\mathrm{DET} < 1$ when they do not. Therefore, computing DET as a bifurcation parameter is varied, will allow us to detect the transition from SNAs to chaotic attractors.

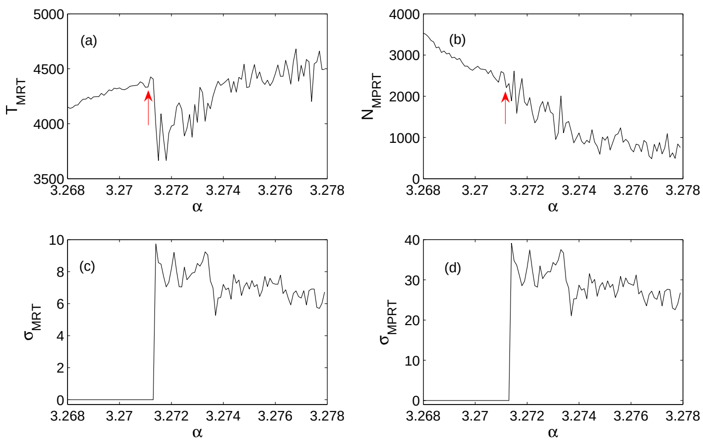

Figure 1. Collision of tori in the Heagy–Hammel route to SNAs in the logistic map (eq. (8)). (**a**) Behaviour of T_{MRT}; (**b**) behaviour of N_{MPRT}; (**c**) variance of T_{MRT} and (**d**) variance of N_{MPRT}. The critical value $\alpha_{\mathrm{c}} = 3.271383$.

3. The transition from quasiperiodic dynamics to SNAs

As already mentioned, SNAs appear to be typical in quasiperiodically forced systems, i.e., if one chooses a single parameter value of the system at random, then the probability that this parameter value yields a SNA is not zero. Many studies have reported different mechanisms through which SNAs could appear in a system. In this section we apply the previously introduced recurrence approach to detect the transition to SNAs through the Heagy–Hammel and fractalization routes. The system that we consider is the quasiperiodically driven logistic map, given by

$$\begin{aligned} x_{n+1} &= \alpha[1 + \epsilon\cos(\phi_n)]x_n(1 - x_n), \\ \phi_{n+1} &= \phi_n + 2\pi\omega \bmod 2\pi. \end{aligned} \tag{8}$$

The mechanism of creation of SNAs described by Heagy and Hammel [14] occurs in the entire class of quasiperiodically forced systems possessing a period doubling cascade without forcing. Following [14], we consider eq. (8) beyond the first period-doubling bifurcation with $\epsilon = 0.1$ and $\omega = (\sqrt{5}-1)/2$. For $\alpha = 3.246$, the attractor in the (ϕ, x) phase has two smooth branches forming a period-2 attractor (T2) and an unstable torus with only one branch. This period-1 repellor (T1) is located in the middle between the two branches of the T2 attractor. For $\alpha = 3.271$, the two branches of the T2 attractor become more and more wrinkled. As α increases, the two branches come closer and closer to the T1 torus, which then cross at $\alpha = 3.272$. From this collision, a SNA is created. The merging of attractors occurs at the critical value $\alpha_{\mathrm{c}} = 3.271383$. We compute the recurrence measures T_{MRT} and N_{MPRT} using a normalized time series consisting of $N = 10{,}000$ data points.

To compute the variances σ_{MRT} and σ_{MPRT}, we use $N = 300{,}000$ data points in total and $N = 2{,}000$ data points as the length of each segment. The threshold for

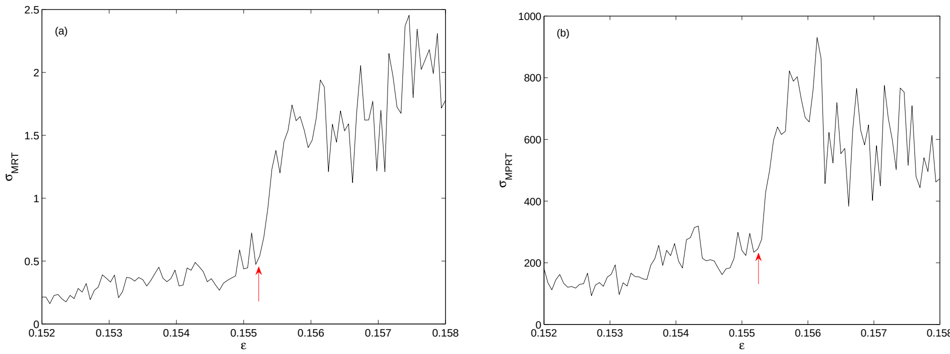

Figure 2. Fractalization route in the logistic map (eq. (9)). (**a**) Variance of T_{MRT}; (**b**) variance of N_{MPRT}; variances computed using the threshold $\delta = 0.05\sigma$, $N = 750{,}000$ data points as the whole trajectory and $N = 3{,}000$ as the length of each segment. The critical value of the bifurcation parameter $\epsilon_{\mathrm{c}} \approx 0.1553$.

the computation of RPs is $\delta = 0.001\sigma$. Note that, in this case, we have used original coordinates which have been normalized leading the standard deviation of the data to unity. Except the measure N_{MPRT}, the three other recurrence measures present a drastic change at the critical value $\alpha_{\mathrm{c}} = 3.271383$, indicating the transition to SNA. This is the value at which Heagy and Hammel identified the merging of attractors leading to the creation of SNAs [14]. Before the critical value, T_{MRT} increases rather smoothly and σ_{MRT} and σ_{MPRT} are almost constant. After the merging of the T2 and T1 tori, the recurrence measures fluctuate strongly, confirming a new regime, which is the SNA one (see figure 1).

We now turn to another transition to SNAs called the fractalization route, which was first described by Kaneko [15]. It is a very common route to SNAs, in which a torus becomes more and more wrinkled until it breaks and gives birth to a SNA without any interaction with a nearby unstable orbit. The usual techniques of nonlinear analysis, such as the Lyapunov exponents, are not able to detect this transition. We study this fractalization route in the logistic map using the same parameters as in [16]:

$$\begin{aligned} x_{n+1} &= a x_n (1 - x_n) + \epsilon \sin(2\pi\theta_n), \\ \theta_{n+1} &= \theta_n + \omega \bmod 1. \end{aligned} \tag{9}$$

The parameter a is fixed to $a = 3$, $\omega = (\sqrt{5} - 1)/2$ and ϵ is considered as a bifurcation parameter. Nishikawa *et al* [16] found that at $\epsilon = 0$ the attractor in phase space is a straight-line torus. Some oscillations of the torus happen as ϵ is increased. The torus becomes fractal at $\epsilon \approx 0.1553$. To compute the recurrence measures, we used original coordinates which we normalized. The threshold used is $\delta = 0.05\sigma$. Before the critical value the measures of complexity (namely the variances σ_{MRT} and σ_{MPRT}) vary slightly. At the critical value $\epsilon_{\mathrm{c}} \approx 0.1553$, the recurrence measures increase suddenly. For the values of the bifurcation parameter in the SNA regime, the measures fluctuate more strongly (figures 2). The measures of complexity are again able to clearly detect this transition.

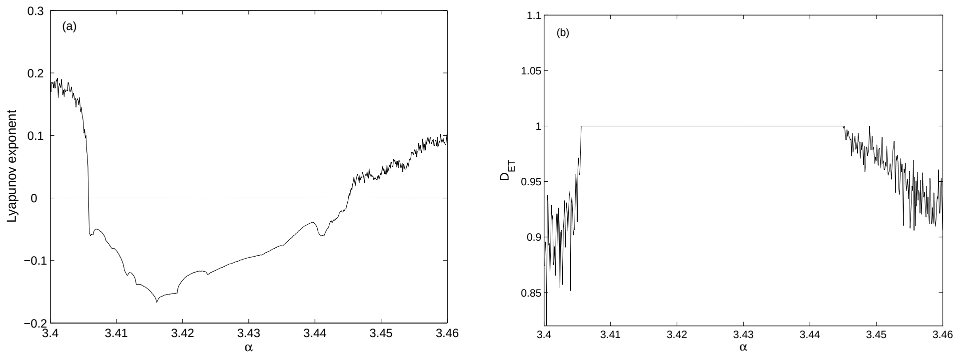

Figure 3. Transitions from SNAs to chaotic attractors in the logistic map (eq. (8)). (**a**) Maximal Lyapunov exponent; (**b**) measure of complexity DET computed using the threshold $\delta = 0.2\sigma$, where σ is the average of the individual standard deviations of the two time series. The embedding dimension is $m = 3$ and the delay is 1.

4. The transition from SNAs to chaotic attractors

To detect this transition, we use the idea of synchronization of trajectories on SNAs introduced in [11]. A complete synchronization of SNAs is possible because their largest Lyapunov exponents are negative. In contrast, in the chaotic regime, both systems have positive Lyapunov exponents and therefore, they show an exponential divergence and do not become synchronized. The transition from SNAs to chaotic attractors is a purely dynamical one: the structure of the attractor remains essentially unchanged, while the largest Lyapunov exponent becomes positive. The recurrence measure which we use to identify this transition is based on the determinism DET of the cross-recurrence matrix defined in eq. (6) for two different time series generated by the same initial phase θ_0.

We exemplify this transition in the quasiperiodically forced logistic map (eq. (8)) rewritten with a rescaled driving parameter $\epsilon' = \epsilon/(4/\alpha - 1)$ fixed to $\epsilon' = 1$ and varying α. The trajectories are reconstructed by delay coordinates with embedding dimension $m = 3$ and delay 1. From figures 3 it can be seen that DET can detect clearly the transition from a SNA to a chaotic attractor. When the Lyapunov exponent changes from negative to positive, DET also changes and becomes smaller than one. DET and the others proposed measures of complexity that are found to be robust to added external noise [9], which is crucial in the context of the analysis of experimental data.

5. Summary

The main objective of this work has been to use measures of complexity based on recurrence plots in order to detect transitions to or from SNAs. Four measures: T_{MRT}, N_{MPRT} and their variances, have been introduced in order to detect the transitions from quasiperiodic dynamics to SNAs. These measures are derived from the distribution of the lengths of vertical distances between recurrence points in the recurrence plots. The determinism DET computed on the main diagonal line of cross-recurrence plots was introduced to detect the transitions from SNAs to chaos. All these measures are able to detect clearly the different transitions. The measures were able to detect the collision of attractors in the route to SNAs introduced by Heagy and Hammel. Furthermore, they detect the critical value at which a torus becomes fractal in the fractalization route to SNAs. This result is important because the usual methods of nonlinear analysis fail here. Another advantage of these measures – which are robust against noise – is that, they can detect the transitions even when the orbits are not very long, in contrast to Lyapunov exponent-based measures. Furthermore, they do not require the knowledge of the equations governing the system under study. Therefore, they are very appropriate for the analysis of experimental data.

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