Lower bound theorem for normal pseudomanifolds

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Abstract : In this paper we present a self-contained combinatorial proof of the lower bound theorem for normal pseudomanifolds, including a treatment of the cases of equality in this theorem. We also discuss McMullen and Walkup's generalised lower bound conjecture for triangulated spheres in the context of the lower bound theorem. Finally, we pose a new lower bound conjecture for non-simply connected triangulated manifolds.

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1 Introduction

The lower bound theorem (LBT) provides the best possible lower bound for the number of faces of each dimension (in terms of the dimension and the number of vertices) for any normal pseudomanifold. When the dimension is at least three, equality holds precisely for stacked spheres. (This is Theorem 3 in Section 8 below.)

Walkup, Barnette, Klee, Gromov, Kalai and Tay proved various special cases of the LBT, with Tay providing the first proof in the entire class of normal pseudomanifolds (cf. [3, 8, 10, 11, 14, 15]). However, Tay's proof rests on Kalai's, and that in turn depends on the theory of rigidity of frameworks.

Kalai showed in [10] that for $d \geq 3$, the edge graph of any connected triangulated d-manifold without boundary is "generically $(d+1)$-rigid" in the sense of rigidity of frameworks. Namely, a particular embedding of a graph in the $(d+1)$-dimensional Euclidean space is *rigid* if it can't be moved to a nearby embedding without distorting the edge-lengths (except trivially by bodily moving the entire embedded graph by applying a rigid motion of the ambient space). A graph is *generically* $(d+1)$*-rigid* if the set of its rigid embeddings in $(d+1)$-space is a dense open subspace in the space of all its embeddings. The LBT for triangulated manifolds without boundary is an immediate consequence of Kalai's rigidity theorem. Kalai also used these ideas to settle the equality case of LBT. Actually, he proved this theorem in the somewhat larger class of normal pseudomanifolds whose two-dimensional links are spheres. In [14], Tay showed that Kalai's argument extends almost effortlessly to the class of all normal pseudomanifolds. This class has the advantage of being closed under taking links, so that an induction on dimension is facilitated. Further, the so

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called M-P-W reduction (after McMullen, Perles and Walkup) works in a link-closed class of pseudomanifolds and this reduces the proof of the general LBT to proving the lower bound only for the number of edges.

The interesting application of the LBT found in [2] led us to take a close look at Kalai's proof. However, we found it difficult to follow Kalai's proof in its totality because of our lack of familiarity with the rigidity theory of frameworks, which in turn is heavily dependent on analytic considerations that seem foreign to the questions at hand. We have reasons to suspect that many experts in Combinatorial Topology share our desire to see a self-contained combinatorial proof of this fundamental result of Kalai. For instance, in a relatively recent paper [5], Blind and Blind present a combinatorial proof of the LBT in the class of polytopal spheres, even though much more general versions were available. These authors motivate their paper by stating that "no elementary proof of the LBT including the case of equality is known so far". One objective of this paper is to rectify this situation. It may be noted that Blind and Blind use the notion of shelling to prove the LBT for polytopal spheres. Shelling orders do not exist in general triangulated spheres (let alone normal pseudomanifolds), so that the proof presented here is of necessity very different.

A pointer to a combinatorial proof of LBT for triangulated closed manifolds was given by Gromov in [8, pages 211–212]. There he introduced a combinatorial analogue of rigidity (which we call *Gromov-rigidity*, or simply *rigidity* in this paper) and sketched an induction argument on the dimension to show that triangulated d-manifolds without boundary are $(d+1)$-rigid in his sense for $d \geq 2$. However, there was an error at the starting point $d = 2$ of his argument. Reportedly, Connelly and Whiteley filled this gap, but it seems that their work remained unpublished. In [14], Tay gave a proof of Gromov 3-rigidity of 2-manifolds. Here we present an independent proof of this result, based on the notion of generalised bistellar moves introduced below. It is easy to see that if all the vertex-links of a d-pseudomanifold are Gromov d-rigid, then the d-pseudomanifold is $(d+1)$-rigid in the sense of Gromov. Therefore, $(d+1)$-rigidity of d-dimensional normal pseudomanifolds follows. Now, it is an easy consequence of Gromov's definition that any n-vertex $(d+1)$-rigid simplicial complex of dimension d satisfies the lower bound $(d+1)n - \binom{d+2}{2}$ on its number of edges, as predicted by LBT. However, Gromov himself never considered the case of equality in LBT. Here we refine Gromov's theory to tackle the case of equality. It may be pointed out that in the concluding remark of [10], Kalai suggested that it should be possible to prove his theorem using Gromov's ideas. However, the details of such an elementary argument were never worked out in the intervening twenty years. It is true that Tay uses Gromov's definition of rigidity in his proofs. But, to tackle the case of equality, Tay shows that when equality holds in LBT for a normal pseudomanifold, it must actually be a triangulated manifold, so that Kalai's initial argument (based on rigidity of frameworks) applies.

We should note that the notion of generic rigidity pertains primarily to graphs and Kalai calls a simplicial complex generically q-rigid if its edge graph is generically q-rigid. On the other hand, Gromov's definition pertains to simplicial complexes. For this reason, it is not possible to compare these two notions in general. However, such a comparison is possible when the dimension d of the simplicial complex is $\geq q-1$ (and we are interested in the case $d = q-1$). In these cases, Gromov's notion of rigidity is weaker than the notion of generic rigidity. From the theory of rigidity of frameworks, it is known that if an n-vertex graph G is minimally generically q-rigid (i.e., G is generically q-rigid but no proper spanning subgraph of G is generically q-rigid) then either G is a complete graph on at most $q+1$ vertices, or else G has $n \geq q+1$ vertices and has exactly $nq - \binom{q+1}{2}$ edges, and any induced subgraph of G (say, with $p \geq q$ vertices) has at most $pq - \binom{q+1}{2}$ edges (cf. [7]. By a theorem of Laman, this fact characterizes minimally generically q-rigid graphs for $q \leq 2$). Using this result, it is easy to deduce that generic q-rigidity (of the edge graph) implies Gromov's q-rigidity for any simplicial complex of dimension $\geq q-1$.

Apart from the pedagogic/esthetic reason for providing an elementary proof of the LBT for normal pseudomanifolds (surely an elementary statement deserves an elementary proof!), we also hope that the arguments developed here should extend to yield a proof of the generalised lower bound conjecture (GLBC) for triangulated spheres. Stanley [13] proved this conjecture for polytopal spheres using heavy algebraic tools, but the general case of this conjecture due to McMullen and Walkup [12] remains unproved. Even in Stanley's result, the characterisation of the equality case remains to be done.

This paper is organized as follows. In Section 2, we give the preliminary definitions, including an explanation of most of the technical terms used in this introduction. In the next four sections, we develop the necessary tools for our proofs. Section 3 provides a combinatorial version of the topological operations of cutting or pasting handles and of connected sums. These combinatorial operations were introduced by Walkup in [15]. However, the precise combinatorics of these operations was never worked out. Section 4 introduces the main actors in the game of LBT's, namely stacked spheres and stacked balls. We also present some elementary but useful results on these objects. These are mostly well known, at least to experts. In Section 5, we introduce the notion of generalized bistellar moves (GBM) and establish their elementary properties. As the name suggests, this is a generalization of the usual notion of bistellar moves. It is also shown that any n-vertex triangulated 2-sphere (with $n > 4$) is obtained from an $(n-1)$-vertex triangulated 2-sphere by a GBM. More generally, we show that any triangulated orientable 2-manifold (without boundary) X is either the connected sum of two smaller objects of the same sort, or it is obtained from a similar object of smaller genus by pasting a handle, or else it may be obtained by a GBM from a triangulation $\widetilde{X}$ of the same manifold using one less vertex. (We wonder if similar results are true for triangulated 3-manifolds.) These results for triangulated 2-manifolds without boundary are used to give an inductive proof of their Gromov 3-rigidity in Section 7. Section 6 contains the general theory of Gromov-rigidity, including a careful treatment of the minimal situations. In Section 7, we prove the Gromov $(d+1)$-rigidity of normal d-pseudomanifolds, and show that for $d > 2$ the minimally Gromov $(d+1)$-rigid normal pseudomanifolds are precisely the stacked d-spheres. This is Theorem 2, the main result of this paper. As already indicated, the proof is an induction on d. Cutting handles plays an important role here. In Section 8, we describe the M-P-W reduction and use it to present the routine deduction of the LBT for normal pseudomanifolds from Theorem 2. In the concluding section, we state and discuss the GLBC in a form which brings out its similarity with the LBT (which is the case $k = 1$ of the GLBC). Included in this section is a discussion of the k-stacked spheres which are expected to play a role in the GLBC similar to the role played by the stacked spheres in LBT. We conclude by posing a new lower bound conjecture for non-simply connected triangulated manifolds.

2 Preliminaries

Recall that a *simplicial complex* is a set of finite sets such that every subset of an element is also an element. For $i \geq 0$, an element of size $i+1$ is called a *face* of dimension i (or an i-*face)* of the complex. By convention, the empty set is a face of dimension -1. All simplicial complexes which appear in this paper are finite. The dimension of a simplicial complex X (denoted by $\dim(X)$) is by definition the maximum of the dimensions of its faces. The

1-dimensional faces of a simplicial complex are also called the *edges* of the complex. $V(X)$ denotes the set of vertices of a complex X and is called the *vertex-set* of X.

For a simplicial complex X, $|X|$ is the set of all functions $f : V(X) \to [0,1]$ such that $\sum_{v \in V(X)} f(v) = 1$ and $\mathrm{support}(f) := \{v \in V(X) : f(v) \neq 0\}$ is a face of X. (Such a function f may be thought of as a convex combination of the Dirac delta-functions δ_x as x ranges over the face support(f).) As a subset of the topological space $[0,1]^{V(X)}$, $|X|$ inherits the subspace topology. The topological space $|X|$ thus obtained is called the *geometric carrier* of X. If $|X|$ is a manifold (with or without boundary) then X is said to be a *triangulated manifold*, or a *triangulation* of the manifold $|X|$.

A *graph* is a simplicial complex of dimension at most 1. A set of vertices of a graph G is said to be a *clique* of G if any two of these vertices are adjacent in G (i.e., form an edge of G). For a general simplicial complex X, the *edge graph* (or 1*-skeleton*) $G(X)$ of X is the subcomplex of X consisting of all its faces of dimensions ≤ 1. (More generally, for $0 \leq k \leq \dim(X)$, the *k-skeleton* $\mathrm{skel}_k(X)$ of X is the subcomplex consisting of all the faces of X of dimension $\leq k$.) Notice that each face of X is a clique in the graph $G(X)$.

If X, Y are two simplicial complexes with disjoint vertex sets, then their *join* $X * Y$ is the simplicial complex whose faces are the (disjoint) unions of faces of X with faces of Y. In particular, if X consists of a single vertex x, then we write $x * Y$ for $X * Y$. The complex $x * Y$ is called the *cone* over Y (with cone-vertex x).

If Y is a subcomplex of a simplicial complex X and Y consists of all the faces of X contained in $V(Y)$, then we say that Y is an *induced subcomplex* of X. If $A \subseteq V(X)$, then the induced subcomplex of X with the vertex-set A is denoted by $X[A]$. If α is a k-face of X, then the *closure* $\overline{\alpha}$ of α is the induced subcomplex $X[\alpha]$. Notice that $\overline{\alpha}$ consists of all the subsets of α. Thus $\overline{\alpha}$ is a triangulation of the k-ball and is also denoted by $B^k_{k+1}(\alpha)$.

If $V(X) = A \sqcup B$ is the disjoint union of two subsets A and B, then the induced subcomplexes $X[A]$ and $X[B]$ are said to be *simplicial complements* of each other. If Y is an induced subcomplex of X, then the simplicial complement of Y is denoted by $C(Y, X)$. For a face α of X, the simplicial complement $C(\overline{\alpha}, X)$ is called the *antistar* of α, and is denoted by $\mathrm{ast}(\alpha)$. Thus, $\mathrm{ast}(\alpha)$ is the subcomplex of X consisting of all faces disjoint from α. The *link* of α in X, denoted by $\mathrm{lk}(\alpha)$ (or $\mathrm{lk}_X(\alpha)$) is the subcomplex of $\mathrm{ast}(\alpha)$ consisting of all faces β such that $\alpha \sqcup \beta \in X$. For a vertex v of X, the cone $v * \mathrm{lk}_X(v)$ is called the *star* of v in X and is denoted by $\mathrm{star}(v)$ (or $\mathrm{star}_X(v)$).

A d-dimensional simplicial complex X is said to be *pure* if all the maximal faces of X have dimension d. The maximal faces in a pure simplicial complex are called its *facets*. The *facet graph* $\Lambda(X)$ of a pure d-dimensional simplicial complex X is the graph whose vertices are the facets of X, two such vertices being adjacent in $\Lambda(X)$ if the corresponding facets intersect in a $(d-1)$-face.

A simplicial complex X is said to be *connected* if $|X|$ is connected. Notice that X is connected if and only if its edge graph $G(X)$ is connected (i.e., any two vertices of X are the end vertices of a path in $G(X)$). A pure simplicial complex X is said to be *strongly connected* if its facet graph $\Lambda(X)$ is connected. The connected components of X are the maximal connected subcomplex of X. The strong components of X are the maximal pure subcomplexes of dimension $d = \dim(X)$ which are strongly connected. Notice that the connected components are vertex-disjoint, while the strong components may have faces of codimension two or more in common.

For $d \geq 1$, a d-dimensional pure simplicial complex is said to be a *weak pseudomanifold with boundary* if each $(d-1)$-face is in at most two facets, and it has a $(d-1)$-face contained in only one facet. A d-dimensional pure simplicial complex is said to be a *weak pseudomanifold* *without boundary* (or simply *weak pseudomanifold*) if each $(d-1)$-face is in exactly two facets. If X is a d-dimensional weak pseudomanifold with boundary then its boundary ∂X is defined to be the $(d-1)$-dimensional pure simplicial complex whose facets are those $(d-1)$-faces of X which are in unique facets of X. Clearly, the link of a face in a weak pseudomanifold is a weak pseudomanifold.

A *pseudomanifold* (respectively *pseudomanifold with boundary*) is a strongly connected weak pseudomanifold (respectively weak pseudomanifold with boundary). A d-dimensional weak pseudomanifold (respectively weak pseudomanifold with boundary) is called a *normal pseudomanifold* (respectively *normal pseudomanifold with boundary*) if each face of dimension $\leq d-2$ has a connected link. Since we include the empty set as a face, a normal pseudomanifold is necessarily connected. But we actually have:

Lemma 2.1. *Every normal pseudomanifold* (*respectively, normal pseudomanifold with boundary*) *is a pseudomanifold* (*respectively pseudomanifold with boundary*)*.*

Proof. Let X be a normal pseudomanifold of dimension $d \geq 1$. We have to show that its facet graph $\Lambda(X)$ is connected. If not, choose two facets σ_1, σ_2 from different components of $\Lambda(X)$ for which $\dim(\sigma_1 \cap \sigma_2)$ is maximum. Then $\dim(\sigma_1 \cap \sigma_2) \leq d-2$ but $\mathrm{lk}(\sigma_1 \cap \sigma_2)$ is disconnected, a contradiction. $\Box$

Lemma 2.2. *If X is a weak pseudomanifold with boundary then ∂X is a weak pseudomanifold without boundary.*

Proof. Take a vertex x outside X and set $\widetilde{X} = X \cup (x * \partial X)$. Clearly $\widetilde{X}$ is a weak pseudomanifold without boundary. Since $\partial X = \mathrm{lk}_{\widetilde{X}}(x)$, the result follows. $\Box$

From the definitions, it is clear that any d-dimensional weak pseudomanifold (respectively weak pseudomanifold with boundary) has at least $d+2$ (respectively $d+1$) vertices, with equality if and only if it is the simplicial complex S^d_{d+2} (respectively B^d_{d+1}) whose faces are all the proper subsets of a set of size $d+2$ (respectively, all subsets of a set of size $d+1$). Clearly, S^d_{d+2} and B^d_{d+1} triangulate the d-sphere and the d-ball, respectively. They are called the *standard d-sphere* and the *standard d-ball* respectively.

A simplicial complex X is called a *combinatorial d-sphere* (respectively, *combinatorial d-ball*) if $|X|$ (with the induced pl structure from X) is pl homeomorphic to $|S^d_{d+2}|$ (respectively, $|B^d_{d+1}|$).

If α is a face of a simplicial complex X, then the number of vertices in $\mathrm{lk}_X(\alpha)$ is called the *degree* of α in X and is denoted by $\deg_X(\alpha)$ (or $\deg(\alpha)$). So, the degree of a vertex v in X is the same as the degree of v in the edge graph $G(X)$. Since the link of an i-face α in a d-dimensional weak pseudomanifold X without boundary is a $(d-i-1)$-dimensional weak pseudomanifold, it follows that $\deg_X(\alpha) \geq d-i+1$, with equality only if $\mathrm{lk}_X(\alpha)$ is the standard sphere S^{d-i-1}_{d-i+1}.

If X is a d-dimensional simplicial complex then, for $0 \leq j \leq d$, the number of its j-faces is denoted by $f_j = f_j(X)$. The vector $(f_0, \dots, f_d)$ is called the *face-vector* of X and the number $\chi(X) := \sum_{i=0}^{d}(-1)^i f_i$ is called the *Euler characteristic* of X. As is well known, $\chi(X)$ is a topological invariant, i.e., it depends only on the homeomorphic type of $|X|$.

3 Cutting and pasting handles

Definition 3.1. Let σ_1, σ_2 be two facets in a pure simplicial complex X. Let $\psi : \sigma_1 \to \sigma_2$ be a bijection. We shall say that ψ is *admissible* if (ψ is a bijection and) the distance between x and $\psi(x)$ in the edge graph of X is ≥ 3 for each $x \in \sigma_1$ (i.e., if every path in the edge graph joining x to $\psi(x)$ has length ≥ 3). Notice that if σ_1, σ_2 are from different connected components of X then any bijection between them is admissible. Also note that, in general, for the existence of an admissible map $\psi : \sigma_1 \to \sigma_2$, the facets σ_1 and σ_2 must be disjoint.

Definition 3.2. Let X be a weak pseudomanifold with disjoint facets σ_1, σ_2. Let $\psi\colon \sigma_1 \to \sigma_2$ be an admissible bijection. Let X^{ψ} denote the weak pseudomanifold obtained from $X\setminus\{\sigma_1, \sigma_2\}$ by identifying x with $\psi(x)$ for each $x \in \sigma_1$. Then X^{ψ} is said to be obtained from X by an *elementary handle addition*. If X_1, X_2 are two d-dimensional weak pseudomanifolds with disjoint vertex-sets, σ_i a facet of X_i $(i = 1, 2)$ and $\psi\colon \sigma_1 \to \sigma_2$ any bijection, then $(X_1 \sqcup X_2)^{\psi}$ is called an *elementary connected sum* of X_1 and X_2, and is denoted by $X_1\#_{\psi}X_2$ (or simply by $X_1\#X_2$). Note that the combinatorial type of $X_1\#_{\psi}X_2$ depends on the choice of the bijection ψ. However, when X_1, X_2 are connected triangulated d-manifolds, $|X_1\#_{\psi}X_2|$ is the topological connected sum of $|X_1|$ and $|X_2|$ (taken with appropriate orientations). Thus, $X_1\#_{\psi}X_2$ is a triangulated manifold whenever X_1, X_2 are triangulated d-manifolds.

Lemma 3.1. *Let N be a $(d-1)$-dimensional induced subcomplex of a d-dimensional simplicial complex M. If both M and N are normal pseudomanifolds then*

- (a) *for any vertex u of N and any vertex v of the simplicial complement $C(N, M)$, there is a path P (in M) joining u to v such that u is the only vertex in $P \cap N$, and*
- (b) *the simplicial complement $C(N, M)$ has at most two connected components.*

Proof. Part (a) is trivial if $d = 1$ (in which case, $N = S^0_2$ and $M = S^1_n$). So, assume $d > 1$ and we have the result for smaller dimensions. Clearly, there is a path P (in the edge graph of M) joining u to v such that $P = x_1x_2\cdots x_ky_1\cdots y_l$ where $x_1 = u$, $y_l = v$ and x_i's are the only vertices of P from N. Choose k to be the smallest possible. We claim that $k = 1$, so that the result follows. If not, then $x_{k-1} \in \mathrm{lk}_N(x_k) \subset \mathrm{lk}_M(x_k)$ and $y_1 \in C(\mathrm{lk}_N(x_k), \mathrm{lk}_M(x_k))$. Then, by induction hypothesis, there is a path Q in $\mathrm{lk}_M(x_k)$ joining x_{k-1} and y_1 in which x_{k-1} is the only vertex from $\mathrm{lk}_N(x_k)$. Replacing the part $x_{k-1}x_ky_1$ of P by the path Q, we get a path P' from u to v where only the first $k-1$ vertices of P' are from N. This contradicts the choice of k.

The proof of Part (b) is also by induction on the dimension d. The result is trivial for $d = 1$. For $d > 1$, fix a vertex u of N. By induction hypothesis, $C(\mathrm{lk}_N(u), \mathrm{lk}_M(u))$ has at most two connected components. By Part (a) of this lemma, every vertex v of $C(N, M)$ is joined by a path in $C(N, M)$ to a vertex in one of these components. Hence the result. $\square$

Let N be an induced subcomplex of a simplicial complex M. One says that N is *two-sided* in M if $|N|$ has a (tubular) neighbourhood in $|M|$ homeomorphic to $|N| \times [-1, 1]$ such that the image of $|N|$ (under this homeomorphism) is $|N| \times \{0\}$.

Lemma 3.2. *Let M be a normal pseudomanifold of dimension $d \geq 2$ and A be a set of vertices of M such that the induced subcomplex $M[A]$ of M on A is a $(d-1)$-dimensional normal pseudomanifold. Let G be the graph whose vertices are the edges of M with exactly one end in A, two such vertices being adjacent in G if the union of the corresponding edges is a 2-face of M. Then G has at most two connected components. If, further, $M[A]$ is two-sided in M then G has exactly two connected components.*

Proof. Let $E = V(G)$ be the set of edges of M with exactly one end in A. For $x \in A$, set $E_x = \{e \in E : x \in e\}$, and let $G_x = G[E_x]$ be the induced subgraph of G on E_x. Note that G_x is isomorphic to the edge graph of $C(\mathrm{lk}_{M[A]}(x), \mathrm{lk}_M(x))$. Therefore, by Lemma 3.1 (b), G_x has at most two components for each $x \in A$. Also, for an edge xy in $M[A]$, there is a d-face σ of M such that xy is in σ. Since the induced complex $M[A]$ is $(d-1)$-dimensional, there is a vertex $u \in \sigma \setminus A$. Then $e_1 = xu \in E_x$ and $e_2 = yu \in E_y$ are adjacent in G. Thus, if x, y are adjacent vertices in $M[A]$ then there is an edge of G between E_x and E_y. Since $M[A]$ is connected and $V(G) = \cup_{x\in A} E_x$, it follows that G has at most two connected components.

Now suppose $S = M[A]$ is two-sided in M. Let U be a tubular neighbourhood of $|S|$ in $|M|$ such that $U \setminus |S|$ has two components, say U^+ and U^-. Since $|S|$ is compact, we can choose U sufficiently small so that U does not contain any vertex from $V(M) \setminus A$. Then, for $e \in E$, $|e|$ meets either U^+ or U^- but not both. Put $E^{\pm} = \{e \in E : |e| \cap U^{\pm} \neq \emptyset\}$. Then no element of E^+ is adjacent in G with any element of E^-. From the previous argument, one sees that each $x \in A$ is in an edge from E^+ and in an edge from E^-. Thus, both E^+ and E^- are non-empty. So, G is disconnected. □

Lemma 3.3. *Let X be a normal d-pseudomanifold with an induced two-sided standard $(d-1)$-sphere S. Then there is a d-dimensional weak pseudomanifold $\widetilde{X}$ such that X is obtained from $\widetilde{X}$ by elementary handle addition. Further,*

- (a) *the connected components of $\widetilde{X}$ are normal d-pseudomanifolds,*
- (b) *$\widetilde{X}$ has at most two connected components,*
- (c) *if $\widetilde{X}$ is not connected, then $X = Y_1 \# Y_2$, where Y_1, Y_2 are the connected components of $\widetilde{X}$, and*
- (d) *if $C(S, X)$ is connected then $\widetilde{X}$ is connected.*

Proof. As above, let E be the set of all edges of X with exactly one end in S. Let E^+ and E^- be the connected components of the graph G (with vertex-set E) defined above (cf. Lemma 3.2). Notice that if a facet σ intersects $V(S)$ then σ contains edges from E, and the graph G induces a connected subgraph on the set $E_\sigma = \{e \in E : e \subseteq \sigma\}$. (Indeed, this subgraph is the line graph of a complete bipartite graph.) Consequently, either $E_\sigma \subseteq E^+$ or $E_\sigma \subseteq E^-$. Accordingly, we say that the facet σ is positive or negative (relative to S). If a facet σ of X does not intersect $V(S)$ then we shall say that σ is a neutral facet.

Let $V(S) = W$ and $V(X) \setminus V(S) = U$. Take two disjoint sets W^+ and W^-, both disjoint from U, together with two bijections $f_\pm \colon W \to W^\pm$. We define a pure simplicial complex $\widetilde{X}$ as follows. The vertex-set of $\widetilde{X}$ is $U \sqcup W^+ \sqcup W^-$. The facets of $\widetilde{X}$ are: (i) W^+, W^-, (ii) all the neutral facets of X, (iii) for each positive facet σ of X, the set $\widetilde{\sigma} := (\sigma \cap U) \sqcup f_+(\sigma \cap W)$, and (iv) for each negative facet τ of X, the set $\widetilde{\tau} := (\tau \cap U) \sqcup f_-(\tau \cap W)$. Clearly, $\widetilde{X}$ is a weak pseudomanifold. Let $\psi = f_- \circ f_+^{-1} \colon W^+ \to W^-$. It is easy to see that ψ is admissible and $X = (\widetilde{X})^\psi$.

Since the links of faces of dimension up to $d-2$ in X are connected, it follows that the links of faces of dimension up to $d-2$ in $\widetilde{X}$ are connected. This proves (a).

As X is connected, choosing two vertices $f_\pm(x_0) \in W^\pm$ of $\widetilde{X}$, one sees that each vertex of $\widetilde{X}$ is joined by a path in the edge graph of $\widetilde{X}$ to either $f_+(x_0)$ or $f_-(x_0)$. Hence $\widetilde{X}$ has at most two components. This proves (b). This arguments also shows that when $\widetilde{X}$ is disconnected, W^+ and W^- are facets in different components of $\widetilde{X}$. Hence (c) follows.

Observe that $C(S,X) = C(W^+ \sqcup W^-, \widetilde{X})$. Assume that $C(S,X)$ is connected. Now, for any $(d-1)$-simplex $\tau \subseteq W^+$, there is a vertex x in $C(S,X)$ such that $\tau \cup \{x\}$ is a facet of $\widetilde{X}$. So, $C(S,X)$ and W^+ are in the same connected component of $\widetilde{X}$. Similarly, $C(S,X)$ and W^- are in the same connected component of $\widetilde{X}$. This proves (d). $\square$

Definition 3.3. If S is an induced two-sided S^{d-1}_{d+1} in a normal d-pseudomanifold X, then the pure simplicial complex $\widetilde{X}$ constructed above is said to be obtained from X by an *elementary handle deletion* over S.

Remark 3.1. In Lemma 3.3, if X is a triangulated manifold then it is easy to see that $\widetilde{X}$ is also a triangulated manifold.

Example 3.1. It is well known that the real projective plane has a unique 6-vertex triangulation, denoted by $\mathbb{R}P^2_6$. It is obtained from the boundary complex of the icosahedron by identifying antipodal vertices. The simplicial complement of any facet in $\mathbb{R}P^2_6$ is an S^1_3. But, it is not possible to obtain a triangulated 2-manifold M by deleting the handle over this S^1_3. Such a 2-manifold would have face vector $(9, 18, 12)$ and hence Euler characteristic $\chi = 3$. But, arguing as in the proof of Lemma 3.3 (d), one can see that M must be connected - and any connected closed 2-manifold has Euler characteristic ≤ 2, a contradiction. Thus the hypothesis "two-sided" in Definition 3.3 is essential. Indeed, in this example, the graph G of Lemma 3.2 is connected: it is a 9-gon.

4 Stacked spheres

Let X be a pure d-dimensional weak pseudomanifold and σ be a facet of X. Take a symbol v outside $V(X)$, and Y be the pure simplicial complex with vertex set $V(X) \cup \{v\}$ whose facets are facets of X other than σ and the $(d+1)$-sets $\tau \cup \{v\}$ where τ runs over the $(d-1)$-faces in σ. Clearly, Y is a weak pseudomanifold and $|X|$ and $|Y|$ are homeomorphic topological spaces. This Y is said to be the weak pseudomanifold obtained from X by *starring* the new vertex v in the facet σ. (In the literature, this is also known as the *bistellar* 0*-move*.) Notice that the new vertex v is of (minimal) degree $d+1$ in Y. Conversely, let Y be a d-dimensional weak pseudomanifold with a vertex v of degree $d+1$. Let $\sigma = V(\mathrm{lk}_Y(v))$. If σ is not a face of Y (which is automatically true if Y is a pseudomanifold other than the standard d-sphere S^d_{d+2}) then consider the pure simplicial complex X with vertex-set $V(Y) \setminus \{v\}$ whose facets are the facets of Y not containing v and the $(d+1)$-set σ. Clearly, X is a weak pseudomanifold. This X is said to be obtained from Y by *collapsing* the vertex v. (This is also called a *bistellar d-move* in the literature.) Obviously, the operations of starring a vertex in a facet and collapsing a vertex of minimal degree are inverses of each other.

Definition 4.1. A simplicial complex X is said to be a *stacked d-sphere* if X is obtained from the standard d-sphere S^d_{d+2} by a finite sequence of bistellar 0-moves. Clearly, any stacked d-sphere is a combinatorial d-sphere.

Lemma 4.1. *Let X be a triangulated d-sphere and x be a vertex of X. If $\mathrm{lk}_X(x)$ is a triangulated sphere then $\mathrm{ast}_X(x)$ is a triangulated d-ball. In particular, if X is a combinatorial d-sphere then the antistar of every vertex of X is a triangulated ball.*

Proof. Note that $|\mathrm{ast}_X(x)|$ is the closure of a component of $|X|\setminus|\mathrm{lk}_X(x)|$. Also, $|\mathrm{lk}_X(x)|$ has a neighbourhood in $|X|$ which is homeomorphic to $|\mathrm{lk}_X(x)| \times [-1, 1]$ via a homeomorphism mapping $|\mathrm{lk}_X(x)|$ onto $|\mathrm{lk}_X(x)| \times \{0\}$. Therefore, by the generalized Schönflies theorem (cf. [6, Theorem 5]), $|\mathrm{ast}_X(x)|$ is a d-ball. If X is a combinatorial d-sphere, then each vertex link is a triangulated (indeed combinatorial) sphere, so that this argument applies to each vertex of X. □

Definition 4.2. A *stacked d-ball* is by definition the antistar of a vertex in a stacked d-sphere. Thus if X is a stacked d-sphere and x is a vertex of X, then the simplicial complex Y, whose faces are the faces of X not containing x, is a stacked d-ball. Lemma 4.1 implies that stacked d-balls are indeed triangulated balls. It's not hard to see that they are actually combinatorial balls.

From the above discussion, we see that any stacked d-sphere is a triangulation of the d-dimensional sphere. Since an n-vertex stacked d-sphere is obtained from S^d_{d+2} by $(n-d-2)$ starring and each starring induces $\binom{d+1}{j}$ new j-faces and retains all the old j-faces for $1 \leq j < d$ (respectively, kills only one old j-face for $j = d$), it follows that it has $(n-d-2)\binom{d+1}{j}+\binom{d+2}{j+1}$ j-faces for $1 \leq j < d$, and $(n-d-2)d + (d+2)$ facets. On simplifying, we get :

Lemma 4.2. *The face-vector of any d-dimensional stacked sphere satisfies*

$$f_j = \begin{cases} \binom{d+1}{j} f_0 - j\binom{d+2}{j+1}, & \text{if} \quad 1 \leq j < d \\ df_0 - (d+2)(d-1), & \text{if} \quad j = d. \end{cases}$$

Lemma 4.3. *Let X be a normal pseudomanifold of dimension $d \geq 2$.*

- (a) *If $X \neq S^d_{d+2}$ then any two vertices of degree $d+1$ in X are non-adjacent.*
- (b) *If X is a stacked d-sphere then X has at least two vertices of degree $d+1$.*

Proof. Let x_1, x_2 be two adjacent vertices of degree $d+1$ in X. Thus, $\mathrm{lk}(x_1) = S^{d-1}_{d+1}$, so that all the vertices in $V = V(\mathrm{st}(x_1))$ are adjacent. It follows that $V \setminus \{x_2\}$ is the set of neighbours of x_2. Hence all the facets through x_2 are contained in the $(d+2)$-set V. Since there must be a facet containing x_2 but not containing x_1, such a facet must be $V \setminus \{x_1\}$. Thus, X induces a standard d-sphere on V. Since X is a d-dimensional normal pseudomanifold, it follows that $X = S^d_{d+2}(V)$. This proves Part (a).

We prove (b) by induction on the number n of vertices of X. If $n = d+2$ then $X = S^d_{d+2}$ and the result is trivial. So assume $n > d+2$, and the result holds for all the smaller values of n. Since X is a stacked sphere, X is obtained from an $(n-1)$-vertex stacked sphere Y by starring a new vertex x in a facet σ of Y. Thus, x is a vertex of degree $d+1$ in X. If Y is the standard d-sphere then the unique vertex y in $V(Y) \setminus \sigma$ is also of degree $d+1$ in X. Otherwise, by induction hypothesis, Y has at least two vertices of degree $d+1$, and since any two of the vertices in σ are adjacent in Y - Part (a) implies that at least one of these degree $d+1$ vertices of Y is outside σ. Say $z \notin \sigma$ is of degree $d+1$ in Y. Then z (as well as x) is a vertex of degree $d+1$ in X. □

Lemma 4.4. *Let X, Y be d-dimensional normal pseudomanifolds. Suppose Y is obtained from X by starring a new vertex in a facet of X. Then Y is a stacked sphere if and only if X is a stacked sphere.*

Proof. The "if" part is immediate from the definition of stacked spheres. We prove the "only if" part by induction on the number $n \geq d+3$ of vertices of Y. The result is trivial for $n = d+3$. So, assume $n > d+3$. Let Y be obtained from X by starring a vertex x in a facet σ of X. Suppose Y is a stacked sphere. Then Y is obtained from some stacked sphere Z by starring a vertex y in a facet τ of Z. If $x = y$ then Z is obtained from Y by collapsing x, so that $X = Z$ is a stacked sphere, hence we are done. On the other hand, if $x \neq y$, then both x and y are of degree $d+1$ in Y, so that by Lemma 4.3, x and y are non-adjacent. Therefore, x is a vertex of degree $d+1$ in Z. Let W be obtained from Z by collapsing the vertex x. By induction hypothesis, W is a stacked sphere. But, X is obtained from W by starring the vertex y. Hence by the "if" part, X is a stacked sphere. □

Lemma 4.5. *The link of a vertex in a stacked sphere is a stacked sphere.*

Proof. Let X be a d-dimensional stacked sphere and v be a vertex of X. We prove the result by induction on the number n of vertices of X. The result is trivial for $n = d+2$. So, assume $n \geq d+3$ and the result is true for all stacked spheres on at most $n-1$ vertices. Let X be obtained from an $(n-1)$-vertex stacked sphere Y by starring a vertex x in a facet σ of Y. If $v = x$ then $\mathrm{lk}_X(v)$ is a standard $(d-1)$-sphere and hence is a stacked sphere. So, assume that $v \neq x$. Since the number of vertices in Y is $n-1$, by induction hypothesis, $\mathrm{lk}_Y(v)$ is a stacked sphere. Clearly, either $\mathrm{lk}_X(v) = \mathrm{lk}_Y(v)$ or $\mathrm{lk}_X(v)$ is obtained from $\mathrm{lk}_Y(v)$ by starring x in a facet of $\mathrm{lk}_Y(v)$. In either case, $\mathrm{lk}_X(v)$ is a stacked sphere. □

Lemma 4.6. *Let X be a stacked d-sphere with edge graph G and $d > 1$. Let $\overline{X}$ denote the simplicial complex whose faces are all the cliques of G. Then $\overline{X}$ is a stacked $(d+1)$-ball and $X = \partial\overline{X}$.*

Proof. We prove the result by induction on the number n of vertices of X. If $n = d+2$ then $X = S^d_{d+2}$ and $\overline{X} = B^{d+1}_{d+2}$, so that the result is obviously true. So assume that $n > d+2$ and the result is true for $(n-1)$-vertex stacked d-spheres. Let x be a vertex of degree $d+1$ in X, and let X_0 be the $(n-1)$-vertex stacked d-sphere obtained from X by collapsing the vertex x. Note that, since $d \geq 2$, the edge graph G_0 of X_0 is the induced subgraph on the vertex-set $V(G_0) = V(G) \setminus \{x\}$, and G may be recovered from G_0 by adding the vertex x and making it adjacent to the vertices in a $(d+1)$-clique σ of G_0 (which formed a facet of X_0, i.e., a boundary d-face of the stacked $(d+1)$-ball $\overline{X}_0$). Thus the simplicial complex $\overline{X}$ is obtained from the stacked $(d+1)$-ball $\overline{X}_0$ by adding the $(d+1)$-face $\tilde{\sigma} := \sigma \cup \{x\}$. Since $\overline{X}_0$ is a stacked $(d+1)$-ball, it is the antistar of a (new) vertex y in a stacked $(d+1)$-sphere Y_0 with vertex set $V(X_0) \sqcup \{y\}$. Since σ is a boundary face of $\overline{X}_0$, it follows that $\hat{\sigma} := \sigma \sqcup \{y\}$ is a facet of Y_0. Let Y be the $(n+1)$-vertex stacked $(d+1)$-sphere obtained from Y_0 by starring the vertex x in the facet $\hat{\sigma}$. Clearly, $\overline{X}$ is the antistar in Y of the vertex y. Therefore, $\overline{X}$ is a stacked $(d+1)$-ball. Now, $\mathrm{lk}_Y(y)$ is obtained from $\mathrm{lk}_{Y_0}(y)$ by starring the vertex x in the d-face σ. Since $\mathrm{lk}_Y(y) = \partial\overline{X}$ and $\mathrm{lk}_{Y_0}(y) = \partial\overline{X}_0 = X_0$, it follows that $\partial\overline{X}$ is obtained from X_0 by starring the vertex x in the facet σ. That is, $\partial\overline{X} = X$. This completes the induction and hence proves the lemma. □

Lemma 4.7. *Any stacked sphere is uniquely determined by its edge graph.*

Proof. Let G be the edge-graph of a stacked d-sphere X. If $d = 1$ then $X = G$, and there is nothing to prove. If $d > 1$, then Lemma 4.6 shows that G determines $\overline{X}$ (by definition) and $\overline{X}$ determines X via the formula $X = \partial\overline{X}$. □

Remark 4.1. (a) From the definition and Lemma 4.5, it follows that the boundary of any stacked ball is a stacked sphere. Conversely, from Lemma 4.6, every stacked d-sphere X is the boundary of a stacked $(d+1)$-ball $\overline{X}$ canonically constructed from X for $d \geq 2$. Indeed, $\overline{X}$ is the unique triangulated ball such that $\text{skel}_{d-1}(X) = \text{skel}_{d-1}(\overline{X})$. Thus, any stacked sphere is a 1-stacked sphere as defined in Section 9.
(b) Lemma 4.2 implies that any stacked d-ball with n boundary vertices and m interior vertices has exactly $n + (m-1)d$ facets. In particular, if X is an n-vertex stacked d-sphere, then the stacked $(d+1)$-ball $\overline{X}$ constructed above has n boundary vertices and no interior vertices, so that X has exactly $n-d-1$ cliques of size $d+2$. Of course, this may be directly verified by induction on n.

Lemma 4.8. *Let X_1, X_2 be d-dimensional normal pseudomanifolds. Then (a) $X_1 \# X_2$ is a triangulated 2-sphere if and only if both X_1 and X_2 are triangulated 2-spheres; and (b) $X_1 \# X_2$ is a stacked d-sphere if and only if both X_1, X_2 are stacked d-spheres.*

Proof. Let $d = 2$. Then X_1, X_2 are connected triangulated 2-manifolds and hence $X_1 \# X_2$ is a connected triangulated 2-manifold. For $0 \leq i \leq 2$, $1 \leq j \leq 2$, let $f_i(X_j)$ denote the number of i-faces in X_j. Then, from the definition, $\chi(X_1 \# X_2) = (f_0(X_1) + f_0(X_2) - 3) - (f_1(X_1) + f_1(X_2) - 3) + (f_2(X_1) + f_2(X_2) - 2) = \chi(X_1) + \chi(X_2) - 2$. Part (*a*) now follows from the fact that the Euler characteristic of a connected closed 2-manifold M is ≤ 2 and equality holds if and only if M is a 2-sphere.

We prove Part (*b*) by induction on the number $n \geq d+3$ of vertices in $X_1 \# X_2$. If $n = d+3$ then both X_1, X_2 must be standard d-spheres (hence stacked spheres) and then $X_1 \# X_2 = S^0_2 * S^{d-1}_{d+1}$ is easily seen to be a stacked sphere. So, assume $n > d+3$, so that at least one of X_1, X_2 is not the standard d-sphere. Without loss of generality, say X_1 is not the standard d-sphere. Of course, $X = X_1 \# X_2$ is not a standard d-sphere. Let X be obtained from $X_1 \sqcup X_2 \setminus \{\sigma_1, \sigma_2\}$ by identifying a facet σ_1 of X_1 with a facet σ_2 of X_2 by some bijection. Then, $\sigma_1 = \sigma_2$ is a clique in the edge graph of X, though it is not a facet of X. Notice that a vertex $x \in V(X_1) \setminus \sigma_1$ is of degree $d+1$ in X_1 if and only if it is of degree $d+1$ in X. If either X_1 is a stacked sphere or X is a stacked sphere then, by Lemma 4.3, such a vertex x exists. Let $\widetilde{X}_1$ (respectively, $\widetilde{X}$) be obtained from X_1 (respectively, X) by collapsing this vertex x. Notice that $\widetilde{X} = \widetilde{X}_1 \# X_2$. Therefore, by induction hypothesis and Lemma 4.4, we have: X is a stacked sphere $\Longleftrightarrow$ $\widetilde{X}$ is a stacked sphere $\Longleftrightarrow$ both $\widetilde{X}_1$ and X_2 are stacked spheres $\Longleftrightarrow$ both X_1 and X_2 are stacked spheres. $\square$

Definition 4.3. For $d \geq 2$, $\mathcal{K}(d)$ will denote the family of all d-dimensional normal pseudomanifolds X such that the link of each vertex of X is a stacked $(d-1)$-sphere. Since all stacked spheres are combinatorial spheres, it follows that the members of $\mathcal{K}(d)$ are triangulated d-manifolds.

Lemma 4.9 (Walkup [15]). *Let X be a normal d-pseudomanifold and $\psi \colon \sigma_1 \to \sigma_2$ be an admissible bijection, where σ_1, σ_2 are facets of X. Then (a) X^{ψ} is a triangulated 3-manifold if and only if X is a triangulated 3-manifold; and (b) $X^{\psi} \in \mathcal{K}(d)$ if and only if $X \in \mathcal{K}(d)$.*

Proof. For a vertex v of X, let $\bar{v}$ denote the corresponding vertex of X^{ψ}. Observe that $\text{lk}_{X^{\psi}}(\bar{v})$ is isomorphic to $\text{lk}_X(v)$ if $v \in V(X) \setminus (\sigma_1 \cup \sigma_2)$ and $\text{lk}_{X^{\psi}}(\bar{v}) = \text{lk}_X(v) \# \text{lk}_X(\psi(v))$ if $v \in \sigma_1$. The results now follow from Lemma 4.8. $\square$

Notice that, Lemma 4.5 says that all stacked d-spheres belong to the class $\mathcal{K}(d)$. Indeed, we have the following characterization of stacked spheres of dimension ≥ 4. This is essentially a result from Kalai [10].

Lemma 4.10 *For $d \geq 4$, every member of $\mathcal{K}(d)$, excepting S^d_{d+2}, has an S^{d-1}_{d+1} as an induced subcomplex.*

Proof. Let $X \in \mathcal{K}(d)$, $X \neq S^d_{d+2}$. Then X has a vertex of degree $\geq d+2$. Fix such a vertex x, and let σ be an interior $(d-1)$-face in the stacked d-ball $\overline{\mathrm{lk}_X(x)}$. (If there was no such $(d-1)$-face, then we would have $\overline{\mathrm{lk}_X(x)} = B^d_{d+1}$, and hence $\deg(x) = d+1$, contrary to the choice of x.) We claim that X induces an S^{d-1}_{d+1} on $\sigma \cup \{x\}$. In other words, the claim is that $\sigma \in X$.

Choose any vertex $y \in \sigma$, and let $\sigma' = (\sigma \cup \{x\}) \setminus \{y\}$. Since $\mathrm{lk}_X(x)$ and $\overline{\mathrm{lk}_X(x)}$ have the same $(d-2)$-skeleton and σ is a $(d-1)$-face of the latter, it follows that every proper subset of $\sigma' \cup \{y\} = \sigma \cup \{x\}$ which contains x is a face of X. Since $d \geq 4$, it follows in particular that σ' is a clique of the edge graph of $\mathrm{lk}_X(y)$. Hence $\sigma' \in \overline{\mathrm{lk}_X(y)}$. Thus every proper subset of σ' is in $\mathrm{lk}_X(y)$. Since $\sigma \subset \sigma' \cup \{y\}$ and $y \in \sigma$, it follows that $\sigma \in X$. $\square$

Theorem 1. *Let X be a normal pseudomanifold of dimension $d \geq 4$. Then X is a stacked sphere if and only if $X \in \mathcal{K}(d)$ and X is simply connected.*

Proof. If X is a stacked sphere of dimension $d \geq 2$ then X is simply connected and $X \in \mathcal{K}(d)$ by Lemma 4.5. Conversely, let $X \in \mathcal{K}(d)$ be simply connected and $d \geq 4$. We prove that X is a stacked sphere by induction on the number n of vertices of X. If $n = d+2$ then $X = S^d_{d+2}$ is a stacked sphere. So, assume $n > d+2$, and we have the result for all smaller values of n. Now, take an induced standard $(d-1)$-sphere S in X (Lemma 4.10). Let $\widetilde{X}$ be obtained from X by deleting the handle over S (Lemma 3.3). Clearly, since X is simply connected, $\widetilde{X}$ must be disconnected. If X_1, X_2 are the connected components of $\widetilde{X}$, then we have $X = X_1 \# X_2$. Clearly, X_1, X_2 are also simply connected. Also, by Lemma 4.9 (b), $X_1, X_2 \in \mathcal{K}(d)$. Hence by the induction hypothesis, X_1, X_2 are stacked spheres. Therefore, by Lemma 4.8, X is a stacked sphere. $\square$

We shall not use this theorem in what follows. It is included only for completeness.

5 Generalized bistellar moves (GBMs)

Definition 5.1. Let X be a d-dimensional weak pseudomanifold. Let B_1, B_2 be two combinatorial d-balls such that B_1 is a subcomplex of X and $\partial B_1 = \partial B_2 = B_2 \cap X$. Then the pure d-dimensional simplicial complex $\widetilde{X} = (X \setminus B_1) \cup B_2$ is said to be obtained from X by a *generalised bistellar move* (GBM) with respect to the pair (B_1, B_2). Observe that $\widetilde{X}$ is also a d-dimensional weak pseudomanifold. [Let τ be a $(d-1)$-face of $\widetilde{X}$. If $\tau \in B_2 \setminus \partial B_2$ then τ is in two facets in B_2. If $\tau \in \widetilde{X} \setminus B_2$ then τ is in two facets in $X \setminus B_1 = \widetilde{X} \setminus B_2$. If $\tau \in \partial B_1 = \partial B_2$ then τ is in one facet in $X \setminus B_1 = \widetilde{X} \setminus B_2$ and in one facet in B_2.] Notice that we then have $\partial B_2 = \partial B_1 = B_1 \cap \widetilde{X}$, and X is obtained from $\widetilde{X}$ by the (reverse) generalised bistellar move with respect to the pair (B_2, B_1). In case both B_1 and B_2 are d-balls with at most $d+2$ vertices (and hence at least one has $d+2$ vertices) then this construction reduces to the usual bistellar move. Clearly, if $\widetilde{X}$ is obtained from X by a generalised bistellar move then $|\widetilde{X}|$ is homeomorphic to $|X|$ and if the dimension of X is at most 3 then $|\widetilde{X}|$ is pl homeomorphic to $|X|$.

Lemma 5.1. *If $\widetilde{X}$ is obtained from X by a GBM, then $\widetilde{X}$ is a normal pseudomanifold if and only if X is a normal pseudomanifold.*

Proof. Let X be a normal pseudomanifold. We prove that $\widetilde{X}$ is a normal pseudomanifold by induction on the dimension d of X. If $d = 1$ then the result is trivial. Assume that the result is true for all normal pseudomanifolds of dimension $< d$ and X is a normal pseudomanifold of dimension $d \geq 2$. Let $\widetilde{X}$ be obtained from X by a GBM with respect to the pair (B_1, B_2). Since X is connected, it follows that $\widetilde{X}$ is connected. We have observed that $\widetilde{X}$ is a weak pseudomanifold. Let α be a face of dimension at most $d-2$. If $\alpha \in B_2 \setminus \partial B_2$ then $\mathrm{lk}_{\widetilde{X}}(\alpha) = \mathrm{lk}_{B_2}(\alpha)$ is connected. If $\alpha \in \widetilde{X} \setminus B_2$ then $\mathrm{lk}_{\widetilde{X}}(\alpha) = \mathrm{lk}_X(\alpha)$ is connected. If $\alpha \in \partial B_1 = \partial B_2$ then $\mathrm{lk}_{\widetilde{X}}(\alpha)$ is obtained from $\mathrm{lk}_X(\alpha)$ by the GBM with respect to the pair $(\mathrm{lk}_{B_1}(\alpha), \mathrm{lk}_{B_2}(\alpha))$. Since $\mathrm{lk}_X(\alpha)$ is a normal pseudomanifold of dimension $< d$, by induction hypothesis, $\mathrm{lk}_{\widetilde{X}}(\alpha)$ is a normal pseudomanifold. In particular, $\mathrm{lk}_{\widetilde{X}}(\alpha)$ is connected. This implies that $\widetilde{X}$ is a normal pseudomanifold. Since X is obtained from $\widetilde{X}$ by the reverse GBM, the converse follows. $\square$

Lemma 5.2. *Let X be an n-vertex connected oriented triangulated 2-manifold. Then one of the following four cases must arise : (i) $X = S^2_4$, (ii) $X = X_1 \# X_2$ where X_1, X_2 are connected orientable triangulated 2-manifolds, (iii) X is obtained from a connected orientable triangulated 2-manifold Y by an elementary handle addition or (iv) for each $u \in V(X)$, there exists a ball B_u with $V(B_u) = V(\mathrm{lk}_X(u))$, $\partial B_u = B_u \cap X = \mathrm{lk}_X(u)$ so that X is obtained from the $(n-1)$-vertex connected orientable triangulated 2-manifold $Y := (X \setminus \mathrm{star}_X(u)) \cup B_u$ by the GBM with respect to the pair $(B_u, \mathrm{star}_X(u))$.*

Proof. Assume that $X \neq S^2_4$. Take a vertex x of X. If $\mathrm{lk}_X(x)$ has a diagonal yz which is an edge of X, then the set $\{x, y, z\}$ induces an S^1_3 in X. Since X is orientable, this S^1_3 is two sided. Let Y be obtained from X by a handle deletion over this S^1_3 (Y exists by Lemma 3.3). Clearly, Y is also orientable. If Y is connected then we are in Case (*iii*) of this lemma. Otherwise, by Lemma 3.3, $X = Y_1 \# Y_2$, where Y_1, Y_2 are the connected components of Y. Here we are in Case (*ii*) of the lemma.

Finally, assume that none of the diagonals of the cycle $\mathrm{lk}_X(x)$ are edges of X for each $x \in V(X)$. Then, for each $x \in V(X)$, X is obtained from an $(n-1)$-vertex triangulated 2-manifold Y by a GBM with respect to $(B_x, \mathrm{star}_X(x))$, where B_x is any 2-ball with $V(B_x) = V(\mathrm{lk}_X(x))$ and $\partial B_x = \mathrm{lk}_X(x)$. Then we are in the Case (*iv*) of the lemma. $\square$

Remark 5.1. Lemma 5.2 shows, in particular, that any minimal triangulation of a connected, orientable 2-manifold of positive genus must arise as the connected sum of two triangulated 2-manifolds or by handle addition over a triangulated 2-manifold of smaller genus. This fact should be useful in the explicit classification of minimal triangulations of orientable 2-manifolds of small genus. Lemma 5.2 also shows that any triangulated 2-sphere on n (> 4) vertices arises from an $(n-1)$-vertex triangulated 2-sphere by a GBM. This should help in simplifying the existing classifications and obtaining new classifications of triangulated 2-spheres with few vertices.

6 Gromov's combinatorial notion of rigidity

Throughout this section, we use the following definition due to Gromov (except that Gromov does not include connectedness as a requirement for rigidity; but it seems anathema to call a disconnected object rigid!). Thus q-rigidity hitherto refers to Gromov's q-rigidity, without further mention.

Definition 6.1. Let X be a d-dimensional simplicial complex and q be a positive integer. We shall say that X is *q-rigid* if X is connected and, for any set $A \subseteq V(X)$ which is disjoint from at least one d-face of X, the number of edges of X intersecting A is $\geq mq$, where $m = \#(A)$.

Lemma 6.1. *Let X be an n-vertex d-dimensional simplicial complex. If X is q-rigid then the number of edges of X is $\geq (n-d-1)q + \binom{d+1}{2}$.*

Proof. Let e be the number of edges of X. Fix a d-face σ of X and put $A = V(X) \setminus \sigma$. Then $\#(A) = n-d-1$ and exactly $e - \binom{d+1}{2}$ edges intersect A. $\square$

Definition 6.2. Let X be an n-vertex d-dimensional simplicial complex and q a positive integer. We shall say that X is *minimally q-rigid* if X is q-rigid and has exactly $(n-d-1)q + \binom{d+1}{2}$ edges (i.e., if the lower bound in Lemma 6.1 is attained by X).

Lemma 6.2. *A connected simplicial complex is q-rigid if and only if the cone over it is $(q+1)$-rigid. It is minimally q-rigid if and only if the cone over it is minimally $(q+1)$-rigid.*

Proof. Let X be an n-vertex d-dimensional simplicial complex and $C(X) = x * X$ be the cone over X with cone-vertex x. Note that all the $(d+1)$-faces of $C(X)$ pass through x, so that $A \subseteq V(C(X))$ is disjoint from a $(d+1)$-face if and only if $A \subseteq V(X)$ and A is disjoint from a d-face of X. Also $C(X)$ has exactly $m = \#(A)$ more edges than X which intersect A (viz., the edges joining x with the vertices of A). In consequence, the number of edges of X intersecting A is $\geq mq$ if and only if the number of edges of $C(X)$ intersecting A is $\geq m(q+1)$. This proves the first part. The second part follows since $C(X)$ has one more vertex and n more edges than X. $\square$

Lemma 6.3. *Let X_1, X_2 be subcomplexes of a simplicial complex X such that $X = X_1 \cup X_2$ and* $\dim(X_1 \cap X_2) = \dim(X)$*. If X_1, X_2 are both q-rigid then X is q-rigid. If, further, X is minimally q rigid then both X_1, X_2 are minimally q-rigid.*

Proof. Since X_1, X_2 are both connected, our assumption implies that X is connected. Let $\dim(X) = d$. Since $\dim(X_1 \cap X_2) = \dim(X)$, it follows that $\dim(X_1) = \dim(X_2) = \dim(X_1 \cap X_2) = d$. Let $A \subseteq V(X)$ be disjoint from some d-face $\sigma \in X = X_1 \cup X_2$. Without loss of generality, $\sigma \in X_1$. Write $A_1 = A \cap V(X_1)$ and $A_2 = A \setminus V(X_1)$. Say $m = \#(A)$, $m_i = \#(A_i)$, $i = 1, 2$. Thus, $m = m_1 + m_2$. Note that $A_1 \subseteq V(X_1)$ is disjoint from the d-face σ of X_1. Also, if τ is a d-face of $X_1 \cap X_2$, then τ is a d-face of X_2 disjoint from A_2 (since $\tau \subseteq V(X_1)$ and A_2 is disjoint from $V(X_1)$). Since, X_1, X_2 are q-rigid, we have at least $m_1 q$ edges of X_1 meeting A_1 and at least $m_2 q$ edges of X_2 meeting A_2. Also, as $V(X_1)$ and A_2 are disjoint, no edge of X_1 meets A_2. Therefore, we have at least $m_1 q + m_2 q = mq$ distinct edges of X meeting A. This proves that X is q-rigid.

Now, if X is minimally q-rigid, then taking A to be the complement in $V(X)$ of a d-face of X_1, one gets exactly mq edges of X meeting A. Since we have equality in the above argument, it follows that exactly $m_1 q$ edges of X_1 intersect $A_1 = A \cap V(X_1)$. Since A_1 is the complement in $V(X_1)$ of a d-face of X_1, this shows that X_1 is then minimally q-rigid. Since the assumptions are symmetric in X_1 and X_2, in this case X_2 is also minimally q-rigid. $\square$

Lemma 6.4. *Let $\{X_\alpha : \alpha \in I\}$ be a finite family of q-rigid subcomplexes of a simplicial complex X. Suppose there is a connected graph H with vertex set I such that whenever $\alpha, \beta \in I$ are adjacent in H, we have* $\dim(X_\alpha \cap X_\beta) = \dim(X)$*. Also suppose $\cup_{\alpha \in I} X_\alpha = X$. Then X is q-rigid. If, further, X is minimally q-rigid, then each X_α is minimally q-rigid.*

Proof. Induction on $\#(I)$. If $\#(I) = 1$ then the result is trivial. For $\#(I) = 2$, the result is just Lemma 6.3. So suppose $\#(I) > 2$ and we have the result for smaller values of $\#(I)$. Since H is a connected graph, there is $\alpha_0 \in I$ such that the induced subgraph of H on the vertex set $I \setminus \{\alpha_0\}$ is connected (for instance, one may take α_0 to be an end vertex of a spanning tree in H). Applying the induction hypothesis to the family $\{X_\alpha : \alpha \neq \alpha_0\}$, one gets that $Y_1 = \cup_{\alpha \neq \alpha_0} X_\alpha$ is q-rigid. Since $Y_2 = X_{\alpha_0}$ is also q-rigid, $X = Y_1 \cup Y_2$, and $\dim(Y_1 \cap Y_2) = \dim(X)$ (if α_0 is adjacent to α_1 in H then $\dim(X) \geq \dim(Y_1 \cap Y_2) \geq \dim(X_{\alpha_1} \cap Y_2) = \dim(X)$), induction hypothesis (or Lemma 6.3) implies that X is q-rigid. Now, if X is minimally q-rigid then, by Lemma 6.3, so are Y_1 and Y_2. Since Y_1 is minimally q-rigid, induction hypothesis then implies that X_α is minimally q-rigid for $\alpha \neq \alpha_0$ (and also for $\alpha = \alpha_0$ since $X_{\alpha_0} = Y_2$). □

Lemma 6.5. *Let X be a connected pure d-dimensional simplicial complex.* (*a*) *If each vertex link of X is q-rigid then X is $(q+1)$-rigid.* (*b*) *If, further, X is minimally $(q+1)$-rigid then all the vertex links of X are minimally q-rigid.*

Proof. Let $I = V(X)$ and H be the edge graph of X. Since X is connected, so is H. For $\alpha \in I$, $\mathrm{st}(\alpha)$ is a cone over the q-rigid complex $\mathrm{lk}(\alpha)$, and hence by Lemma 6.2, $\mathrm{st}(\alpha)$ is $(q+1)$-rigid for each $\alpha \in I$. Since X is pure, the family $\{\mathrm{st}(\alpha) : \alpha \in I\}$ satisfies the hypothesis of Lemma 6.4. Hence X is $(q+1)$-rigid. If it is minimally $(q+1)$-rigid, then by Lemma 6.4, each $\mathrm{st}(\alpha)$ is minimally $(q+1)$-rigid, and hence, by Lemma 6.2, $\mathrm{lk}(\alpha)$ is minimally q-rigid for all $\alpha \in I$. □

Lemma 6.6. *Let X_1, X_2 be d-dimensional normal pseudomanifolds. If X_1, X_2 are $(d+1)$-rigid then their elementary connected sum $X_1 \# X_2$ is $(d+1)$-rigid. If, further, $X_1 \# X_2$ is minimally $(d+1)$-rigid then both X_1 and X_2 are minimally $(d+1)$-rigid.*

Proof. Since X_1, X_2 are both connected, so is $X_1 \# X_2$. Let σ_i be a facet of X_i $(i = 1, 2)$ and $f : \sigma_1 \to \sigma_2$ be a bijection, such that $X = X_1 \# X_2$ is obtained from $X_1 \sqcup X_2 \setminus \{\sigma_1, \sigma_2\}$ via an identification through f. We view $V(X_i)$ as a subset of $V(X)$ in the obvious fashion. Put $\widetilde{X} = (X_1 \# X_2) \cup \{\sigma_1 = \sigma_2\}$. Then X_1, X_2 are subcomplexes of $\widetilde{X}$ satisfying the hypothesis of Lemma 6.3 with $q = d+1$. Hence, by Lemma 6.3, $\widetilde{X}$ is $(d+1)$-rigid. Since $X_1 \# X_2$ is a subcomplex of $\widetilde{X}$ of the same dimension with the same set of edges, it follows that $X_1 \# X_2$ is $(d+1)$-rigid.

If $X_1 \# X_2$ is minimally $(d+1)$-rigid, then so is $\widetilde{X}$ and hence, by Lemma 6.3, so are X_1, X_2. □

Lemma 6.7. *Let Y be a d-dimensional normal pseudomanifold which is obtained from a d-dimensional normal pseudomanifold X by an elementary handle addition. If X is $(d+1)$-rigid then Y is $(d+1)$-rigid.*

Proof. Let $Y = X^\psi$, where $\psi : \sigma_1 \to \sigma_2$ is an admissible bijection between two disjoint facets σ_1, σ_2 of X. Thus Y is obtained from $X \setminus \{\sigma_1, \sigma_2\}$ by identifying x with $\psi(x)$ for each $x \in \sigma_1$ (cf. Definition 3.2). Let's identify $V(Y)$ with $V(X) \setminus \sigma_2$ via the quotient map

$V(X) \to V(Y)$. Let $A \subseteq V(Y)$ be an m-set disjoint from a facet σ of Y. Then, under this identification $A \subseteq V(X)$ is disjoint from σ and it follows from the definition of X^ψ that σ is a facet of X. This implies, by $(d+1)$-rigidity of X, that at least $m(d+1)$ edges of X meet A. Since $A \cap \sigma_2 = \emptyset$, these edges corresponds to distinct edges of Y under our identification. Hence Y is $(d+1)$-rigid. $\Box$

Lemma 6.8. *Let X be a triangulated 2-manifold. Suppose for each vertex u of X, there is a triangulated 2-manifold X_u with vertex-set $V(X) \setminus \{u\}$, and a triangulated 2-ball $B_u \subseteq X_u$ with vertex-set $V(\mathrm{lk}_X(u))$ such that X is obtained from X_u by the GBM with respect to the pair $(B_u, \mathrm{star}_X(u))$. If X_u is 3-rigid for all $u \in V(X)$, then X is 3-rigid.*

Proof. Take any set $A \subseteq V(X)$ which is disjoint from at least one 2-face σ of X. Say $\#(A) = m$. Fix a vertex $x \in A$, say of degree k. Take a 2-ball B with vertex set $V(B) = V(\mathrm{lk}(x))$ as in the hypothesis. Note that B is a k-vertex 2-ball with k edges in the boundary (viz., the edges of $\mathrm{lk}_X(x)$), hence it has $k-3$ edges in the interior: these are not edges of X. By assumption $X_x = (X \setminus \mathrm{st}(x)) \cup B$ is 3-rigid, so that at least $3(m-1)$ edges of X_x intersect $\widetilde{A}$, and hence also A. Of these edges, at most $k-3$ edges are not in X. Thus at least $3(m-1)-(k-3)$ edges of X (not passing through x) meet A. Also, all the k edges of X through x meet A. Thus we have a total of at least $3(m-1)-(k-3)+k = 3m$ edges of X meeting A. Hence X is 3-rigid. $\Box$

7 $(d+1)$-rigidity of normal d-pseudomanifolds

Lemma 7.1. *Let X be a 2-dimensional normal pseudomanifold. Then X is 3-rigid. X is minimally 3-rigid if and only if X is a triangulated 2-sphere.*

Proof. Since X is 2-dimensional normal pseudomanifold, it follows that X is a connected triangulated 2-manifold.

First assume that X is orientable. Recall that the connected orientable closed 2-manifolds are classified up to homeomorphism by their genus g. The genus is related to the Euler characteristic χ by the formula $\chi = 2 - 2g$. With any X as above, we associate the parameter (g, n), where g is the genus of $|X|$ and n is the number of vertices of X. Let's well order the collection of all possible parameters by the lexicographic order $\prec$. That is, $(g_1, n_1) \prec (g_2, n_2)$ if either $g_1 < g_2$ or else $g_1 = g_2$ and $n_1 < n_2$. We prove the 3-rigidity of X by induction with respect to $\prec$. Notice that the smallest parameter is $(0, 4)$ corresponding to $X = S^2_4$, which is trivially 3-rigid. This starts the induction. If $(g, n) \succ (0, 4)$, then X is as in Case (ii), (iii) or (iv) of Lemma 5.2.

If X is as in (ii), then $X = X_1 \# X_2$ where X_1, X_2 are connected orientable 2-manifold with small parameters. Hence by induction hypothesis, X_1, X_2 are 3-rigid. Hence by Lemma 6.6, X is 3-rigid. If X is as in Case (iii), then X is obtained from a connected orientable triangulated 2-manifold Y of smaller genus, by elementary handle addition. By induction hypothesis, Y is 3-rigid, and hence by Lemma 6.7, X is 3-rigid. If X is as in Case (iv) of Lemma 5.2, then it satisfies the hypothesis of Lemma 6.8, and hence is 3-rigid. This completes the induction.

Now suppose X is non-orientable. Let $\widehat{X}$ be the orientable double cover of X. By the above, $\widehat{X}$ is 3-rigid. Since the covering map $V(\widehat{X}) \to V(X)$ is a two-to-one simplicial map, it is immediate that X is 3-rigid.

Finally, X is minimally 3-rigid $\Longleftrightarrow$ number of edges in X is $3(n-2)$ $\Longleftrightarrow$ the Euler characteristic of X is 2 $\Longleftrightarrow$ X is a triangulated 2-sphere. $\Box$

Proposition 7.1. *Let X be a d-dimensional normal pseudomanifold. If $d \geq 2$ then X is $(d+1)$-rigid. If, further, $d \geq 3$ and X is minimally $(d+1)$-rigid, then all the vertex links of X are minimally d-rigid.*

Proof. The proof is by induction on d. For $d = 2$ this is Lemma 7.1. For $d \geq 3$, all the vertex links of X are $(d-1)$-dimensional normal pseudomanifolds and hence, by the induction hypothesis, all vertex links of X are d-rigid. So the result follows from Lemma 6.5. □

Lemma 7.2. *Let X be a minimally $(d+1)$-rigid normal pseudomanifold of dimension $d \geq 3$. Then every clique of size $\leq d$ in the edge graph of X is a face of X.*

Proof. Let $I = V(X)$ and let H be the edge graph of X. For $\alpha \in I$, let H_α be the induced subgraph of H on the vertex-set $V(\mathrm{lk}(\alpha))$ and put $X_\alpha = \mathrm{st}(\alpha) \cup H_\alpha$. By Lemma 6.2 and Theorem 7.1, $\mathrm{st}(\alpha)$ is $(d+1)$-rigid and hence so is X_α. Thus $\{X_\alpha : \alpha \in I\}$ satisfies the hypothesis of Lemma 6.4. Since X is minimally $(d+1)$-rigid, it follows that X_α is minimally $(d+1)$-rigid for each $\alpha \in I$. But $X_\alpha \supseteq \mathrm{st}(\alpha)$, $V(X_\alpha) = V(\mathrm{st}(\alpha))$ and $\mathrm{st}(\alpha)$ is $(d+1)$-rigid. Therefore, X_α and $\mathrm{st}(\alpha)$ have the same edge graph. That is, $H_\alpha \subseteq \mathrm{st}(\alpha)$. Thus, each clique of size ≤ 3 through α is a face of X. Since this holds for each $\alpha \in I$, it follows that each clique of size ≤ 3 in H is a face of X.

Now, by an induction on k, one sees that for $k \leq d$, any k-clique of H is a face of X: if C is a k-clique (and $k \geq 4$ and hence $d \geq 4$), then for any $x \in C$, $C \setminus \{x\}$ is a $(k-1)$-clique of $\mathrm{lk}(x)$ and $\dim(\mathrm{lk}(x)) = d - 1 \geq 3$. Therefore, $C \setminus \{x\}$ is a face of $\mathrm{lk}(x)$ and hence C is a face of X. □

Lemma 7.3. *Let X be a minimally $(d+1)$-rigid normal pseudomanifold of dimension $d \geq 3$. Then the edge graph of X has a clique of size $d+2$.*

Proof. If we have the result for $d = 3$ then the result follows for all $d \geq 3$ by a trivial induction on dimension (using the second statement in Proposition 7.1). So, we may assume $d = 3$.

Let $n \geq 5$ be the number of vertices of X. Since X is minimally 4-rigid, it has $4n - 10$ edges and hence the average degree of the vertices is $\frac{2(4n-10)}{n} < 8$. Therefore, X has a vertex x of degree ≤ 7. Then, by Lemmas 6.5 and 7.1, $\mathrm{lk}(x)$ is a triangulated 2-sphere on ≤ 7 vertices. If possible, suppose $\mathrm{lk}(x)$ has no vertex of degree 3. It is easy to see that up to isomorphism there are only two such S^2, namely $S^0_2 * S^1_m$ with $m = 4$ or 5. Thus $\mathrm{lk}(x)$ is one of these two spheres, say $\mathrm{lk}(x) = S^0_2(\{y, z\}) * S^1_m(A)$. Since xyz is not a 2-face, by Lemma 7.2, yz is not an edge of X. Put $B_1 = \mathrm{st}_X(x)$, $B_2 = B^1_2(\{x, y\}) * S^1_m(A)$. Set $\widetilde{X} = (X \setminus B_1) \cup B_2$. Then $\widetilde{X}$ is obtained from X by a GBM. Hence $\widetilde{X}$ is a 3-dimensional normal pseudomanifold with $n - 1$ vertices and $4n - 10 - (m+2) + 1 = 4n - 11 - m < 4(n-1) - 10$ edges (as $m \geq 4$). This is impossible since $\widetilde{X}$ is 4-rigid by Proposition 7.1. This proves that $\mathrm{lk}(x)$ has a vertex y of degree 3. Then the vertex-set of $\mathrm{st}(xy)$ is a 5-clique. This completes the proof. □

Lemma 7.4. *Let X be an n-vertex minimally $(d+1)$-rigid d-dimensional normal pseudomanifold. If $d \geq 3$ and $n > d+2$ then X contains a standard $(d-1)$-sphere S as an induced subcomplex.*

Proof. By Lemma 7.3, there is a $(d+2)$-set $C \subseteq V(X)$ which is a clique of the edge graph of X. If all the $(d+1)$-subsets of C were facets of X then the induced subcomplex of X on the vertex-set C would be a proper subcomplex which is a (standard) d-sphere. This is not possible since X is a d-dimensional normal pseudomanifold. So, there is a $(d+1)$-set $C_0 \subseteq C$ such that C_0 is not a facet of X. But C_0 is a $(d+1)$-clique of the edge graph of X, so by Lemma 7.2, all proper subsets of C_0 are faces of X. Thus the induced subcomplex S of X on the vertex-set C_0 is a standard $(d-1)$-sphere. □

Lemma 7.5. *If X is a minimally 4-rigid 3-dimensional normal pseudomanifold then X is a stacked 3-sphere.*

Proof. By Theorem 7.1, all the vertex links are minimally 3-rigid. Therefore, by Lemma 7.1, X is a triangulated 3-manifold. Let the number of vertices in X be n. We wish to prove by induction on n that X must be a stacked 3-sphere. This is trivial for $n = 5$, so that we may assume that $n > 5$ and we have the result for smaller values of n.

By Lemma 7.4, X contains a standard 2-sphere S as an induced subcomplex. Since S is a 2-sphere, S is two-sided in X. Let Y be the simplicial complex obtained from X by deleting the "handle" over S. Since X is a triangulated 3-manifold, by Lemma 4.9 (a), Y is a triangulated 3-manifold. Also, Y has $n+4$ vertices and $4n - 10 + \binom{4}{2} < 4(n+4) - \binom{5}{2}$ edges. Therefore Y is not 4-rigid and hence, by Theorem 7.1, Y must be disconnected. Since X is connected, Lemma 3.3 implies that $X = Y_1 \# Y_2$, where Y_1, Y_2 are 3-dimensional normal pseudomanifolds. Since X is minimally 4-rigid, Lemma 6.6 implies that Y_1, Y_2 are both minimally 4-rigid. Let Y_i have n_i vertices ($i = 1, 2$). Since $n_1 + n_2 = n + 4$, $n_1 > 4$, $n_2 > 4$, it follows that $n_1 < n$, $n_2 < n$. Therefore, by induction hypothesis, Y_1, Y_2 are stacked 3-spheres. Since X is an elementary connected sum of Y_1 and Y_2, Lemma 4.8 (b) implies that X is a stacked 3-sphere. □

Proposition 7.2. *For $d \geq 3$, the stacked d-spheres are the only minimally $(d+1)$-rigid d-dimensional normal pseudomanifolds.*

Proof. If X is an n-vertex stacked d-sphere then (cf. Lemma 4.2) the number of edges of X is $(d+1)n - \binom{d+2}{2}$, so that X is minimally $(d+1)$-rigid by Theorem 7.1.

For the converse, let X be a minimally $(d+1)$-rigid d-dimensional normal pseudomanifold, with $d \geq 3$. We prove by induction on d that X is a stacked d-sphere. The $d = 3$ case is Lemma 7.5. So, assume $d > 3$ and we have the result for smaller values of d. By Theorem 7.1 and induction hypothesis, all the vertex links of X are stacked $(d-1)$-spheres. That is, X is in the class $\mathcal{K}(d)$ (cf. Definition 4.3). In particular, X is a triangulated d-manifold.

Let the number of vertices in X be n. We wish to prove by induction on n that X must be a stacked d-sphere. This is trivial for $n = d+2$, so that we may assume that $n > d+2$ and we have the result for smaller values of n.

By Lemma 7.4 (also by Lemma 4.10), X contains a standard $(d-1)$-sphere S as an induced subcomplex. Since $d > 3$, S is two-sided in X. Let Y be the simplicial complex obtained from X by deleting the "handle" over S. Since X is in the class $\mathcal{K}(d)$, by Lemma 4.9 (b), Y is in the class $\mathcal{K}(d)$. In particular, Y is a triangulated d-manifold. Also, Y has $n+d+1$ vertices and $((d+1)n - \binom{d+2}{2}) + \binom{d+1}{2} = (n+d+1)(d+1) - (d+1)(d+2) < (n+d+1)(d+1) - \binom{d+2}{2}$ edges. Therefore Y is not $(d+1)$-rigid and hence, by Theorem 7.1, Y must be disconnected. Since X is connected, Lemma 3.3 implies that $X = Y_1 \# Y_2$, where Y_1, Y_2 are d-dimensional normal pseudomanifolds. Since X is minimally $(d+1)$-rigid, Lemma 6.6 implies that Y_1, Y_2 are both minimally $(d+1)$-rigid. Let Y_i have n_i vertices

$(i = 1, 2)$. Since $n_1 + n_2 = n + d + 1$, $n_1 > d + 1$, $n_2 > d + 1$, it follows that $n_1 < n$, $n_2 < n$. Therefore, by induction hypothesis, Y_1, Y_2 are stacked d-spheres. Since X is an elementary connected sum of Y_1 and Y_2, Lemma 4.8 (b) implies that X is a stacked d-sphere. □

Theorem 2. *For $d \geq 2$, all d-dimensional normal d-pseudomanifold are $(d+1)$-rigid. For $d \geq 3$, the stacked d-spheres are the only minimally $(d+1)$-rigid d-dimensional normal pseudomanifolds.*

Proof. Immediate from Propositions 7.1 and 7.2. □

8 LBT for normal pseudomanifolds

Now we are ready to state and prove the main result of this paper:

Theorem 3. *Let X be any d-dimensional normal pseudomanifold. Then the face-vector of X satisfies;*

$$f_j(X) \geq \begin{cases} \binom{d+1}{j} f_0(X) - j\binom{d+2}{j+1}, & \text{if} \quad 1 \leq j < d, \\ d f_0(X) - (d+2)(d-1), & \text{if} \quad j = d. \end{cases}$$

Further, for $d \geq 3$, equality holds here for some j if and only if X is a stacked sphere.

Proof. This is trivial for $d = 1$. So, assume $d > 1$. For $j = 1$, the result is immediate from Lemma 6.1, Definition 6.2 and Theorem 2. So let $1 < j \leq d$. Counting in two ways the incidences between vertices and j-faces of X, we obtain

$$f_j(X) = \frac{1}{j+1} \sum_{v \in V(X)} f_{j-1}(\mathrm{lk}_X(v)).$$

Since $\mathrm{lk}_X(v)$ is a $(d-1)$-dimensional normal pseudomanifold with $\deg(v)$ vertices, induction hypothesis (on the dimension) implies that

$$f_{j-1}(\mathrm{lk}_X(v)) \geq \begin{cases} \binom{d}{j-1} \deg(v) - (j-1)\binom{d+1}{j}, & \text{if} \quad 1 < j < d, \\ (d-1)\deg(v) - (d+1)(d-2), & \text{if} \quad j = d. \end{cases}$$

Adding this inequality over all vertices v, and noting that $\sum_{v \in V(X)} \deg(v) = 2f_1(X)$, we conclude:

$$f_j(X) \geq \begin{cases} \frac{1}{j+1}\left(2\binom{d}{j-1} f_1(X) - (j-1)\binom{d+1}{j} f_0(X)\right), & \text{if} \quad 1 < j < d, \\ \frac{1}{d+1}\left(2(d-1)f_1(X) - (d+1)(d-2)f_0(X)\right), & \text{if} \quad j = d. \end{cases}$$

But, by the $j = 1$ case of the theorem, $f_1(X) \geq (d+1)f_0(X) - \binom{d+2}{2}$. Hence we get:

$$f_j(X) \geq \begin{cases} \frac{1}{j+1}\left(\left(2\binom{d}{j-1}(d+1) - (j-1)\binom{d+1}{j}\right) f_0(X) - 2\binom{d}{j-1}\binom{d+2}{2}\right), & \text{if } 1 < j < d, \\ \frac{1}{d+1}\left((2(d-1)(d+1) - (d+1)(d-2))f_0(X) - 2(d-1)\binom{d+2}{2}\right), & \text{if } j = d. \end{cases}$$

Since $(d+1)\binom{d}{j-1} = j\binom{d+1}{j}$ and $\binom{d}{j-1}\binom{d+2}{2} = \binom{d+2}{j+1}\binom{j+1}{2}$, this inequality simplifies to the one stated in the theorem. From this argument, it is clear that if the equality holds for some j, then it also holds with $j = 1$, so that (when $d \geq 3$) X is a stacked sphere in the case of equality. The converse is immediate from Lemma 4.2. □

Remark 8.1. The argument in the above proof (reducing the inequality for arbitrary j to the case $j = 1$) is known as the M-P-W reduction - after its independent inventors McMullen, Perles and Walkup.

9 Some more lower bound conjectures

Definition 9.1. For $0 \le k \le d$, a triangulated d-sphere X is said to be a *k-stacked sphere* if there is a triangulated $(d+1)$-ball B such that $\partial B = X$ and $\mathrm{skel}_{d-k}(B) = \mathrm{skel}_{d-k}(X)$. Recall that $\mathrm{skel}_{d-k}(X)$, for instance, is the subcomplex of X consisting of all its faces of dimension at most $d-k$.

Definition 9.2. Let X be a d-dimensional pseudomanifold and u be a vertex of X. Then, for a new symbol $v \notin V(X)$, the $(d+1)$-dimensional pseudomanifold $\Sigma_{u,v}(X) := (u * \mathrm{ast}_X(u)) \cup (v * X)$ is called an *one point suspension* of X. The geometric carrier of $\Sigma_{u,v}(X)$ is the suspension of $|X|$. In particular, $\Sigma_{u,v}(X)$ is a triangulated $(d+1)$-sphere if X is a triangulated d-sphere (cf. [1]).

Lemma 9.1. *If X is a triangulated d-sphere then there is a triangulated $(d+1)$-ball $\widetilde{X}$ such that $V(\widetilde{X}) = V(X)$ and $\partial \widetilde{X} = X$.*

Proof. Fix a vertex u of X, and let $X_u = u * \mathrm{ast}_X(u)$. Since X is a triangulated d-sphere, it follows that $\Sigma_{u,v}(X)$ is a triangulated $(d+1)$-sphere. Thus, X_u is the antistar of the vertex v in the triangulated $(d+1)$-sphere $\Sigma_{u,v}(X)$ and the link of v in $\Sigma_{u,v}(X)$ is X. Therefore Lemma 4.1 implies that X_u is a triangulated $(d+1)$-ball. Clearly, $V(X_u) = V(X)$ and $\partial X_u = X$. Thus $\widetilde{X} = X_u$ works for any vertex u of X. □

Remark 9.1. Trivially, for $0 \le k < l \le d$, every k-stacked d-sphere is also l-stacked. Further, the standard sphere S^d_{d+2} is the only 0-stacked d-sphere, while Lemma 9.1 shows that all triangulated d-spheres are d-stacked. Remark 4.1 (a) shows that every stacked sphere is 1-stacked. Conversely, the case $k = 1$ of the following proposition shows that the face-vector of any 1-stacked sphere satisfies the LBT with equality, so that 1-stacked spheres are precisely the stacked spheres.

Proposition 9.1. *Let $k \ge 0$. Then for $d \ge 2k+1$, the k components $f_0, \dots, f_{k-1}$ of the face-vector of any k-stacked d-sphere determines the rest of its face-vector by the formulae*

$$f_j = \begin{cases} \displaystyle\sum_{i=-1}^{k-1} (-1)^{k-i+1} \binom{j-i-1}{j-k} \binom{d-i+1}{j-i} f_i, & \text{if} \quad k \le j \le d-k, \\ \displaystyle\sum_{i=-1}^{k-1} (-1)^{k-i+1} \left[\binom{j-i-1}{j-k} \binom{d-i+1}{j-i} - \binom{k}{d-j+1} \binom{d-i}{d-k+1} \right. \\ \qquad\qquad \left. + \displaystyle\sum_{l=d-j}^{k-1} (-1)^{k-l} \binom{l}{d-j} \binom{d-i}{d-l+1} \right] f_i, & \text{if} \quad d-k+1 \le j \le d. \end{cases}$$

(Here $f_{-1} = 1$, consistent with the convention that the empty face is the only face of dimension -1 in any simplicial complex.)

Sketch of proof. Let X be a k-stacked d-sphere. Let B be a $(d+1)$-ball as in Definition 9.1. Put $\widetilde{X} = B \cup (x * X)$, where x is a new symbol. Thus $\widetilde{X}$ is a triangulated $(d+1)$-sphere. Let $(f_0, f_1, \dots, f_d)$ and $(\tilde{f}_0, \tilde{f}_1, \dots, \tilde{f}_{d+1})$ be the face-vectors of X and $\widetilde{X}$, respectively. From the relation between X and B, we get

$$\tilde{f}_j = f_j + f_{j-1}, \quad 0 \le j \le d-k. \tag{1}$$

Being triangulated spheres of dimension d and $d+1$ respectively, X and $\widetilde{X}$ satisfy the following Dehn-Sommerville equations (cf. [9, 9.2.2, Page 148]):

$$\sum_{i=-1}^{j-1}(-1)^{d-i-1}\binom{d-i}{d-j+1}f_i=\sum_{i=-1}^{d-j}(-1)^i\binom{d-i}{j}f_i,\quad 0\le j\le\lfloor\tfrac{d}{2}\rfloor,$$
$$\sum_{i=-1}^{j-1}(-1)^{d-i}\binom{d-i+1}{d-j+2}\tilde{f}_i=\sum_{i=-1}^{d-j+1}(-1)^i\binom{d-i+1}{j}\tilde{f}_i,\quad 0\le j\le\lfloor\tfrac{d+1}{2}\rfloor. \tag{2}$$

Substituting (1) in (2), we get a system of $\lfloor\frac{d}{2}\rfloor+\lfloor\frac{d+1}{2}\rfloor+2=d+2$ independent linear equations in the $(d-k+1)+(k+1)=d+2$ unknowns $f_k,\dots,f_d$, $\tilde{f}_{d-k+1},\dots,\tilde{f}_{d+1}$. Solving these equations, we get the result (in terms of $f_0,\dots,f_{k-1}$, which are regarded as "known" quantities in this calculation). Notice that this calculation shows that $\tilde{f}_j$ is given by the same formula as f_j (with $d+1$ in place of d and $\tilde{f}_i=f_i+f_{i-1}$ in place of f_i). This is no surprise: putting $\widetilde{B}=x*B$, one sees that $\widetilde{B}$ is a $(d+2)$-ball with $\partial\widetilde{B}=\widetilde{X}$ and $\mathrm{skel}_{d+1-k}(\widetilde{B})=\mathrm{skel}_{d+1-k}(\widetilde{X})$. Thus, $\widetilde{X}$ is also a k-stacked sphere. □

Now we are ready to state the generalized lower bound conjecture:

Conjecture 1 (GLBC). *For $d\ge 2k+1$, the face-vector $(f_0,\dots,f_d)$ of any triangulated d-sphere X satisfies*

$$f_j\ge\begin{cases}\displaystyle\sum_{i=-1}^{k-1}(-1)^{k-i+1}\binom{j-i-1}{j-k}\binom{d-i+1}{j-i}f_i, & \text{if}\quad k\le j\le d-k,\\ \displaystyle\sum_{i=-1}^{k-1}(-1)^{k-i+1}\left[\binom{j-i-1}{j-k}\binom{d-i+1}{j-i}-\binom{k}{d-j+1}\binom{d-i}{d-k+1}\right. & \\ \displaystyle\qquad\left.+\sum_{l=d-j}^{k-1}(-1)^{k-l}\binom{l}{d-j}\binom{d-i}{d-l+1}\right]f_i, & \text{if}\quad d-k+1\le j\le d.\end{cases}$$

Equality holds here for some j if and only if X is a k-stacked d-sphere.

Remark 9.2. The $k=1$ case of this conjecture is precisely the LBT (for triangulated spheres). The $j=k$ case of this conjecture was first stated by McMullen and Walkup [12] for the smaller class of polytopal spheres (i.e., boundary complexes of convex $(d+1)$-polytopes). Note that, when X is a combinatorial sphere, all its vertex links are spheres, so that using the $j=k$ case of the conjecture (if settled), one may deduce the general case by an obvious extension of the M-P-W reduction.

However, note that the vertex links of triangulated spheres need not be simply connected. (Björner and Lutz [4] have constructed a 16-vertex triangulation Σ^3_{16} of the Poincaré homology 3-sphere. Then $S^1_3*\Sigma^3_{16}$ is an example of a triangulated 5-sphere some of whose vertex-links are not simply connected. Note that the face-vector of Σ^3_{16} is $(16,106,180,90)$, and hence the face-vector of the triangulated 5-sphere $S^1_3*\Sigma^3_{16}$ is $(19,157,546,948,810,270)$, which does satisfy Conjecture 1 with $d=5$, $k=2$.) Moreover, the cases of larger j (the case $j=d$, for instance) of the conjecture may be easier to settle. In [13], Stanley proved the inequality in Conjecture 1 for polytopal spheres (in the case $j=k$, but as the vertex links of polytopal spheres are again polytopal, this settles the inequalities for all j). However, even for polytopal spheres, the case of equality remains unsolved. It has been suggested that Conjecture 1 holds for all simply connected triangulated manifolds.

We end with a conjecture on non-simply connected triangulated manifolds.

Conjecture 2 (LBC for the non-simply connected manifolds). *For $d \geq 3$, the face-vector of any connected and non-simply connected triangulated d-manifold X satisfies*

$$f_j(X) \geq \begin{cases} \binom{d+1}{j} f_0(X), & \text{if } 1 \leq j < d, \\ d f_0(X), & \text{if } j = d. \end{cases}$$

Equality holds here for some j if and only if X is obtained from a stacked d-sphere by an elementary handle addition.

Remark 9.3. Notice that Conjecture 2 would imply, in particular, that the face-vector of any connected and non-simply connected manifold of dimension $d \geq 3$ must satisfy $\binom{f_0}{2} \geq f_1 \geq (d+1)f_0$, so that any such triangulation requires $f_0 \geq 2d+3$ vertices, and the triangulation must be 2-neighbourly when $f_0 = 2d+3$. Indeed, in [2], we proved that any non-simply connected triangulated d-manifold requires at least $2d+3$ vertices, and there is a unique such $(2d+3)$-vertex triangulated d-manifold for $d \geq 3$. It is 2-neighbourly, and does arise from a stacked sphere by an elementary handle addition. Thus, the main theorem of [2] would be a simple consequence of Conjecture 2. The special case $f_0 = 2d+4$ of this conjecture was posed in [2]. In [15], Walkup proved that this conjecture holds for $d = 3$.

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