Umklapp Processes for Electrons and Their Renormalisation Group Flow

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Abstract

We study Umklapp couplings and their renormalisation group flow for electrons in a two-dimensional lattice. It is shown that the effective low energy Hamiltonian involves not only forward scattering, but also considerable non-forward Umklapp scattering, when the electrons fill a substantial fraction of the valence band. The behavior of these couplings under the renormalisation group is studied at the tree and one-loop level. It is shown that they remain marginal. We conclude with the possible consequences of this on properties of such electronic systems and their Fermi liquid theory.

I. INTRODUCTION.

In recent years the renormalisation group (RG) method has been used to give a simple microscopic justification of the postulates of the Landau theory of Fermi liquids [1–6]. In particular it has been shown that for purposes of the low-energy excitations near the Fermi surface, the fermion interaction can be characterised by just a few relevant couplings. For normal Fermi liquids in two dimensions without BCS interactions, it has been argued that, the only interaction that is marginal corresponds to forward scattering and its exchange. Roughly speaking , this happens because (a) low-energy physics involves only states lying arbitrarily close to the Fermi surface and (b) for such states on the Fermi surface momentum conservation permits only forward scattering and its exchange. Careful RG analysis supports this conclusion as shown in Shankar's lucid review of this topic [1]. It is this forward scattering coupling constant that forms the microscopic basis for Landau's phenemenological Fermi liquid parameter in 2 dimensions [7–9].

While this argument is good at sufficiently low densities, at higher densities when the Fermi surface approaches the Brilluoin zone boundary, additional lattice effects come into play. These are related to Umklapp processes, where momentum is conserved only modulo reciprocal lattice vectors. In such a process, a pair of particles lying on or near the Fermi surface can scatter in specific non-forward directions, in addition to the usual forward scattering [10].

In this paper we will analyse the conditions under which such Umklapp couplings arise to a significant extent and study the RG flow of such couplings. We will see that circumstances where substantial Umklapp scattering takes place are by no means extraordinary. For typical systems, Umklapp processes will shown to be permitted for some electrons on or near the Fermi surface already at densities well below half-filling. Around half-filling, almost all pairs of states on the Fermi surface will be permitted some Umklapp scattering, in addition to forward scattering. Systems with such filling factors in the valence band are not uncommon. For example in Alkali metals, which are nice examples of one band free electron-like Fermi

surfaces, the ratio of Fermi surface radius to the distance of the zone face closest to the origin is 0.88 [11].

We will also study the flow of these Umklapp couplings under the renormalisation group process, both at the tree level and the one-loop level. We will see that at the tree level they are marginal under the RG flow and thus survive. But they do not make any further contribution at the one-loop level to either their own RG flow equation or to the flow equation of either the forward or BCS couplings.

This paper is organised as follows: In the next section we will set up the preliminaries and study the RG flow of the couplings at the tree level. For simplicity, we will work in two space-dimensions although it will be clear that our analysis and conclusions can be generalised to 3-dimensions as well. To be specific we will work with a square lattice and since spin is not an essential complication to our arguments, we will study spinless one-band Hamiltonians, just as was done in ref 1. The kinetic and the quartic interaction terms in the starting Hamiltonian will be taken to be quite general. The Fermi surface need not be isotropic (circular) ; it is sufficient for our analysis if it is convex and time-reversal invariant – which includes a fairly large class of systems. Following this we do the RG analysis at the tree level and show that even in the presence of Umklapp, the only relevant couplings are those that satisfy (lattice) momentum conservation on the Fermi surface.

Armed with this result, in sec.III, we will discuss the kinematics of Umklapp processes, for particles lying <u>on</u> the Fermi surface. A pictorial criterion will be given of the fraction of initial states that can suffer Umklapp scattering. This will establish our basic claim that significant Umlapp can take place at densities near half-filling. In sec. IV, we will study the one-loop RG flow of Umklapp couplings. It will be shown that while they remain marginal at the tree level, at one-loop level, they do not to contribute to either their own flow equation or those of the forward or BCS couplings. This is followed by a discussion on the physical implications of our result in Sec.V.

II. THE MODEL AND TREE LEVEL RG

The aim of this section is to obtain conditions , upto tree level, on 4-point Umklapp couplings so that they may remain marginal under RG flow. The analysis in this section closely follows that of Shankar [1] , adapted to handle anisotropic Fermi surfaces and Umklapp processes. We will omit intermediate steps but the major steps will be shown, with all the definitions needed for completeness. We begin with a fairly general one-band system of spinless electrons in a two dimensional lattice with chemical potential μ and a Hamiltonian given by

$$
\begin{aligned}
\mathcal{H} & = \sum_{\vec{K} \epsilon S_\Lambda} (E(\vec{K}) - \mu)C^\dagger_{\vec{K}}C_{\vec{K}} \; + \; \frac{1}{N} \sum_{\vec{K}_1,\vec{K}_2,\vec{K}_3,\vec{K}_4 \epsilon S_\Lambda} U(\vec{K}_1, \vec{K}_2, \vec{K}_3, \vec{K}_4) \\
& \times C^\dagger_{\vec{K}_4} C^\dagger_{\vec{K}_3} C_{\vec{K}_2} C_{\vec{K}_1} [\sum_i \delta(\vec{K}_4 + \vec{K}_3 - \vec{K}_2 - \vec{K}_1 - \vec{G}_i) \,]
\end{aligned}
\tag{2.1}
$$

where the notation is as follows:

$C^\dagger_{\vec{K}}$ and $C_{\vec{K}}$ denote the creation and annhilation operators for the single particle eigenstates of wavevector $\vec{K}$. Though in principle the wavevector $\vec{K}$ ranges over the entire first Brillouin zone of the lattice, we will be interested only in the low energy excitations of the system near the Fermi surface $E(\vec{K}) = \mu$. Therefore, as is usually done the $\vec{K}$-sums are restricted in the model to an energy shell S_Λ defined by $\mid E(\vec{K}) - \mu \mid \leq \Lambda$. This is what is meant by S_Λ in the sums in eq.(2.1).

Given these energy shell conditions and the fact that we want to consider non-isotropic $E(\vec{K})$ as well, we will sometimes find it convenient to change variables on the $\vec{K}$-plane from $\vec{K} = (K_x, K_y)$ to (ϵ, θ) where $\epsilon(\vec{K}) \equiv E(\vec{K}) - \mu$, and $\theta \equiv tan^{-1}\frac{K_y}{K_x}$, with $J(\epsilon, \theta)$ as the Jacobian of this tranformation. (We will ignore possible Van Hove singlarities [11] in the Jacobian J as well as nesting effects since these are not directly related to our primary interest on Umklapp processes. Those effects will have to be handled in addition if needed.) The function $U(\vec{K}_1, \vec{K}_2, \vec{K}_3, \vec{K}_4)$ denotes the 4-point interaction coupling in the single particle wavevector basis. In the second term of Eq.(1), momentum conservation δ-functions are inserted as appropriate for the periodic lattice, i.e. where $\vec{G}_i$'s are all possible reciprocal

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lattice vectors, including the null vector. When $\vec{G}_i = 0$, we have the usual normal process, but Umklapp processes with $\vec{G}_i \neq 0$, are also included wherever allowed. In fact our main interest in this paper lies in such Umklapp processes. We place no restriction on the form of $E(\vec{K})$ other than that it ,

(i) be time-reversal invariant, i.e.

$$E(\vec{K}) = E(-\vec{K}), \tag{2.2}$$

and, (ii) give rise to a Fermi surface [FS] defined by $E(\vec{K}) = \mu$ which is convex.

The renormalisation group [RG] procedure has been described in ref. [1] in detail. For completeness we will outline the major steps and concentrate on what it does to Umklapp couplings. The zero temperature grand canonical partition function Z is first written as a path integral over Grassmann variables. One has,

$$Z = \int \left(\prod_{|\epsilon| < \Lambda} d\bar{\psi}(\omega, \epsilon, \theta) \ d\psi(\omega, \epsilon, \theta) \right) \ exp(-(S_0 + S_I)) \tag{2.3}$$

where,

$$S_0 = \int_{-\Lambda}^{\Lambda} \frac{d\epsilon}{2\pi} \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \int_0^{2\pi} \frac{d\theta}{2\pi} \ \bar{\psi}(\omega, \epsilon, \theta) \ (i\omega - \epsilon) \ \psi(\omega, \epsilon, \theta) \ J(\epsilon, \theta) \tag{2.4}$$

$$S_I = \frac{1}{4} \sum_i \left[\prod_{j=1}^{3} \int_{-\Lambda}^{\Lambda} \frac{d\epsilon_j}{2\pi} \int_{-\infty}^{\infty} \frac{d\omega_j}{2\pi} \int_0^{2\pi} \frac{d\theta_j}{2\pi} \ J(\epsilon_j, \theta_j) \ U(\vec{K}_1, \vec{K}_2, \vec{K}_3, \vec{K}_{4i}) \right.$$
$$\left. \times \bar{\psi}(\omega_4, \epsilon_{4i}, \theta_4) \ \bar{\psi}(\omega_3, \epsilon_3, \theta_3) \ \psi(\omega_2, \epsilon_2, \theta_2) \ \psi(\omega_1, \epsilon_1, \theta_1) \ \Theta(\Lambda - |\epsilon_{4i}|) \right] \tag{2.5}$$

In eq(2.5), all momenta $\vec{K}_j$ are $\vec{K}_j(\epsilon_j, \theta_j)$. ω_4 and $\vec{K}_{4i}$ are understood to be fixed by conservation conditions

$$\omega_4 = \omega_1 + \omega_2 - \omega_3 \tag{2.6}$$

and

$$\vec{K}_{4i} = \vec{K}_1 + \vec{K}_2 + \vec{G}_i - \vec{K}_3 \tag{2.7}$$

for each reciprocal vector $\vec{G}_i$.

The requirement that particle 4 also lie in the energy shell S_Λ has been implemented in (2.5), following ref. [1] by the step-function $\Theta(\Lambda - |\epsilon_{4i}|)$.

Clearly , with $\vec{K}_{4i}$ already fixed by (2.7), this shell condition on particle 4 in fact becomes a condition on particle 3. For a given $\vec{G}_i$, $\vec{K}_1$ and $\vec{K}_2$, some or no values of $\vec{K}_3$ may exist which place $\vec{K}_{4i}$ within the shell S_Λ.

The steps of the RG procedure are to first integrate out field variables in the subshells between S_Λ and $S_{\Lambda/s}$, where $s > 1$ is the scaling parameter. Then we use rescaled variables $\omega' = s\omega, \epsilon' = s\epsilon$, and $\psi'(\omega', \epsilon', \theta) = (s)^{\frac{-3}{2}}\psi(\omega, \epsilon, \theta)$, so that the shell thickness is restored to Λ and the kinetic term $(\bar{\psi}i\omega\psi)$ in S_0 is left invariant. Under this rescaling of energy variables, the corresponding rescaled momenta will be $\vec{K}'_j \equiv \vec{K}_j(\epsilon'_j, \theta_j)$. One also replaces the shell condition by an exponential damping factor $exp \frac{-|\epsilon_{4i}|}{\Lambda}$. See ref. [1] for a full description.

Under these RG transformations, in terms of rescaled variables, the shell condition returns to S_Λ. The renormalised quartic interaction becomes at the tree level,

$$S'_I = \frac{1}{4}\sum_i \left[\prod_{j=1}^{3} \left(\int_{-\Lambda}^{\Lambda} \frac{d\epsilon'_j}{2\pi} \int_{-\infty}^{\infty} \frac{d\omega'_j}{2\pi} \int_{0}^{2\pi} \frac{d\theta'_j}{2\pi} \ J\left(\frac{\epsilon'}{s}, \theta\right)\right) U'(\vec{K}'_1, \vec{K}'_2, \vec{K}'_3, \vec{K}'_{4i}) \right.$$
$$\left. \times \bar{\psi}'(\omega'_4, \epsilon'_{4i}, \theta_4) \ \bar{\psi}'(\omega'_3, \epsilon'_3, \theta_3) \ \psi'(\omega'_2, \epsilon'_2, \theta_2) \ \psi'(\omega'_1, \epsilon'_1, \theta_1) \ exp \left(-\frac{|\epsilon'_{4i}|}{\Lambda}\right)\right] \qquad (2.8)$$

where the renormalised coupling U' is given by,

$$U'(\vec{K}'_1, \vec{K}'_2, \vec{K}'_3, \vec{K}'_{4i}) \equiv U(\vec{K}_1, \vec{K}_2, \vec{K}_3, \vec{K}_{4i}) exp \left(\frac{-(s-1)|\epsilon_{4i}|}{\Lambda}\right) \qquad (2.9)$$

The following results hold in the $s \to \infty$ limit : (i) Only the ϵ'-independent term in $J(\frac{\epsilon'}{s}, \theta)$ will remain marginal allowing us to replace $J(\frac{\epsilon'}{s}, \theta)$ by $J(\theta)$ in subsequent discussion.

(ii) More importantly, in (2.9), the survival of the renormalised coupling U' requires in the $s \to \infty$ limit that $\epsilon_{4i} = 0$. Recall that ϵ_{4i} depends on $\vec{K}_{4i}$ which in turn is determined by lattice momentum conservation (2.7). Hence $\epsilon_{4i} = 0$ forces a condition on $\vec{K}_3$ for a given $\vec{K}_1$ $\vec{K}_2$ and $\vec{G}_i$. To unravel this condition on $\vec{K}_3$ it is useful to define

$$\vec{k} \equiv \vec{K} - \vec{K}_F(\theta)$$
$$\epsilon(\vec{K}) = E(\vec{K}) - \mu \equiv v_F(\theta) \ k \qquad (2.10)$$

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for each $\vec{K}$, where $\vec{K}_F(\theta)$ is a vector <u>on</u> the Fermi surface in the same direction θ as $\vec{K}$, and $v_F(\theta)$ is the Fermi velocity at that angle. Under rescaling we have $\epsilon' = s\epsilon$, $k' = sk$. Given all this,

$$
\begin{aligned}
|\epsilon_{4i}| &= v_F(\theta_{4i})|K_{4i} - K_F(\theta_{4i})| \\
&= v_F(\theta_{4i})||\vec{Q}_i + \vec{q}| - K_F(\theta_{4i})| \\
&= 0
\end{aligned}
\tag{2.11}
$$

where

$$
\begin{aligned}
\vec{Q}_i &= \vec{K}_F(\theta_1) + \vec{K}_F(\theta_2) - \vec{K}_F(\theta_3) + \vec{G}_i \\
\vec{q} &= \vec{k}_1 + \vec{k}_2 - \vec{k}_3
\end{aligned}
\tag{2.12}
$$

Since $q = q'/s$, (2.11) reduces to

$$
|Q_i - K_F(\theta_4)| = O(1/s)
\tag{2.13}
$$

In the $s \to \infty$ limit, this gives

$$
\vec{K}_F(\theta_{4i}) = \vec{K}_F(\theta_1) + \vec{K}_F(\theta_2) - \vec{K}_F(\theta_3) + \vec{G}_i
\tag{2.14}
$$

Note that this equation looks like a (lattice) momentum conservation equation for any given reciprocal lattice vector $\vec{G}_i$. Of course, $\vec{K}_F(\theta_j)$ is not the same vector as $\vec{K}_j$, but it has the same direction and lies on the Fermi surface. Thus eq.(2.14) is to be viewed as a condition on the final angles θ_3 and θ_4 given the initial angles θ_1 and θ_2 and some $\vec{G}_i$.

As far as the dependence on energy is concerned, once (2.11) is satisfied, the exponential factor in (2.9) becomes unity, and the remaining expresion for the renormalised coupling U' obeys just,

$$
U'(\{\vec{K}_j'\}) = U(\{\vec{K}_j\})
\tag{2.15}
$$

which when written as a function of (ϵ, θ) variables, becomes

$$U'(\{\epsilon_j, \theta_j\}) \;=\; U(\{\epsilon_j, \theta_j\})$$

$$\;=\; U(\{\epsilon'_j/s, \theta_j\}) \tag{2.16}$$

When expanded in powers of $1/s$, this shows that in the RG ($s \to \infty$) the energy-dependent parts of U' becomes irrelevant, and $U' \;=\; U'(\theta_1, \theta_2, \theta_3, \theta_4)$. All one needs to do is to find the allowed dependence of U' on the angles, using (2.14).

Such results had already been obtained in [1] for the isotropic non-Umklapp case. There it was shown that only forward scattering (and its exchange) remain marginal for normal liquids. The forward coupling will then depend only on the initial angles and was denoted in ref.[1] by the function $F(\theta_1, \theta_2)$. What we have done so far is only to ensure , for completeness, that a similar on-Fermi-surface condition on the angles obtains more generally. Our new results on Umklapp couplings come starting with the next section where we find the allowed values (if any) of θ_3 and θ_4 when $\vec{G}_i \neq 0$, by solving (2.14) <u>on</u> the Fermi surface.

III. KINEMATICS OF UMKLAPP PROCESSES.

In this section let us explore the angle condition (2.14). We recall that this is just a lattice momentum conservation condition on four momentum vectors <u>on</u> the Fermi surface for any given lattice vector $\vec{G}_i$. To be specific, let us take a square lattice. Its reciprocal lattice vectors are $G_{\pm 1} \;=\; (\pm 2\pi, 0)$ and $G_{\pm 2} \;=\; (0, \pm 2\pi)$ in units of inverse lattice spacing. In 2-D, the Fermi surface can be taken to be a closed curve, described in polar coordinates by $r \;=\; f(\theta)$, with $f(\theta) \;=\; f(\theta + \pi))$ by virtue of the time-reversal condition (2.2).

To start with, let us prove a small Lemma about the domain of the total momentum of a pair of particles on the Fermi surface.

<u>Lemma</u>

Consider a pair of particles with momenta $\vec{K}_1$ and $\vec{K}_2$ lying <u>on</u> the Fermi surface $r \;=\; f(\theta)$, corresponding to vectors OC and OD in figure 1a. Let their total momentum $\vec{P} \;=\; \vec{K}_1 \;+\; \vec{K}_2$ lie at some angle θ_P with magnitude $|\vec{P}| = $ OP as shown. Then by the assumed convexity,

$$OP = 2(OB) \leq 2f(\theta_P) \qquad (3.1)$$

Hence the sum of any two momenta on the Fermi surface always lies in the area bounded by the curve $r = 2f(\theta)$. Conversely, consider any vector $\vec{P}$ lying on or inside the curve $r = 2f(\theta)$, and denoted by OP in fig 1b. Draw another Fermi surface centered at P. Since $OP \leq 2f(\theta_P) = 2(OL) = OL + PA$, these two Fermi surfaces will generically intersect at two points C and D. By time-reversal symmetry, the vectors OD and CP are equal. Hence $\vec{P}$ will be the sum of two momenta $\vec{K}_1$ and $\vec{K}_2$ on the original Fermi surface denoted by OC and OD respectively. Altogether, we see that the domain of the sum of all possible pairs of vectors on the Fermi surface is the entire area bounded by $r = 2f(\theta)$. For convenience let us henceforth call this curve $r = 2f(\theta)$ the "double Fermi surface" (DFS) and the area enclosed by it as the "double Fermi volume" (DFV).

[Although our discussion will mostly deal with 2 dimensional systems, it is useful to note that a similar statement about the domain of $\vec{P}$ can also be made for a 3 dimensional system with a convex time-reversal invariant Fermi surface $r = f(\theta, \phi)$, using the same arguments. The total momentum of any pair of particles on the Fermi surface will once again lie in the DFV bounded by the DFS $r = 2f(\theta, \phi)$. Conversely, any $\vec{P}$ in this volume will be a sum of some pair of momenta on the Fermi surface. In fact in 3-D, the analogue of fig 1b will have two Fermi surfaces intersecting on a whole curve and not just 2 points, so that any given $\vec{P}$ will correspond to a whole family of pairs of momenta $\vec{K}_1$ and $\vec{K}_2$ on the original Fermi surface .]

Returning to 2 dimensions, it will be helpful to first recall the familiar case of non-umklapp processess, with all momenta lying on the Fermi surface. In a non-umklapp scattering process momentum conservation requires

$$\vec{K}_1 + \vec{K}_2 = \vec{P} = \vec{K}_3 + \vec{K}_4 \qquad (3.2)$$

In the absence of nesting, a generic non-zero total momentum $\vec{P}$ will be the sum of a unique pair of momenta on the Fermi surface, such as the pair $\vec{K}_1$ and $\vec{K}_2$ in figures 1a and 1b. Hence, the final pair of momenta have to be the same as the initial pair. Therefore,

only forward or its exchange (backward) scattering is allowed on the Fermi surface. The corresponding tree level renormalised coupling U' is some function $F(\theta_1, \theta_2)$ where recall that θ_i refers to the polar angle of the $\vec{K}_i$. The exception to this rule is the case $\vec{P} = 0$. Then the two Fermi surfaces in Fig 1b will coalesce, and the final momenta could be any pair of equal and opposite vectors on the Fermi surface. This corresponds to the "BCS" coupling denoted by $V(\theta_1, \theta_3)$. In the absence of Umklapp, these two functions $F(\theta_1, \theta_2)$ and $V(\theta_1, \theta_3)$ are the only tree-level four-fermion couplings that remain marginal as has been shown in [1].

Our interest in this paper is to discuss the additional Umklapp four fermion coupling functions that arise when the lattice vector $\vec{G}_i \neq 0$ in eq.(2.14). These satisfy

$$\vec{P} \equiv \vec{K}_1 + \vec{K}_2$$
$$= \vec{P}_i$$
$$\equiv \vec{K}_3 + \vec{K}_4 \pm G_i \quad , \qquad i = \pm 1, \pm 2 \tag{3.3}$$

with all four $\vec{K}_j$ lying on the Fermi surface.

This requirement is most easily illustrated pictorially, as shown in fig 2. Draw the DFS at $r = 2f(\theta)$, around the origin O in momentum space. As argued above, any pair of initial momenta on the Fermi surface will add up to a total momentum $\vec{P}$ which corresponds to some point P lying in the DFV enclosed by this curve. Now, draw four more images of this DFV displaced respectively by the reciprocal lattice vectors G_i. They will be centered respectively around O_1, O_2, O_3 and O_4, which are the images of O in the neighbouring Brillouin zones. For sufficiently low density and small $K_F(\theta)$, none of these image DFV will overlap with the original DFV. At such low densities, as is well known, no umklapp is possible.

But, for sufficiently high density such overlap will develop, as depicted in fig. 2a. [The threshold density for such overlap to begin will depend on the details of the single-particle energy $E(\vec{K})$, but typically it should begin to happen well before half-filling.] When such overlap exists, consider a total initial momentum $\vec{P} = \vec{OP}$ which lies in the region of overlap between the original DFV and one of its displaced images. Then the correspondingly Brillouin displaced image of P, denoted by R in fig.2a will also lie inside the original DFV,

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and hence be an acceptable total momentum for the final pair of states. Thus, an umklapp process can happen in such a case. As illustrated in fig.2b, if

$$\vec{OP} \equiv \vec{P} = \vec{K}_1 + \vec{K}_2 \tag{3.4}$$

then,

$$\vec{OR} \equiv \vec{P}_1 = \vec{P} + \vec{G}_1$$
$$= \vec{K}_3 + \vec{K}_4 \tag{3.5}$$

Note that in this umklapp process the final pair $\vec{K}_3$ and $\vec{K}_4$ will be distinct from (although uniquely determined by) the initial pair $\vec{K}_1$ and $\vec{K}_2$. This in turn implies , for these momenta, an umklapp coupling $F_u(\theta_1, \theta_2)$, which equals $U(\vec{K}_1, \vec{K}_2, \vec{K}_3, \vec{K}_4)$ with momenta as given in fig.2b. This is in addition to the usual forward scattering coupling $F(\theta_1, \theta_2)$ (which formed the microscopic basis of the Fermi liquid parameter in 2-d).

Such umklapp processes exist only for those values of the initial total momentum $\vec{P}$ which lie in the region of overlap between the original DFV and one of its Brillouin displaced images. Such regions are shown dotted in the example of fig.2a. In the shaded regions where the original DFV overlaps with two different displaced images, each initial $\vec{P}$ will permit two different umklapp processes. In the un-dotted, unshaded region in the interior of the original DFV in fig.2a, only the forward coupling is allowed except for $\vec{P} = 0$ where the so-called BCS coupling V is also allowed. As the electron filling fraction increases the overlap regions will increase. A stage will be reached when every point in DFV, or equivalently, every pair of initial momenta on the Fermi surface will yield one or more umklapp processes, in addition to forward scattering. The threshold density at which this begins to happen will depend on the precise shape of the Fermi surface, but generically it should happen around half-filling.

To illustrate this let us take the example of free fermions half-filling a square lattice with nearest neighbour hopping (see fig.3). The spectrum is

$$E(\vec{K}) = -cos(K_x) - cos(K_y) \tag{3.6}$$

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The reciprocal lattice vectors are $(0, \pm 2\pi)$ and $(\pm 2\pi, 0)$. The Brillouin zone is a square of side 2π not shown in the figure but its corners $E = (-\pi.\pi), F = (\pi, \pi), G = (\pi, -\pi)$ and $H = (-\pi, -\pi)$ are marked in fig.3. The Fermi surface at half-filling is the smaller tilted square shown in the figure whose vertices touch the Brillouin zone . The corresponding DFV is the larger tilted square ABCD with vertices at $(\pm 2\pi$, 0) and $(0$, $\pm 2\pi)$. Now, consider this DFV and its image shifted by one of the reciprocal lattice vectors, say $\vec{G}_1 \equiv (2\pi, 0)$ as shown by the dashed tilted square . They overlap in the shaded region OFCG. For any generic point P in this overlap region, its Brillouin shifted image P_1 also lies in the original DFV and therfore a valid total momentum. The initial pair $\vec{K}_1 + \vec{K}_2$ on the Fermi surface whose total momentum is $\vec{OP}$ as well as the distinct final pair $\vec{K}_3 + \vec{K}_4$ whose total momentum is $\vec{OP}_1$ are also shown in fig.3. Thus any initial total momentum lying in OFGC will permit an umklapp process, where momentum conservation is violated by the reciprocal vector $(-2\pi, 0)$. This overlap region OFGC corresponding to the reciprocal vector $(-2\pi, 0)$ occupied one-fourth (the right quadrant) of the original DFV. By similarly considering overlaps with the other 3 DFV images shifted respectively by $(-2\pi, 0)$ and $(0, \pm 2\pi)$, one can see that every initial momentum will lie in the overlap with one or the other DFV image and can permit an umklapp final state. Therefore, in this example, for every generic initial pair of paticles on the Fermi surface, there will be an umklapp process in addition to the familiar forward scattering coupling. If the filling is somewhat less than half, there will be a small region near the origin (corresponding to $\vec{P} \approx 0$ where the original DFV does not overlap with any of its Brillouin shifted cousins. Barring this region, Umklapp will happen for all other initial total momentum. [We chose this example because its geometry is simple. We are aware it carries other problems like Van Hove singularities and nesting vectors . One can avoid these problems by choosing slightly more than half filling, or anisotropic hopping but the presentation will be more cumbersome.]

In summary, we have given in this section a pictorial discussion of when umklapp couplings exist. It is evident from our discussion that for typical systems at a filling fraction of about one-half, a substantial fraction of , if not all initial states on the Fermi surface can

yield umklapp processes in addition to the forward scattering process. Put together with the tree level RG analysis of sec. 2, we see that these umklapp couplings will remain marginal and can therefore potentially play a role in the low energy properties of such systems. Let us next extend their study to the one-loop level.

IV. RG FLOWS AT HIGHER LOOPS

This section is concerned with the RG analysis at one loop level. The forward coupling $F(\theta_1, \theta_2)$ and the BCS coupling $V(\theta_1, \theta_3)$ have been studied in ref.(1). We follow the same procedure to study the role of $F_u(\theta_1, \theta_2)$. Since the analysis is more transparent in the wave-vector space, we present the case of isotropic Fermi surfaces in detail and indicate the steps for generalisation to noncircular FS.

The contribution of F_u to S_0 is easily disposed of, as it comes from the diagram (4a). Clearly only forward coupling is allowed in this diagram, hence F_u cannot contribute to the renormalisation of any quadratic couplings.

Next we consider the quartic couplings. We need study at one-loop level only those couplings which survived at the tree-level in our sec 2. The one-loop diagrams that contribute to these are shown in Figs.(4b) to (4d). Referring to these diagrams note that all the internal lines have momenta in the shell $S_{d\Lambda}$ and for the outer legs the magnitude of all the momenta are set equal to K_F and their frequencies to zero. Set $\vec{Q} = \vec{K}_3 - \vec{K}_1$, $\vec{Q'} = \vec{K}_4 - \vec{K}_1$, and $\vec{P} = \vec{K}_1 + \vec{K}_2$ (where we have used the three-vector notation to write both wave-vector and frequency). Note that the forward coupling $F(\theta_1, \theta_2)$ is obtained in the limit $Q \to 0$, the forward exchange coupling $F(\theta_2, \theta_1)$ is obtained in the limit $Q' \to 0$, the BCS coupling $V(\theta_1, \theta_3)$ is obtained in the limit $P \to 0$, and finally $F_u(\theta_1, \theta_2)$ is obtained when $\vec{K}_3 + \vec{K}_4 = \vec{P} + \vec{G}_i$. Now the contribution of the diagram (4b) to the quartic coupling is given by

$$dU\,(\vec{K}_4, \vec{K}_3, \vec{K}_2, \vec{K}_1) \;=\; \int_{S_{d\Lambda}} \frac{d^2K}{2\pi} \int \frac{d\omega}{2\pi} \; \frac{U(\vec{K} - \vec{Q},\, \vec{K}_3, \vec{K}, \vec{K}_1)\, U(\vec{K}_4, \vec{K}, \vec{K}_2, \vec{K} - \vec{Q})}{(i\omega - \epsilon(\vec{K}))\,(i\omega - \epsilon(\vec{K} - \vec{Q}))}$$

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$$+ \int_{S_{d\Lambda}} \frac{d^2K}{2\pi} \int \frac{d\omega}{2\pi} \frac{U(\vec{K} - \vec{Q} + \vec{G},\ \vec{K}_3, \vec{K}, \vec{K}_1)\ U(\vec{K}_4, \vec{K}, \vec{K}_2, \vec{K} - \vec{Q} + \vec{G})}{(i\omega - \epsilon(\vec{K}))(i\omega - \epsilon(\vec{K} - \vec{Q} + \vec{G}))} \qquad (4.1)$$

where the first term corresponds to the normal process, and the second term corresponds to the Umklapp process. When $\vec{Q} \to 0$, as required for the contribution to $F(\theta_1, \theta_2)$, the integration over ω yields a zero answer for the first term, as both the poles have the same sign. For the second term the contributing phase space is shown in Fig.(5) where the FS and its shell are drawn with origins at $\vec{0}$ and at $\vec{R} = -\vec{G}$. The requirement that the two poles have opposite signs restricts the phase space to regions I, II, III and IV. All these are of order $\frac{1}{s^2}$ and hence make no contribution to the RG flow equation.

In order to consider the contribution of Eq.(4.1) to other the couplings V and F_u, one notes that as long as $\vec{K}' - \vec{K} \equiv \vec{R}$ with $R \neq 0$, the contributing phase space is of order $\frac{1}{s^2}$ and thereby makes no contribution to the RG flow equations. For V-coupling $\vec{P} = 0$ and $\vec{Q}$ is generically nonzero which makes $\vec{R} = \vec{Q}$ or $\vec{Q} - \vec{G}$ for the first and the second terms of eq.(4.1) respectively. It should also be noted that for less than a certain filling fraction depending upon the geometry of the FS the region around $\vec{P} = 0$ is not subject to Umklapp scattering. For F_u coupling, for a given $\vec{K}_1$ and $\vec{K}_2$, one one obtains $\vec{K}_3$ and $\vec{K}_4$ using the construction of Fig.(2b). One again sees barring an exception $\vec{Q} \neq 0$. Thus this diagram make no contribution to the flows of V and F_u. The diagram (4c) is similar to (4b) except that $\vec{K}' = \vec{K} - \vec{Q}'$. It is easily verified that contributions to F, V, F_u are all of order $\frac{1}{s^2}$

Now we come to diagram (4d). Here $\vec{K}' = \vec{R} - \vec{K}$ with $\vec{R} = \vec{P}$ or $\vec{P} + \vec{G}$. The contribution of the diagram may be written as

$$dU\ (\vec{K}_4, \vec{K}_3, \vec{K}_2, \vec{K}_1) = \frac{1}{2} \int_{S_{d\Lambda}} \frac{d^2K}{2\pi} \int \frac{d\omega}{2\pi} \frac{U(\vec{P} - \vec{K},\ \vec{K}, \vec{K}_2, \vec{K}_1)\ U(\vec{K}_4, \vec{K}_3, \vec{K}, \vec{P} - \vec{K})}{(i\omega - \epsilon(\vec{K}))\ (-i\omega - \epsilon(\vec{P} - \vec{K}))}$$

$$+ \frac{1}{2} \int_{S_{d\Lambda}} \frac{d^2K}{2\pi} \int \frac{d\omega}{2\pi} \frac{U(\vec{P} - \vec{K} + \vec{G},\ \vec{K}, \vec{K}_2, \vec{K}_1)\ U(\vec{K}_4, \vec{K}_3, \vec{K}_1, \vec{P} - \vec{K} + \vec{G})}{(i\omega - \epsilon(\vec{K}))(-i\omega - \epsilon(\vec{P} - \vec{K} + \vec{G}))} \qquad (4.2)$$

The frequency summation yields non-zero contribution only when $\epsilon(\vec{K})$ and $\epsilon(\vec{K}')$ have the same sign. The scattering phase space is again determined in the same fashion. One replaces in fig (5) $\vec{K} - \vec{R}$ by its negative and sets $\vec{R} = \vec{P}$, or $\vec{P} + \vec{G}$. Contributions come from four regions like I' which are of order $\frac{1}{s^2}$. Now clearly for couplings F and F_u, $\vec{P}$ is nonzero

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and one gets no contribution. Finally consider the V-coupling for which $\vec{P} = 0$. Here as discussed in Ref.(1), a finite contribution is obtained from the first integral of Eq.(4.2), but the second integral does not contribute. Thus the F_u coupling does not affect the flow equation for V either. To summarise, one find that upto one-loop, couplings F and F_u are marginally relevent, while V is relevent and its flow is not affected by either F or F_u.

Our discussion so far has been up to the one loop level. We have no rigorous results about higher loop diagrams or the convergence of the loop expansion of the beta function, unlike references 4 to 6. However, using the techniques of $\frac{1}{N}$ expansion one can offer a heuristic argument, similar to the one used in ref 1, to the effect that the umklapp coupling F_u will continue to _not_ contribute even in higher loop diagrams to the beta functions of the forward, BCS, or umklapp processes. Recall that in the RG limit, this theory can be mapped into a N flavour theory (with $N \to \infty$) by discretising the angular integration into cells of width

$$\Delta\theta \;=\; \frac{2\Lambda}{K_F} \;=\; \frac{2\pi}{N} \tag{4.3}$$

The interaction in this mapping becomes of order $\frac{1}{N}$. Then, by counting powers of N, one can argue (see ref. 1) that only certain sequences of "bubble" diagrams will survive in the RG limit ($\Lambda \to 0, N \to \infty$).

The one-bubble diagram which we had just studied has two vertices each bringing a power of $\frac{1}{N}$. Then there was integration over one internal momentum $\vec{K}$ as in eq. 4.1 or 4.2. If the angle of this momentum can run freely over a finite range as $\Lambda \to 0$, that introduces a power of N, giving the diagram a $\frac{1}{N}$ dependence altogether, of the same order as the tree diagram. This can happen only if the vector $\vec{R}$ of fig. 5 is zero. On the other hand , when $\vec{R}$ is not zero, i.e. when the internal momentum angles are constrained by the external momenta, they do not bring another power of N. Thus, for the forward coupling F the sequence that survives is a product of bubbles with purely F vertices, while for the BCS couplin V, it is a product of bubbles with purely V vertices. Whereas, whenever a F_u vertex is present, the vector $\vec{R}$ is non-zero and the diagram is down by a power of $\frac{1}{N}$.

Hence, the umklapp coupling F_u does not contribute at any loop order to the beta

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function of any of the couplings , including itself. However, at the tree level, we saw that it is marginal and survives. Admittedly, these arguments are only heuristic. We have not examined questions of convergence of the perturbation series unlike references 4 to 6.

From these arguments the generalisation to noncircular Fermi surfaces possessing the time-reversal symmetry is quite straight forward. The contribution in this case also depends on the value of the momenta $\vec{R}$. Whenever $\vec{R}$ determined by external legs is nonzero the scattering phase space is again of order $\frac{1}{s^2}$ .It is easily verified that the above conclusion holds for such surfaces too.

V. CONCLUSIONS

We have developed above a method for determining when and to what extent umklapp couplings will survive as marginally relevant interactions for electrons in 2 dimensional one-band systems on a lattice. We found that for densities of the order of one-half per site, most (if not all) initial pairs of particles near the Fermi surface are kinematically permitted to undergo umklapp scattering. We also showed that under RG flow these umklapp couplings , if present in the original starting Hamiltonian , will remain marginal , along with forward scattering and the BCS coupling. Therefore , unless neglected for some reason in the original microscopic Hamiltonian, umklapp couplings will very much be copiously present in the effective low energy Hamiltonian obtained in the $s \to \infty$ limit of the RG flow, for densities near one-half.

These results were obtained in 2 dimensions, but as mentioned in sec.III, they can be generlised to 3 dimensions in a straightforward way. Given a pair of particles on a convex time-reversal invariant Fermi surface $r = f(\theta, \phi)$, the domain of their sum $\vec{P}$ will be the DFV enclosed by $r = 2f(\theta, \phi)$. The overlap of this DFV with its partners shifted by various reciprocal lattice vectors will again give $\vec{P}$-domains of umklapp processes. Whereas in 2D, generically there will be a unique pair of final momentum directions for umklapp, given an initial pair, in 3D there will be a whole cone of allowed final directions for umklapp, because

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the intersection of two Fermi surfaces will be closed curve rather than a pair of points. This azimuthal degree of freedom for the final states will be available for umklapp for the same reason as happened for normal (non-umklapp) couplings [1],

Given that umklapp couplings will survive in the low energy effective Hamiltonian, it is bound to affect some physical properties of such systems such as resistivity, thermal conductivity etc. For instance, in the case of one band Hamiltonians such as ours, electron-electron interaction cannot lead to resistance unless some umklapp process takes place. The well known T^2 dependence of resistance can arise for such systems only because of umklapp. As far as the Landau Fermi liquid theory is concerned, its existing formulation seems to involve only the forward scattering amplitude. In that theory, low energy excitations of the system are described in terms of quasi-particles with a weak residual interaction of the form $f(\vec{p}, \vec{p'})\delta n_{\vec{p}}\delta n_{\vec{p'}}$, where $\delta n_{\vec{p}}$ is the change in the occupation of the $\vec{p}$-state from the ground state. This form of the interaction involves only the occupation numbers and hence only the forward scattering amplitude. Indeed in the detailed many-body perturbative analysis $f(\vec{p}, \vec{p'})$ has been identified with the scattering amplitude $T(p+q, p'-q, p, p')$ under the zero momentum transfer limit $(\vec{q}, \omega) \to 0$ such that $\frac{q}{\omega} \to 0$. Now, as shown in sec.IV , the higher order contributions of F_u to the forward coupling F vanishes in the $s \to \infty$ limit of the RG flow. Hence the Landau interaction $f(\vec{p}, \vec{p'})$ will not , in and of itself, be affected by the presence of umklapp. However, the interaction F_u is marginal. It will have to be included by modifying the Fermi liquid Hamiltonian.

Similarly, whenever there is only a soft response $R(Q \to 0, \omega \to 0)$ to a soft probe, only the forward coupling matters. But, in a lattice, as compared to a spatial continuum, a soft probe $J(Q \to 0, \omega \to 0))$ can lead to a response $R(Q = G, \omega \to 0))$ in the presence of umklapp. In such cases, the fact that umklapp coupling remains marginal will affect the physics.

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FIG. 1. Construction of the Double Fermi Volume (DFV). The DFV is the $\vec{K}$-space region allowed to vector $\vec{P} = \vec{K}_1 + \vec{K}_2$, where $\vec{K}_1$ and $\vec{K}_2$ are any pair of vectors lying on the Fermi surface. The FS, given in polar coordinates by $r = f(\theta)$ is denoted by the solid curve. DFV is the region inside the dotted curve $r = 2f(\theta)$. Fig.(a) supports the argument that all vectors $\vec{P}$ must lie in DFV provided FS is a convex surface. Fig.(b) illustrates the converse argument that any point P inside the DFV can be sum of two vectors on the FS.

FIG. 2. Regions of total momentum $\vec{P}$ (of two particles) for which Umklapp processes are allowed. The diagram (a) shows DFV and its four images, which are DFV's drawn with their origins shifted by four reciprocal vectors of the square lattice. The lines (not drawn) OO_1, OO_2, OO_3, and OO_4 denote the four reciprocal vectors. The dotted regions for which DFV and any of its images overlap are those values of $\vec{P}$ for which an Umklapp process is allowed. For the shaded regions the DFV overlaps with two of its images, and for these two Umklapp processes are permitted. The diagram (b) shows the construction of outgoing momenta $\vec{K}_3$ and $\vec{K}_4$, when an Umklapp process occurs. Also shown are the incoming momenta $\vec{K}_1$ and $\vec{K}_2$ and their sum $\vec{P}$.

FIG. 3. Fermi surface (inner diamond) and DFV (shown as ABCD) for a half-filled square lattice with $E(\vec{K})$ given by Eq.(3.6). The Brillouin zone is the square whose vertices are EFGH. The dashed square is one of the images of DFV and the shaded region is the P-region for which Umklapp by vector (-2π,0) is allowed. Clearly the other 3 images will overlap with the remaining portions of DFV, thus permitting Umklapp for every point of DFV. Also shown are $\vec{K}_3$ and $\vec{K}_4$ for a typical set of $\vec{K}_1$ and $\vec{K}_2$.

FIG. 4. The diagrams contributing to couplings at one loop level. The internal lines carry momenta from the shells $S_{d\Lambda}$. Fig.(a) gives the contribution to two-point couplings, while (b), (c) and (d) denote the three diagrams contributing to four-point couplings.

FIG. 5. This diagram gives the momentum phase space regions that are allowed for processes depicted in diagrams from Fig.(4b) to (4d). For Fig.(4b) the vector $\vec{R}$ takes values $\vec{Q}$ and $\vec{Q} + \vec{G}$. For Fig.(4c) $\vec{R} = \vec{Q}'$ and $\vec{Q}' + \vec{G}$. The allowed values of $\vec{K}$ are from regions denoted by I, II, III, and IV. For Fig.(4d) $\vec{R} = \vec{P}$ and $\vec{P} + \vec{G}$ and the sign of $\vec{K} - \vec{R}$ is to be reversed. The allowed values of $\vec{K}$ come from four regions like I'.