ON THE EXISTENCE OF A METRIC AND THE INVERSE VARIATIONAL PROBLEM

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1. An extension of Riemannian Geometry begins with the postulation of existence of n^3 functions of position Γ^i_{jk} which are transformed by the point transformation $(x \leftarrow\rightarrow x')$ in the following manner:

$$\Gamma^i_{jk} = \Gamma^r_{mn} \frac{\partial x'^i}{\partial x^r} \frac{\partial x^m}{\partial x'} \frac{\partial x^n}{\partial x^k} + \frac{\partial x'^i}{\partial x^r} \frac{\partial^2 x^r}{\partial x'^j \; \partial x'^k} \tag{1.1}$$

We here adopt the following notation:—

(*a*) The "dummy index" occurring both as subscript and superscript indicates summation over all values.

$$A_r u^r = A_1 u^1 + A_2 u^2 + \dots\dots\dots\dots + A_n u^n$$

(*b*) But when underlined, the summation is not to take place; also, indices on the same level with a bar over them are to be summed:

$$A_{\underline{r}} u^{r} = \text{the } r\text{th term in } A_c u^r$$

$$A_{\bar{r}} u_{\bar{r}} = A_1 u_1 + A_2 u_2 + \dots A_n u_n .$$

(*c*) A comma followed by an additional subscript denotes partial differentiation. A semicolon in place of the comma will indicate that the variable of differentiation is $\dot{x}^n$ in place of x^k.

$$\frac{\partial A::}{\partial x^k} = A::, k \qquad\qquad \frac{\partial A::}{\partial \dot{x}^k} = A::;\, k$$

(*d*) A vertical bar in place of the comma or semicolon will be used for covariant differentiation, the indices being those of a tensor.

2. Our equations (1. 1) allow us to perform a good many of the operations possible in Riemannian manifolds, without any hypothesis as to the existence of distance, or of a groundform. The usual formulæ for covariant differentiation holds. We have a parallelism, and thence the equation of "paths," *i.e.*, curves having autoparallel tangents:

$$\ddot{x}^i + \Gamma^i_{jk} \dot{x}^j \dot{x}^k = 0 \tag{2.1}$$

$$\ddot{x}^i + \Gamma^i_{jk} \dot{x}^j \dot{x}^k = a\dot{x}^i . \qquad \text{[Schouten]}$$

One operates with the Γ's precisely as with Christoffel symbols of the second kind in a Riemannian space, and thus obtains a mixed curvature tensor

$$(2.2) \qquad R^{h}_{ijk} = \Gamma^{h}_{ik,j} - \Gamma^{h}_{ij,k} + \Gamma^{m}_{ik}\Gamma^{h}_{mj} - \Gamma^{m}_{ji}\Gamma^{n}_{mk}$$

Even more general results as the theorem of Fermi can be extended. The principal difficulty arises in discussing the problems of mathematical physics. There is no way of defining the magnitude of vectors, no way in which an index can be raised or lowered, i.e., covariant and contravariant tensors associated. Laplace's equation

$$(2.3) \qquad \triangle V = \text{div. grad. } V = 0$$

cannot be expressed, as the divergence of a covariant vector cannot be defined here. Similarly, the famous gravitational equations of the older Einsteinian theory

$$R_{ij} - \tfrac{1}{2} R g_{ij} = \lambda T_{ij}$$

lack a basis of deduction, the g_{ij} having no meaning. The restricted case

$$R^{h}_{ijh} = R_{ij} = 0$$

can still be treated, though on a purely formal basis.

Thus, it is not immediately profitable to investigate space-time, and the possibilities of a physical world based on a set of Γ's that differ but little from their Euclidean or Galilean values, which are all null. A first step must therefore investigate the possibilities of measuring "distance," of founding the affine manifold on a Riemannian. If we assume or define the paths of our geometry to be the actual paths of material particles, or of disturbances in space-time, the geodesics of older speech, an added contact is established with material reality.

We take the path equations (2.1) as our starting point. Symmetry of the Γ's

$$(2.6) \qquad \Gamma^{i}_{jk} = \Gamma^{i}_{kj}$$

may be assumed from the algebraic symmetry of (2.1) without loss of generality. To discuss the existence of a Riemannian groundform,

$$(2.7) \qquad \left(\frac{ds}{dt}\right)^2 = g_{ij}\dot{x}^i\dot{x}^j$$

is equivalent to discussing the solutions of

$$(2.8) \qquad g_i \; g_{ij/k} = 0$$

$$g_{ij,k} = g_{ih}\Gamma^{h}_{jk} + g_{hj}\Gamma^{h}_{jk}$$

The above system of partial differential equation being the familiar system of Koenig, is completely integrable if the compatibility conditions

$$(2.9) \qquad g_{ih} R^{h}_{jkl} + g_{jh} R^{h}_{ikl} = 0$$

are fulfilled. If identically, we have $R^h_{ijk} = 0$ and the totally uninteresting case of a flat or Galilean space. If not, further conditions can be derived (Eisenhart).

$$g_{ih} R^h_{jkl/m} + g_{jh} R^h_{ikl/m} = 0 \tag{2.10}$$

by covariant differentiation of (2.9). The process must lead to the sets of solutions, or to the absence of any. The method has been brought to a greater degree of elegance by Graustein, for the case of Einstein spaces.

The contracted curvature is the only fundamental covariant tensor of rank two that enters into our theory. If it does not vanish identically, it is natural to build another space using this as a ground-tensor.

$$\left(\frac{d\sigma}{dt}\right)^2 = R_{ij} \dot{x}^i \dot{x}^j \tag{2.11}$$

In this associated space, the operations that we call physical are possible, if $R = |R_{ij}|$ does not vanish. A co-tensor of rank two can, by the process of finding the normalized co-factors of its square array, be associated with a contravariant tensor of the same rank. The further operations proceed as in any Riemannian space. An Einstein space is one for which the first associate is identically null, or conformal to the original space. One may find a second and further associate spaces, by a continuation of the same process. The difficulty again lies here in the meaninglessness of conformality for affine connections.

I, therefore, propose to approach the problem from another point of view, and to see first the types of metric that can be deduced from a given set of paths. In equations (2.8), the following various properties are expected of the solutions :

(*a*) $g_{ij} = g_{ji}$. If this does not follow from the equations, we can at any rate replace g_{ij} in the groundform by $\frac{1}{2}(g_{ij} + g_{ji})$. It is seen that in general $g_{ij} + g_{ji} \neq 0$.

(*b*) g_{ij} must be a covariant tensor of rank two.

$$g'_{ij} = g_{mn} \frac{\partial x^m}{\partial x'^i} \frac{\partial x^n}{\partial x'^j} \quad (x \longleftrightarrow x')$$

If the equations are taken as invariant under a point transformation, and we assume

$$g'_{ij} = \phi^{mn}_{ij} \; g_{mn}$$

a set of relations will be obtained involving x, the g's, the Γ's, and $\phi^{\alpha\beta}_{\partial\delta}$ as well as $x^i_{,} x'^k$. If it be further demanded that the ϕ's be purely functions of the transformation, $(x \longleftrightarrow x')$, not explicitly dependent on the g's or the Γ's, we have a set of equations satisfied by

$$\phi^{m\,n}_{i\;j} = \frac{\partial x^m}{\partial x'^j} \frac{\partial x^n}{\partial x'^j}$$

It does not transpire that these are the only solutions possible, but I do not intend to develop this possible generalization of the tensor here.

(*c*) $g=|g_{ij}|\neq 0$. This is usually another condition added to the equations. If however (2.8) has a solution, it follows that

$$\frac{\partial}{\partial x^k}\log g=\Gamma^r_{r\,h}$$

The value of g being other than null initially, it will not vanish.

(*d*) Continuity, differentiability, and other analytic properties of the g's depend on the corresponding properties of the Γ's. Uniqueness of the solutions does not in general hold, but the various groundforms must give spaces in geodesic correspondence.

(*e*) The distance s, represented by

$$\int_{p_0}^{p} ds,$$

must be stationary over the paths given by (2.1). This is usually written as $\delta\int\sqrt{g_{ij}\dot{x}^i\dot{x}^j}\,dt$ with the extra condition $g_{ij}\dot{x}^i\dot{x}^j=$const. along the extremals. We replace it by

$$\delta\int g_{ij}\dot{x}^i\dot{x}^j\,dt=0$$

It is intuitively obvious that this will do as well. We can consider the geodesics as the actual trajectories of a particle of unit mass sliding on a smooth hypersurface of the given groundform under a zero potential. The least action formula is precisely the one that we have adopted, and the auxiliary condition is merely the conservation of energy. The Eulerian equations become

$$2\frac{d}{dt}\left(g_{ij}\dot{x}^j\right)-g_{k,ij}\,\dot{x}^k\,\dot{x}^j=0.$$

These reduce to $g_{ij}\left[\ddot{x}^i+\Gamma^i_{jk}\,\dot{x}^j\,\dot{x}^k\right]=0$ if and only if (2.8) is fulfilled and if g is not zero, these will be identical with (2.1).

It is the main purpose of this paper to follow the last condition more closely, and to investigate its full significance.

3. Given a single differential equation of the second order

(3.1) $$\ddot{x}+\alpha(x,\dot{x},t)=0$$

It is asked whether there exist any functions of x, $\dot{x}$, t such that

(3.2) $$\delta\int f(x,\dot{x},t)\,dt=0$$

represents by its extremals the curves that are solutions of (3.1). The Eulerian equation is

(3.3) $$\frac{d}{dt}f_{\dot{x}}-f_x=0$$

$$\ddot{x}\,f_{\dot{x}\dot{x}}+\dot{x}\,f_{\dot{x}x}+f_{\dot{x}t}-f_x=0$$

If then, such an f exists, the equation (3.3) must be reducible to (3.1) with almost a factor of proportionality, $\xi(x, \dot{x}, t) \neq 0$

$$\xi [\ddot{x}+\alpha] = \frac{d}{dt} f_{\dot{x}} - f_x$$

This gives directly,

$$(3.4) \quad \xi = f_{\dot{x}\dot{x}}, \quad \xi \alpha = \dot{x} f_{\dot{x}x} + f_{\dot{x}t} - f_x,$$

$$-\alpha f_{\dot{x}\dot{x}} + \dot{x} f_{\dot{x}x} + f_{\dot{x}t} - f_x = 0$$

And we may state

Theorem 1.— *The equation $\ddot{x} = \alpha (x, \dot{x}, t)$ gives the extremals of $\delta \int f dt = 0$ if and only if f is a solution of*

$$\alpha \frac{\partial^2 f}{\partial \dot{x}^2} + \dot{x} \frac{\partial^2 f}{\partial \dot{x} \partial x} + \frac{\partial^2 f}{\partial \dot{x} \partial t} - \frac{\partial f}{\partial x} = 0$$

such that $\frac{\partial^2 f}{\partial \dot{x}^2} \neq 0$

Since the partial differential equation always has a solution, we could have stated that every second order differential equation can be deduced from a variational principle. The solution is not unique, as the addition of any perfect differential leaves the Euler equations unchanged. As a corollary, we have,

The linear differential equation

$$\ddot{x} + \dot{x}P(t) + x\, Q = 0$$

is equivalent to

$$\delta \int e^{\int P dt} \frac{1}{a} \left[\dot{x}^2 + 2x\dot{x}P - x^2 (Q - \frac{dP}{dt} - P^2) \right] dt = 0$$

This could have been derived from inspection after the equation is put in the normal form and the integrand transformed back again. It must be kept in mind that $cf + \frac{dv(x)}{dt}$ gives the same equation as f.

The same derivation will now be attempted for systems of second order differential equations.

4. We start with the system

$$(4.1) \qquad \ddot{x}^i + \alpha^i (x, \dot{x}, t) = 0 \qquad i = 1, \dots n$$

which is to be deduced from

$$(4.2) \qquad \delta \int f (x^i, \dot{x}^k, t)\, dt = 0.$$

Here the single factor of proportionality will be replaced by $\rho^{ij} (x, \dot{x}, t)$ since both (4.1) and the expanded Euler equations

$$(4.3) \qquad \ddot{x}^j f_{\dot{x}^i \dot{x}^j} + \dot{x}^j f_{\dot{x}^i x^j} + f_{\dot{x}^i t} - f_{x^i} = 0$$

are linear in $\ddot{x}^i$. This leads to

$$\rho_{ij} = f_{;i;j} \qquad \rho_{ij}a^j = \dot{x}^j f_{;i;j} + \frac{\partial}{\partial t} f_{;i} - f_{,i}$$

$$(4.4) \qquad a^i f_{;i;j} - \dot{x}^i f_{;j,i} - \frac{\partial}{\partial t} f_{;j} + f_{,j} = 0 \quad \left| f_{;i;j} \right| \neq 0.$$

We may then state the theorem

Theorem 2. *The system of equations (4.1) is deducible from a variational principle if and only if there exists a solution of the partial system*

$$a^i \frac{\partial^2 f}{\partial \dot{x}^i \partial \dot{x}^j} - \dot{x}^i \frac{\partial^2 f}{\partial x^i \partial \dot{x}^j} - \frac{\partial^2 f}{\partial \dot{x}^j \partial t} + \frac{\partial f}{\partial x^j} = 0$$

such that

$$\triangle = \left| \frac{\partial^2 f}{\partial \dot{x}^i \partial \dot{x}^j} \right| \neq 0.$$

The solutions of (4·4) exist in general, any perfect differential being one. But I am unable to find directly the necessary and sufficient restrictions on the a's for nontriviality represented by $\triangle \neq 0$.

It would seem evident, however, that the desired solutions exist much oftener than a groundform exists for affine connections.

Taking the coefficients of affine connection as usual, we investigate the possibility of a special type of metric f. This f is to be independent of the parameter t and expansible as a sum of, or as an uniformly convergent series of polynomials in $\dot{x}$, whose coefficients are functions of x alone.

$$(4.5) \qquad f = A + A_i \dot{x}^i + A_{i;} \dot{x}^i \dot{x}^j + \ldots + {}^{(n)}A_{i_1 \ldots i_k} \dot{x}^{i_1} \ldots \dot{x}^{i_k}$$

If f is to be an invariant, the coefficients must be tensors of rank k. If we substitute in (4.4) and demand that the result be an identity in $\dot{x}$, we get conditions on each set of coefficients

$$(4.6) \qquad A_{,k} = 0 \qquad\qquad A = \text{Const.}$$

$$A_{i,j}\dot{x}^i \equiv A_{j,i}\dot{x}^i \equiv A_{j,i}\dot{x}^i,$$

$$\therefore \; A_{i,j} = A_{j,i}$$

$$\text{and} \quad A_i \dot{x}^i = \frac{d}{dt}\mu(x).$$

The first two terms are trivial and can be neglected. For the rest

$$(4.7) \quad \dot{x}^i \dot{x}^p \dot{x}^m_{\ldots} \Big[k(k-) \left\{ A_{ihp} \Gamma^h_{lm} + A_{ihl} \Gamma^h_{rm} + \ldots \right\}$$

$$- n \left\{ A_{ilm\ldots,p} + A_{ilp\ldots,m} + \ldots \right\} A_{lmp\ldots,i} \Big] = 0$$

Substituting for the ordinary partial derivatives in terms of covariant derivatives, and keeping always in mind the complete symmetry of the A's in all their subscripts, we have

$$(4.8) \quad n\,[\,A_{ilmr\ldots/p} + A_{ilmp\ldots/r} + \ldots - A_{lmrp\ldots/i}\,] = 0$$

Changing the subscripts in turn with i and adding, this reduces finally to

$$(4.9) \qquad (k)\ A_{ilmp\ldots/r} = 0$$

Theorem 3.—*A necessary and sufficient condition for the existence of an invariant f of the type (4.5) is the vanishing of the covariant derivative of the tensor coefficients of rank higher than two. The first two terms, moreover, must be trivial and* $\triangle \neq 0$

The equations (4.9) again form a system of Koenig, whose conditions of integrability are, on account of the symmetry of the A's,

$$R^{h\ (m)}_{i,jk}\, A_{hi_2i_3\ldots} + R^{h\ (m)}_{i,jr}\, A_{i,hi_4\ldots} + - \ldots = 0$$

and of course any set that might be derived from these as in (2.10) by further covariant differentiation. In a flat space, these conditions are identically fulfilled, but the most general metric is any f in which only the $\dot{x}$ enter. The Galilean metric is the simplest, containing only terms of the lowest degree admissible for non-triviality. Similarly, in the general Riemannian case, we shall in general obtain a wide choice of admissible f for the given paths; the ground-form is only the non-trivial metric of lowest possible degree.

The parametric case, as also a solution for general α^i by means of expansion in series is too cumbersome. The next step to be discussed will be a reduction of (4.4) to a system of partial differential equations of the first order.

5. At the end of the second section, under (e), we found the same extremals for two integrands that had the form f and f^2. If it be demanded that any function $\phi\,(f)$ be a solution of (4.4) with f itself, we have upon substitution in (4.4)

$$(5.1) \qquad \phi'' f;_j\,[\alpha^i f;_i - \dot{x}^i f,_i - f_t] = 0$$

$\phi''=0$ gives $\phi = af + b$. $f;_j = 0$ gives $f=f\,(x, t)$ both being trivial cases. If (5.1) is to be true for all at least twice differentiable ϕ, it follows that

$$(5.2) \qquad \alpha^i \frac{\partial f}{\partial \dot{x}^i} - \dot{x}^i \frac{\partial f}{\partial x^i} - \frac{\partial f}{\partial t} \equiv D\,f = 0$$

This condition is necessary as well as sufficient, and scrutinised closely, is seen to be precisely $f=$*constant* along the paths (4.1).

Theorem 4. *A necessary and sufficient condition that the integral of any at least twice differentiable function* $\phi\,(f)$ *be stationary over the extremals of*

$$\delta \int f\,(x, \dot{x}, t)\, d\,t = 0,\quad x \sim x^i,\quad \dot{x} \sim \dot{x}^j$$

is that f be constant along those extremals. As a rule, the auxiliary condition $f = constant$ along the extremals is a restriction on the choice of parameter, and in no case can it modify the form of the Eulerian differential equations.

The equation (5.2) will therefore be adjoined to the system (4.4) Differentiating (5.2) with respect to $\dot{x}$, and substituting in (4.4), a first order system results

$$(5.3) \qquad \frac{\partial \alpha^i}{\partial \dot{x}^j} \frac{\partial f}{\partial \dot{x}^i} - 2 \frac{\partial f}{\partial x^j} \equiv D_j f = 0.$$

Theorem 5. *The existence of solution of (5.2) and (5.3) is necessary and sufficient for the deduction of (4.1) from a variational principle. Then any, at least twice differentiable function of the integrand is also a solution, and the integrand will in all cases be a constant along the extremals.*

A first condition of compatibility is seen by solving (5.3) for $f_{,j}$ and substituting in (5.2)

$$(5.4) \qquad (\alpha^i - \tfrac{1}{2} \dot{x}^j \alpha^i_{;j}) \frac{\partial f}{\partial \dot{x}^i} - \frac{\partial f}{\partial t} = 0.$$

If this is not identically satisfied, then it must be adjoined to the original system. If the α's are homogeneous of degree two in $\dot{x}$, the solution of the system, if any, is independent of the parameter t though this is not a necessary condition. We might sum up several results in

Theorem 6. *If the solution is independent of t and*

$$\alpha^i (\dot{x}, \lambda \dot{x}, t) \equiv \lambda^2 \alpha^i (x, \dot{x}, t)$$

then (5.2) and (4.4) are consequences of (5.3).

The existence theorems for first partial systems are quite well known, whereas for (4 4), they have yet to be deduced. The conditions for our system (5.2) and (5.3) are seen to be

$$(5.5) \qquad (DD_j - D_j D) f \equiv Q^i_j \frac{\partial f}{\partial x^i} - \frac{\partial \alpha^i}{\partial \dot{x}}{}_{;j} \frac{\partial f}{\partial x^i} = 0$$

or, eliminating $\frac{\partial f}{\partial \dot{x}^i}$ from (5.3), $P^i_j \frac{\partial f}{\partial \dot{x}^i} = 0$

$$(D_j D_k - D_k D_j) f \equiv R^i_{j\,k} \frac{\partial f}{\partial \dot{x}^i} = 0$$

where $f_{,i}$ is eliminated by virtue of (5.3) wherever it occurs and

$$(5.6) \qquad -2 P^i_j = \alpha^r \alpha^i_{;r;j} - \dot{x}^r \alpha^i_{,r;j} - \tfrac{1}{2} \alpha^r_{;j} \alpha^i_{;r} - \frac{\partial \alpha^i}{\partial t}{}_{;j} + 2 \alpha^i_{,j}$$

$$4 R^i_{jk} = 4 R^i_{kj} = \alpha^r_{;k} \alpha^i_{;r;j} 2 \alpha^i_{,k;j} + 2 \alpha^i_{jkj} - \alpha'_{;j} \alpha^i_{ir;k}.$$

These new equations, when identically fulfilled, give us complete integrability of the system under discussion. Otherwise, they must also be adjoined to (5.2), (5.3) and (5.4).

The coefficient of (5.6) are connected with each other by means of the relation

$$\text{(5.7)} \qquad \frac{\partial P^i_j}{\partial \dot{x}^k} - \frac{\partial P^i_k}{\partial \dot{x}^j} = \frac{3}{2} R^i_{gk}$$

And the usual Riemann-Christoffel tensor is given by

$$\text{(5.8)} \qquad \frac{\partial R^i_{jk}}{\partial \dot{x}^c} = R^i_{jk\,;\,e} = R^i_{e;kj}$$

The process of adducing further sets of equations can be further continued. But as we eliminate $\frac{\partial f}{\partial x^j}$ at each step, and there are left only homogeneous equations in not more than n of the equations can be independent. Even if n equations are found to be independent, there can be only the trivial solution $f=f\ (x, t)$ inasmuch as $f\,;\,i=0$

Theorem 7. *A necessary condition that there exist a nontrivial solution of the system (5.2) and (5.3) is that the matrix of coefficients of (5.5) and all other derived equations containing only $f\,;\,i$ be of rank less than n.*

The condition will be seen to be sufficient when f is to be non-parametric, or when the α's are given homogeneous of degree two in $\dot{x}$.

If an invariant and non-degenerate f is found to exist, we have a "space" very similar to the Riemann spaces, and, in fact, the condition of non-triviality suggests a groundform

$$g_{ij} = f_{\dot{x}^i \dot{x}^j} = f;\,i;\,j.$$

This can always be justified, if f satisfies a relation of the form

$$\dot{x}^i\,\dot{x}^j f_{;i;\,j} = \phi\,(f) + \frac{d}{dt}\psi\,(x, t),$$

$\phi\,(f)$ being any at least twice differentiable function of f itself. The invariance of f will necessarily make $f_{;\,i;\,j}$ a tensor of rank two, covariant in the indices. Physical problems can then be discussed, and conformality has a meaning. We get the obvious generalizations of Einstein spaces and of the associate spaces as well that need not be discussed here. The equations (5.3) as also (4.4) are generalizations of the vanishing of the covariant derivatives of the fundamental tensor. The coefficients R^i_{jk} and P^i_j are actually tensors, if the tensor-invariance of the path equations is known.*

Developments and geometrical interpretations of the various fundamental conditions in the calculus of variations such as the conditions of Legendre, Jacobi, Weierstrass, the equations of transversality, and the question of conjugate foci,

* All the differential invariants of the "space" can be had by considering the coefficients of our successive derived equations that contain only $\frac{\partial t}{\partial \dot{x}}$.

all of which should be fundamental in our new geometry, will be left to a later paper, or to abler analysts. Parallelism and covariant differentiation are fundamental concepts in recent differential geometry, which have received no consideration here. I shall leave all of these aside, and conclude the paper with a series of remarks, all compressed into one section :

6. (*a*) The general inverse variational problem can be stated as follows :

GIVEN : A set of differential equations in any number of variables of any given order, partial or ordinary, and a set of auxiliary conditions not a consequence of the differential equations,

TO FIND : Whether or not the manifolds of the solutions of the given equations can be made to coincide for some region with the extremals of a variational problem.

It would seem simpler to discuss the whole problem for ordinary differential equations by means of reduction to a system of ordinary first order differential systems, and then consider the possibility of equating this system to a Plaffian variational principle say, a generalized Hamiltonian principle. This will also give us systems of first order partial differential equations, but unfortunately in several unknowns, for which I have been unable to find any elegant method of solution. When there is to be discussed the problem of fractional differential equations also, no method at all is to be seen. For, the generalized derivative cannot be uniquely defined, as a rule, and may not be real for real variables. The direct problem of the calculus of variations does not seem to have been solved when the generalized derivative enters into the integrand.

(*b*) A space with trivial metric is not necessarily uninteresting. Take for instance, f as a perfect differential. The distance of two points is independent of the path, provided f is non-singular in regions with the proper connectivity, and often, even then. Such a space will have the additive property of distance on a line

$$D(P_1\ P_2) + D(P_2\ P_3) = D(P_1\ P_3)$$

direction has no significance, and the relativitist who attempts to locate his neighbours by means of light-signals will be in some difficulty unless he has more than one origin of observation.

(*c*) Consider the following differential equations that occur so often in mathematical physics :

$$\begin{aligned} &\ddot{x} - \lambda\,\dot{y} - v_x = 0 \\ (6.1) \qquad &\ddot{y} + \lambda\,\dot{x} - v_y = 0 \end{aligned}$$

They are the simplest example of "non-energic" forces in a dynamical system. We see them in the restricted problem of three bodies, the vibrations of an infinite cylinder in a circulating fluid, an etectron in a magnetic field, the gyroscopic pendulum, and so on, even to the Zeemann effect. By inspection, we deduce these when λ is a constant from

$$(6.2) \qquad \delta \int (\overline{T} + \overline{U}) = 0$$

$$\overline{T}=\tfrac{1}{2}(\dot{x}^2+y^2)-\frac{\lambda}{4}(\dot{x}y-x\dot{y}) \quad \overline{U}=U-\frac{\lambda}{4}(\dot{x}y-x\dot{y}).$$

And it is seen that there is still the energy integral in the from $T-U=$ constant. For λ not a constant but a function of position, and parameter, we may apply the methods of the previous paragraph. But as the integrand itself is not constant along the extremals, the general problem comes to that of finding the solutions of two equations of the second order, and not our reducible case. This again calls for a profounder study of the relation between the forms of the integrals and the conditions of compatibility.

(*d*) The method of the paper is also extensible to partial differential equations, and as an example of the most general procedure for a partial differential equation of the second order, we shall show that the equation of wave mechanics, known as Schrödinger's equation, cannot be deduced from a variational principle.

Compare the equation

$$(6.3)\qquad \frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}+\Omega(p,q,r,s,x,y,z,t,u)=0$$

$$p=\frac{\partial u}{\partial x};\ q=\frac{\partial u}{\partial y};\ r=\frac{\partial u}{\partial z};\ s=\frac{\partial u}{\partial t}$$

to the Euler equation for

$$(6.4)\qquad \delta\int_V f(x,y,z,t,p,q,r,s,u)\,dv=0.$$

Using again, a factor of proportionality $\rho\ (p,q,\ldots u)$ we have the following relationships:

$$(6.5)\qquad \begin{aligned} &f_{pp}=f_{zz}=f_{rr}=\rho\\ &f_{ss}=f_{pz}=f_{zr}=f_{rp}=f_{ps}=\ldots\ldots\ldots=0\\ &\rho\,\Omega=pf_{pu}+zf_{zu}+rf_{ru}+sf_{su}+f_{px}+f_{zy}+f_r+f_{st}-f_u\end{aligned}$$

Differentiate the last of these partially with respect to p, q, r, s and using the others, we have again a system of first partial differential equations, which is:

$$(6.6)\qquad \begin{aligned} &\sigma=\log\rho\\ &\frac{\partial\sigma}{\partial x}+p\frac{\partial\sigma}{\sigma u}-\Omega\frac{\partial\sigma}{\partial p}=\Omega_p\\ &\frac{\partial\sigma}{\partial y}+q\frac{\partial\sigma}{\partial u}-\Omega\frac{\partial\sigma}{\partial q}=\Omega_q\\ &\frac{\partial\sigma}{\partial z}+r\frac{\partial\sigma}{\partial u}-\Omega\frac{\partial\sigma}{\partial r}=\Omega_r\\ &\frac{\partial\sigma}{\partial t}=-\frac{\Omega_s}{\Omega}\end{aligned}$$

In the equation of Schrödinger, Ω has the form $\lambda s+u\ V\ (xy)$. The following are the derived equations, easily seen to be incompatible with above :

(a): (6.7) $$(\Omega_y + q\ \Omega_u)\frac{\partial\sigma}{\partial p} - (\Omega_x + p\ \Omega_u)\frac{\partial\sigma}{\partial q} = 0$$

and two others by cyclic rotation of letters.

(b) : $$\frac{\partial\sigma}{\partial p} = \Omega^2(\Omega_x + p\ \Omega_u)$$

and two others by cyclic rotation of the letters.

These are consistent among themselves, but further derived sets give the contradiction.

Theorem 8. *Schrödinger's wave equation in its general from is not derivable from a variational principle.*

It is well to note here that the u in the wave equation is taken to be complex as also that the classic derivation by Pauli and Heisenberg is based on the physical assumption of Eigenwerte.

(e) The analogy between the derivation of ordinary equations from a minimum principle and that of differential equations from a variational principle is easily worked out.

Given the equations

(6.8) $$f^i\ (x^1, x^2 \ldots x^n) = 0 \qquad i = 1 \ldots n,$$

It is desired to equate the whole set to a single minimum principle

$$dF\ (x^1, x^2, \ldots x^n) = 0.$$

Using again our integrating factors $\rho_{ij}\ (x^1 \ldots x^n)$

$$\rho_{ij}\ f^j = \frac{\partial F}{\partial x^i}.$$

That gives the following partial differential equations for the ρ's :

(6.9) $$(\rho_{ij}\ f^j)_{,k} - (\rho_{kj}\ f^j)_{,i} = 0$$

A simple solution is

$$\rho_{ij} = \frac{\partial f^j}{\partial x^i}.$$

Theorem 9. The equations $f^i\ (x^1, x^2\ x^n) 0 =$ can be derived from $dF = 0$, if the Jacobian $\left|\frac{\partial f^i}{\partial x^j}\right|$ does not vanish. One such F is $\sum_{i=1}^{n} (f^i)^2$.

References.

[1] E. P. Eisenhart, *Riemannian Geometry.*
[2] E. P. Eisenhart, *Non-Riemannian Geometry.*
[3] T. Levi-Cività, *The Absolute Differential Calculus.*
[4] G. D. Birkhoff, *Dynamical Systems.*
[5] M. Janet, *Lecons sur les Systèmes d'équations aux dérivées partielles.*
[6] J. A. Schouten, *Der Ricci-Kalkül,*