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Precessions of an Elliptical Orbit.

(Notes on: Vibrating Strings; Planetary Orbits; The Raman Effect.)

By

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I. Any text-book of hydrodynamics will give the following equations for the motion of an infinite cylinder through a perfect incompressible fluid, at rest at infinity:

$$(1)\qquad (M+M')\ddot{\xi}+k\rho\dot{\eta}=X$$

$$(M+M')\ddot{\eta}-k\rho\dot{\xi}=Y.$$

$$(2)\qquad (M+M')\frac{dU}{dt}=P$$

$$(M+M')\,U\frac{d\psi}{dt}=k\rho U+Q.$$

Here, M is the mass of an unit length of the cylinder, M' that of the fluid displaced thereby; ξ, η coordinates of the central axis with respect to axes fixed in space, the whole motion being in a direction perpendicular to the length of the cylinder. The constant of circulation about the body is denoted by k; density of the fluid medium by ρ; components of the external force per unit length by X, Y. In equations (2), P, Q are components of the external force along the tangent and normal to the path of the center; ψ the angle made by the direction of the velocity U, with a fixed direction. (*Cf.* Lamb, Hydrodynamics, 5th ed., p. 76.)

The important characteristic of these equations is that the total energy of the motion represented is exactly the same as when there is no circulation ($k=0$). In fact, the force of circulation is perpendicular to the velocity, and so does no work. These equations may be made also to represent the case of an electron in a magnetic field, a Foucault pendulum, and even the restricted problem of three bodies, as by the equations of Hill. We proceed to consider special cases.

II. If a vibrating string be set in motion by plucking it in the middle, most of its motion will be represented by—

$$(3) \qquad y = A \sin bx \sin bct$$

Seen lengthwise, the string is approximately our infinite cylinder attracted to the center with a force proportional directly to the distance. Due to natural unevenness of the apparatus the actual path will be a flat ellipse rather than a straight line, and so circulation of the air will be set up. Our equations (1) become:

$$(4) \quad \left.\begin{array}{l}\ddot{\xi} + 2\nu\dot{\eta} + \lambda^2\xi = 0 \\ \ddot{\eta} - 2\nu\dot{\xi} + \lambda^2\eta = 0\end{array}\right\} \quad \nu = \frac{k\rho}{2(M+M')}, \quad \lambda^2 = \frac{\text{force at unit distance}}{(M+M')}$$

These may be formally integrated by an ingenious device due to Bromwich (Proceedings of the London Math. Soc. Series 2, Vol. XIII, p. 225). Multiplying the second by $i = \sqrt{-1}$ and adding to the first, we have:

$$(4a) \quad \ddot{s} - 2\nu i\dot{s} + \lambda^2 s = 0, \qquad s = \xi + i\eta, \quad i = \sqrt{-1}$$

$$(5) \quad \textit{whence} \quad s = e^{\nu it}\left(Ae^{ipt} + Be^{-ipt}\right)$$

$$\text{where} \quad p = \sqrt{\nu^2 + \lambda^2}$$

With the initial conditions $z'=0$, $z=a$, when $t=0$, the motion follows an hypocycloid tangent to $|z|=\frac{a\nu}{p}$ with cusps on the circle $|z|=a$.

With another set of initial conditions, say $\xi_0=\eta_0=\dot{\eta}_0=0$ $\dot{\xi}_0=\nu$ when $t=0$ we obtain :

$$(6) \qquad z=\frac{\nu}{p}e^{\nu it}\sin pt$$

In polar co-ordinates, $z=re^{i\theta}$

$$(6a) \quad \theta=\nu t, \qquad r=\frac{\nu}{p}\sin pt \quad or \quad r=\frac{\nu}{p}\sin\frac{p\theta}{\nu}$$

The path is thus a rosette, described by a casual observer as a rotating ellipse, as also in the gyroscopic pendulum. We see that a new period has appeared, that of precession :

$$(7) \qquad T=\frac{2\pi}{\nu}=\frac{4\pi(M+M')}{k\rho}$$

Actually if one tightens the heavy string of a banjo, and plucks it in the middle, the whole motion seems a blurred region to naked eye, and its boundaries, instead of narrowing down uniformly to rest because of air resistance, are seen expanding again, after a little while. The period of this expansion would be just a half of the above. An ink dot on the string apparently follows the "rotating ellipse," and the period of a full rotation would then be T. An experiment seems to be called for with brilliant points and photographic observations.

III. The theory of relativity has still to account for the observed precessions of perihelion in our planets, especially Venus. We can consider the atmosphere of the planet and other light surrounding debris such as the rings of Saturn as constituting a circulatory effect with respect to the space occupied by the planet, since the rotation of the planet is sure to set this matter into motion. Unfortunately, the infinite cylinder cannot yield numerical results applicable to the planetary case, and the three-dimensional analysis presents difficulties. However, a criterion may be obtained as to the direction of rotation of the planet, *i.e.*, as to the circulation set up. Notice in passing that Venus is known to have a dense atmosphere though the question of its period of rotation about its axis is still a point of some doubt.

Instead of taking equations (1) with attraction inversely as the square of the distance, we assume the divergence from the normal state to be small and apply equations (2).

(8) Let $\left.\begin{array}{l} U=U_1+U_2 \\ \psi=\psi_1+\epsilon \end{array}\right\}$ U_1, ψ_1, as in the Keplerian condition.

then $(M+M')\dfrac{dU}{dt}=P=(M+M')\dfrac{dU_1}{dt}\quad\therefore\quad U_2=0$

and $(M+M')U\dfrac{d\psi}{dt}=k\rho U+Q$ and $(M+M')U_1\dfrac{d\psi}{dt}=Q$

whence $\dfrac{d\epsilon}{dt}=\dfrac{k\rho}{M+M'}$ and $\epsilon=\displaystyle\int_0^t\frac{K\rho dt}{M+M'}$

which leads to the result that ϵ in a complete period increases by $\frac{k\rho T}{M+M'}$. If the direction from which ψ is measured be taken as the radius vector to the perihelion, the orbit being nearly circular, the perihelion is retarded by this amount if k be positive. The actual direction of precession depends only on the direction of circulation. An advance would signify that the directions of circulation and of description of the orbit were opposite; a retardation would mean that they were identical. Finally, this precession when small might also be taken as a slight change of period of the moving body: advance of perihelion as an increase of period, retardation as a decrease. In the string, the circulation is set up by the vibration itself and a retardation should always occur.

IV. The solar system is but a step from the atomic model. The two-dimensional fluid motion of our cylinder, of a vortex, of an electric charge in an electromagnetic field are indistinguishable. Thus the change of period just pointed out in the case of a planet might well be seen as an extra line of the spectrum, if all the electrons have a sole direction in describing the orbit; as two lines, symmetrically displaced from the ordinary line, if the constituent electrons have both senses of description. And indeed, this is the usual explanation of the well known Zeeman effect, after Larmor. In the Raman effect, however, a number of electrons are excited by a light wave and send out a certain number of extra waves not yet satisfactorily explained (the Smekal jump is most unattractive). We might extend our analysis, and replace the constant of circulation k by a periodic function of the time which represents the change of electromagnetic intensity. The phenomenon and the problem will not be discussed in the present note, though I hope a suggestion will not be ill received. Quantisation and the critical value method of Schrödinger should be used to obtain the proper numbers corresponding to the new periods caused by the disturbing function. Secondly, the asymmetry of the Raman effect has also to be considered. The Zeeman effect might be observed, for purposes of comparison, in a magnetic field wherein the intensity has a period comparable to that of the light waves used in the more recent discovery. Let it finally be noticed that while electrodynamically unsound models have usually been employed for purposes of illustration and deduction, equations of our type will be applicable wherever a stable periodic motion of any given system is acted on by "non-energic" forces, here forces perpendicular to the displacement. As for the mathematical justification of the assumption that in an arbitrary system, a number of stable recurrent—if not periodic-motions exist, the reader is referred to modern dynamical theorists, such as Birkhoff.

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Note.—The Kármán solution involving rotational motion of the fluid about the infinite cylinder has been neglected; it should most certainly be taken into account in any experiments with vibrating strings.