ON A GENERALIZATION OF THE SECOND THEOREM OF BOURBAKI

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In a paper under publication,[1] I have discussed the existence of covariant derivatives, and proved that there are infinitely many parallelisms connected with the paths:

$$\ddot{x}^i + \alpha^i (x, \dot{x}, t) = 0 \qquad \ldots \qquad \ldots \quad (1)$$

These parallelisms are defined by

$$D\,(u)^i = \dot{u}^i + \gamma^i_k\, u^k + \;\epsilon^i \qquad \ldots \quad (2)$$

where $\dot{x}^k\; \gamma^i_k (x, \dot{x}, t) + \;\epsilon^i\; (x, \;\dot{x}, t) = \alpha^i$

One of these, for which

$$\gamma^i_k = \tfrac{1}{2}\, \alpha^i_{;\,k} \qquad \ldots \qquad \ldots \qquad \ldots \quad (3)$$

is the fundamental parallelism; with this, a covariant derivative independent of the direction exists only for the symmetric affine connection:

$$\alpha^i \;=\; \Gamma^i_{jr}\; \dot{x}^j\;\; \dot{x}^k, \qquad \Gamma^i_{jr} = \Gamma^i_{rj} \;\ldots\; (4)$$

I was not aware that a little-known Russian author, D. Bourbaki, who died of acute lead-poisoning during the Revolution, had anticipated part of these results and pointed out a way to their extension. I shall not go into details here, for an excellent résumé and critique has been published recently by L. Lusternik and L. Schnirelmann.[2] But it will be clear to geometers acquainted with the last-named paper that I merely proceed by discarding all three of the '*Vysokoblagodaren*' axioms. With our notations, this means that a vector-field $u^i(x)$ will have a covariant derivative $u^i_{|r}$ independent of direction, such that :

$$u^i_{|r}\,\dot{x}^r = \mathrm{D}(u)^i \qquad \ldots \qquad \ldots \qquad \ldots \quad (5)$$

We have, therefore :

$$u^i_{|r} = \frac{\partial u^i}{\partial x^r} + \gamma^i_{kr}\,u^k + \epsilon^i_r \qquad \ldots \qquad \ldots \quad (6)$$

where $\gamma^i_{kr}\,\dot{x}^r = \gamma^i_k$ and $\epsilon^i_r\,\dot{x}^r = \epsilon^i$

It follows, with the notation of my first paper, that :

$$\gamma^i_{kr} = \gamma^i_{k\,;\,r} \text{ independent of } \dot{x}$$

$$\epsilon^i_r = \epsilon^i_{\,;\,r}$$

That is :

$$\alpha^i_{\,;\,r} - \dot{x}^r\left[\gamma^i_{kr} + \gamma^i_{rk}\right] = \phi^i_r(x)$$

Thus for the most general α^i, we can have at best:

$$\alpha^i = \gamma^i_{kr}\,\dot{x}^k\,\dot{x}^r + \phi^i_r\,\dot{x}^r \qquad \ldots \qquad \ldots \quad (7)$$

For the principal parallelism, $\alpha^i_{\,;\,k} = 2\,\gamma^i_k$. This gives my former result. For the general γ^i_k, linear in $\dot{x}$, (7) gives the most general form of the α's, and hence of the paths.

It will be noted that the $\phi^i_r(x)$ are precisely the ϵ^i_r. Furthermore, an important consequence of this generalization is the inclusion of Cartan's torsion, which is given by:

$$\Omega^i_{kr} = \gamma^i_{kr} - \gamma^i_{rk} = \gamma^i_{k\,;\,r} - \gamma^i_{r\,;\,k} \qquad \ldots \quad (8)$$

The second is the most general form of torsion, for all possible parallelisms. The quantities

$$\epsilon^i_r = \alpha^i_{\,;\,r} - \left[\gamma^i_{kr} \pm \gamma^i_{rk}\right]\dot{x}^k$$

are used in the new unitary field theories to denote the electromagnetic components of the forces deforming the hyperspace E_4.

References

[1] D. D. Kosambi. *Modern Differential Geometries*, to appear in the *Ind. Journ. of Physics*.

[2] *Topologicheskie Metody v Variatsionnykh Zadachakh*, Math.—mech. Forschungsinstitut, Moskau. 1930, pp. 69 - 73. I am indebted to Dr. A. Weil for this important reference, and for permission to use his private reprint. I understand that Schnirelmann's work is shortly to be published in German, and this will undoubtedly fill a considerable gap in the existing literature. It is also highly desirable that Bourbaki's posthumous papers, at present lodged with the Leningrad Academy, should be published in full. Unofficial reports claim that Bourbaki was shot after the Miakhii Znak affair with other members of the 'Russko-Angliskii Slovar.'