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The effect of finite response–time in coupled dynamical systems

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Abstract. The paper investigates synchronization in unidirectionally coupled dynamical systems wherein the influence of drive on response is cumulative: coupling signals are integrated over a time interval τ. A major consequence of integrative coupling is that the onset of the generalized and phase synchronization occurs at higher coupling compared to the instantaneous ($\tau = 0$) case. The critical coupling strength at which synchronization sets in is found to increase with τ. The systems explored are the chaotic Rössler and limit cycle (the Landau–Stuart model) oscillators. For coupled Rössler oscillators the region of generalized synchrony in the phase space is intercepted by an asynchronous region which corresponds to anomalous generalized synchronization.

Keywords. Drive–response; response time; distributed delay; synchronization.

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1. Introduction

The interaction between two or more systems gives rise to novel collective behaviour such as synchronization [1], hysteresis [2], amplitude death [3], riddling [4] etc. with synchronization [1,5] being the most common of all. The study of chaotic synchronization has been an active area of research for over two decades, and the ubiquity of this phenomenon in various areas of natural science and its potential application in communication systems [6] has led to the establishment of synchronization as a distinct sub-field in nonlinear science, with the need to understand the phenomenon in its most fundamental form. Synchronization is manifest in different forms: Phase synchronization [7], complete synchronization [9], lag synchronization [11], mixed synchronization [8] and generalized synchronization [10]. Phase synchronization (PS) means entrainment of frequencies of coupled oscillators whereas their amplitudes remain uncorrelated. Complete synchronization refers to complete equality of states of coupled units reducing synchronous motion on to an identity hyperplane and appears only when interacting systems are identical or nearly identical. For small mismatch in parameters, the phase space trajectories are similar but shifted in time relative to each other. Such coincidence of shifted-in-time states of two systems is called lag synchronization. Mixed synchronization arises when the rotation of coupled oscillators are in opposite direction. In this form of synchronization, certain variables of the coupled units get synchronized in-phase while others can be out-of-phase.

When the interaction is unidirectional, the nonlinear systems show generalized synchronous behaviour. Generalized synchronization (GS) in directionally coupled systems is defined as the emergence of a functional relation between the state variables of drive ($x_{\rm d}$) and response ($x_{\rm r}$), namely $x_{\rm r} = \phi(x_{\rm d})$. When ϕ is the identity, this corresponds to CS for identical drive–response, but for nonidentical systems the transformation ϕ is richer and the overall synchronous motion lives on a complicated manifold.

The particular kind of dynamical behaviour which emerges in a composite system is determined by the configuration and type of coupling: the coupling is bidirectional or unidirectional, linear or nonlinear, instantaneous or delayed. Recently, several studies have been devoted to explore the effect of time-delayed coupling [12] and its importance in real-world applications. Time delays arise when information exchange between the coupled units require transmission time. Furthermore, delays can be constant (accounting for finite propagation speed) or distributed [13,14], in cases where the time delay is not fixed but spread over a range. Certain physically significant situations [14] which are modelled by distributed delay arise when time delay is imprecisely known, the delay evolves with time or systems have memory.

We choose a particular type of distribution and examine its effect on directionally coupled oscillators. The distribution chosen is such that the driven system responds to the cumulative effect of all information it has received over a time interval τ. The motivation comes from the fact that it is possible for systems to have some inherent response time (time a generic unit takes to respond to a stimulus). During response time period the system is in a state processing all the information it receives during this interval and gives an average effect of it. A possible scenario can be the transmission duration of neurotransmitter molecules in chemical synapses. In these synapses signal transmission is a chemical event and depends on release, diffusion and receptor binding of neurotransmitter molecules [15].

In the response time period receiving unit averages a range of dynamical states, making it a kind of delay distributed uniformly over the time interval $[0, \tau]$. Since distributed delays cover a broad spectrum of configurations [13,14], for ease of communication we address this particular distribution incorporating response time (rather than considering uncertainty in discrete delay [14]) as 'integrative coupling' [17]. In a recent study [16] we explored the effect of such coupling on bidirectionally coupled systems. In this paper the effect has been investigated for unidirectionally coupled oscillators. The work has been organized into three sections. Section 2 gives a general description of the systems coupled and the explicit formulation of coupling. In §3 we present the results for chaotic and periodic drive–response configurations. Section 4 summarizes our results.

2. Integrative coupling

Unidirectionally coupled systems of the following form are considered:

$$\begin{aligned} \dot{X}(t) &= F(X(t)) \\ \dot{Y}(t) &= G(Y(t)) + \epsilon C(X(t), Y(t)). \end{aligned} \tag{1}$$

Here, $X = [x_1, x_2, ..., x_\mathrm{d}]^T$ and $Y = [y_1, y_2, ..., y_\mathrm{r}]^T$ are the respective state vectors of d-dimensional drive space and r-dimensional response space. F and G define vector fields of drive and response respectively in the uncoupled state. C is the coupling function and ϵ characterizes the strength of coupling. When $\epsilon = 0$ drive and response evolve on separate phase spaces. Coupling being unidirectional keeps the dynamics of drive intact and that of response is controlled by drive. In this paper we consider diffusive-type of interaction. Hence,

$$C(X(t), Y(t)) = (\langle X \rangle_\tau - Y(t)), \tag{2}$$

where

$$\langle X \rangle_\tau = \frac{1}{\tau} \int_{t-\tau}^{t} X(t')\mathrm{d}t' \tag{3}$$

τ is the response time of the response system. Clearly, as $\tau \to 0$, $\langle X \rangle_\tau \to X(t)$ and the coupling becomes instantaneous.

3. Results

In the following subsections we analyse the effect of response time when the subsystems are chaotic as well as periodic.

3.1 *Chaotic subsystems*

We consider two nonidentical coupled Rössler oscillators [18]. Parameters are chosen such that the dynamics of drive and response in the uncoupled state is chaotic.

$$\begin{aligned}
\dot{x_1} &= -w_1 x_2 - x_3, \\
\dot{x_2} &= w_1 x_1 + 0.15 x_2, \\
\dot{x_3} &= 0.2 + x_3(x_1 - 10), \qquad (4) \\
\dot{y_1} &= -w_2 y_2 - y_3 + \epsilon(\langle x_1 \rangle - y_1), \\
\dot{y_2} &= w_2 y_1 + 0.15 y_2, \\
\dot{y_3} &= 0.2 + y_3(y_1 - 10), \qquad (5)
\end{aligned}$$

where $w_1 = 0.95$ and $w_2 = 1.05$ are the natural frequencies of drive (eq. (4)) and response (eq. (5)). Phase synchrony and generalized synchrony set in between drive and response as the strength of interaction between them increases. However, the order in which this happens is governed by the extent of parameter mismatch between the coupled units [19].

To detect GS we consider an auxiliary response system, Y' [20], and define an order parameter

$$O = \lim_{t\to\infty} \frac{1}{r} \sum_{i=1}^{r} |y_i - y_i'| \tag{6}$$

which becomes zero when complete synchrony is achieved with response system.

Figure 1a shows the phase diagram of the driven Rössler oscillator as a function of τ and ϵ. Regions in white represent synchronous dynamical states whereas dark regions correspond to asynchronous motion. Areas belonging to only GS or both GS and PS are labelled distinctly. Dashed (red) curve partitions phase space into phase synchronized and nonphase synchronized regions. Looking at the evolution of the onset of GS, it is evident from figure 1a that as delay increases, stronger coupling strengths are required for establishing functional dependence between drive and response. In figure 1a this trend is marked by open circles. Similar behaviour is seen for the PS, whence phase synchrony sets in at larger coupling strength for increasing delay. The onset of PS is confirmed by looking at the difference in frequencies of drive and response. For delay $\tau = 1$, the frequency difference is plotted in figure 1c, where $\Delta\Omega$ denotes the magnitude of frequency difference between drive and response. With increase in coupling strength (ϵ), $\Delta\Omega$ decreases. At $\epsilon = \epsilon_{\mathrm{PS}}$, $\Delta\Omega$ equals zero, marking the set-up of PS.

For small delay, GS region is found to be intercepted by asynchronous region. This is reflected in the plot of maximal conditional Lyapunov exponent (λ_{c_1}) for $\tau = 1$ (figure 1b). For very weak coupling the dynamics is not synchronized. As coupling strength increases, λ_{c_1} decreases and becomes negative in a small window of ϵ. Beyond this window, $\lambda_{c_1} > 0$ finally becoming negative at $\epsilon = 0.1$. Since negativeness of the largest conditional Lyapunov exponent is a necessary condition for GS, the areas of $\lambda_{c_1} < 0$ correspond to generalized synchrony. Figure 1b is a composite of 50 different initial conditions ruling out the possibility of multistability. The island of asynchrony is found to exist even for other values of parameter misfit. Figure 1b shows that λ_{c_1} decreases monotonically till

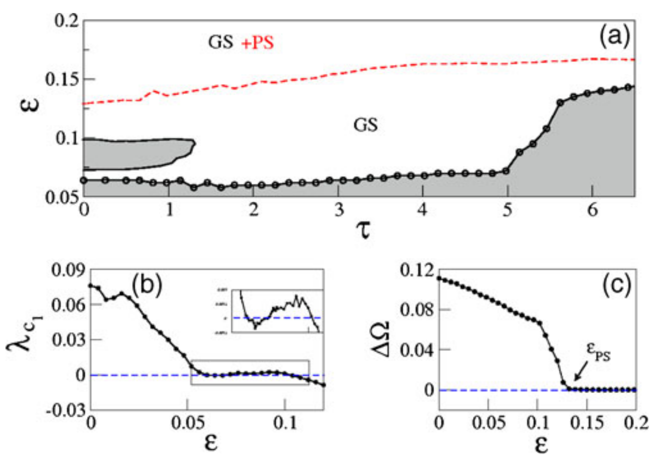

Figure 1. (**a**) Schematic phase diagram for the driven Rössler oscillator. Dark region represents unsynchronized dynamics. Regions in white correspond to GS, PS or both, with respective areas labelled. Black line segregates areas of synchrony and asynchrony. Open circles mark the points of first onset of GS. Red curve is the critical line across which PS occurs. Region below it corresponds to phase asynchrony and above it corresponds to PS, (**b**) largest conditional Lyapunov exponent for $\tau = 1$. This plot is a composite of 50 different initial conditions, (**c**) frequency difference between drive and response for $\tau = 1$. Phase synchronization is established at coupling strength ϵ_{PS}.

$\epsilon = 0.065$, with λ_{c_1} crossing zero at $\epsilon = 0.06$. After reaching the minimum, λ_{c_1} starts increasing. This increase signifies decrease in amplitude correlation between drive and response finally leading to destabilization of the manifold of generalized synchronization at $\epsilon = 0.072$. This region of asynchrony survives only for a small range of intermediate coupling strength values. As coupling becomes stronger, GS is reestablished at $\epsilon = 0.1$.

In coupled systems it has been observed that before phase synchronization there is an increase in natural frequency disorder, which is called anomalous phase synchronization [21]. The effect has also been detected in delay-coupled oscillators where phase synchronized region is intercepted by island of asynchrony in the parameter space of coupling strength and delay [22]. Analogously, here the transition from GS to unsynchronized motion and then back to GS is similar to the effect of anomalous phase synchronization and hence, we term it as anomalous generalized synchronization.

3.2 *Limit cycles*

Next we study the case of periodic drive–response configuration. The system under consideration is Landau–Stuart oscillator [23], which shows simple period-1 limit cycle behaviour. The evolution equations are as follows:

$$\dot{Z}_{\mathrm{d}}(t) = [A_{\mathrm{d}} + i w_{\mathrm{d}} - |Z_{\mathrm{d}}(t)|^2] Z_{\mathrm{d}}(t),$$

$$\dot{Z}_{\mathrm{r}}(t) = [A_{\mathrm{r}} + i w_{\mathrm{r}} - |Z_{\mathrm{r}}(t)|^2] Z_{\mathrm{r}}(t) + \epsilon[\langle Z_{\mathrm{d}}\rangle_\tau - Z_{\mathrm{r}}(t)], \quad (7)$$

where $Z_{\mathrm{d,r}}$ are the complex amplitudes of respective oscillators. In the absence of coupling ($\epsilon = 0$), each oscillator has a stable limit cycle on which it moves with its natural frequency w_{d} or w_{r}. The amplitude of oscillations is directly proportional to A_{d} for drive and A_{r} for

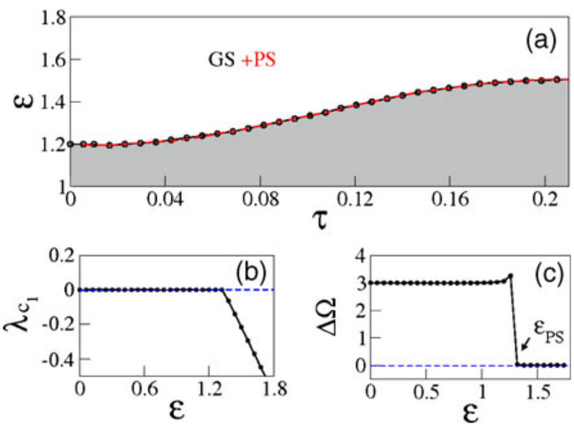

Figure 2. (**a**) Schematic phase diagram for unidirectionally coupled Landau–Stuart oscillators. Dark region represents unsynchronized dynamics. Regions in white correspond to GS and PS. Black line segregates regions of synchrony and asynchrony. Open circles mark the points of onset of GS. Red line is the locus of transition to PS. For fixed τ, GS and PS are built at the same ϵ, hence the curves for GS and PS overlap, (**b**) maximal conditional Lyapunov exponent for $\tau = 0.1$, (**c**) frequency difference between drive and response for $\tau = 0.1$, ϵ_{PS} indicates the onset of PS.

response. For numerical simulations we take parameters as $A_\mathrm{d} = 1$, $A_\mathrm{r} = 1.5$, $w_\mathrm{d} = 30$ and $w_\mathrm{r} = 33$.

Different synchronous regimes as a function of τ and ϵ are shown in figure 2a. The black curve demarks regimes of asynchronous and synchronous dynamical states in the parameter space. The locus of transition to GS (marked by open circles) overlaps with that of PS (marked by red curve), showing clearly distinct regions of synchrony (white) and asynchrony (dark). Figure 2b shows variation of largest conditional Lyapunov exponent (λ_{c_1}) of response for $\tau = 0.1$. At $\epsilon = 1.34$, λ_{c_1} becomes negative indicating the onset of GS. Difference in frequencies of drive and response for the same τ is plotted in figure 2c. At ϵ_PS, frequency of response equals that of drive, their phases get locked and PS is established.

Transition curves for GS and PS show an increasing trend with τ, similar to that obtained for coupled chaotic Rössler oscillators. Results from the analysis of the above two distinct configurations indicate that inclusion of response time makes realization of synchronous relation between drive and response difficult.

4. Conclusions

In nature all the systems have certain inherent response time. They perceive and process all the input received during this time interval. We have explored the effect of response time in unidirectionally coupled oscillators and found that larger delay requires stronger interaction between drive and response for establishing generalized synchronization and phase synchronization. This behaviour is found to hold true for different kinds of dynamical systems coupled in regular as well as chaotic drive–response configurations. The increasing trend of PS and GS locus in directed coupling can be interpreted as the destabilizing effect of response time on the phase and amplitude response of coupled units. In coupled chaotic Rössler oscillators, the regime of GS was found to be intercepted by region of asynchrony. This indicates that the destabilizing effect of response time on amplitude response may reappear even after GS is stabilized. This behaviour is analogous to the anomalous effect in phase synchronization, hence, it can be referred to as anomalous generalized synchronization.

Delays arising as a consequence of response time is common in neuronal systems. For instance, signal transmission at chemical synapses take a finite time to occur. Similarly, electrical and electronic devices have inherent response time due to internal circuitry, or in lasers, there can be finite time for output due to finite cavity length. Another significant area of application of response delays is in the modelling of epidemics, where time taken for the number of infected persons to become sufficiently large for propagation of epidemic is a significant factor. The interaction in these systems, particularly in the case of neurons, population dynamics and signal transmission, can be unidirectional, and thus the results presented here can find application in these practical situations.

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