Gravity Waves in Turbulent Flow

Sujit K. Bose[1] and Subhasish Dey[2]

Abstract: Gravity waves propagating over free surface flows with shallow depth are well known phenomenon. The theory of gravity wave can be derived from the theory of sinusoidal wave based on the Laplace's equation and the theory on cnoidal wave based on the Korteweg-de Vries equation that also admits of the existence of solitary waves. These theories were developed from the consideration of an irrotational flow in an ideal fluid. However, in case of a real Newtonian fluid, the bed resistance and the rapid motion of fluid generate turbulence (eddies) in the medium. The effects of turbulence are taken into account herein by using the continuity equations for the surface elevation η and the depth-averaged flow velocity U developed earlier by Bose and Dey. These equations are based on the Reynolds averaged Navier-Stokes (RANS) equations for the turbulent flow in open channels. It is shown that the wave profile can be approximated by a form $a \cos k\hat{\xi}/(1 - b \cos k\hat{\xi})$, where a, b are the constant amplitudes; k is the wave number; $\hat{\xi}$ is the dimensionless horizontal distance given by $(x - ct)/h$; x is the horizontal distance, c is the wave velocity; t is the time; and h is the undisturbed flow depth. Such a profile has the characteristic that the peaks are narrower but higher compared to wider but shallower troughs. The effects of the stream flow on wave propagations are also considered. It is found that if the waves travel in the direction of the stream flow, there is a lengthening effect on the peaks and troughs; whereas if the waves travel against the direction of stream flow, they become shorter.

CE Database subject headings: Hydraulics, open channel flow, shallow water, surface waves, turbulent flow, unsteady flow, water waves, gravity waves

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Introduction

The propagation of inviscid gravity waves over free surface flows of finite depth is a classical topic of interest in hydrodynamics (Lamb 1932; Stoker 1957). In such a theory, it was shown that a sinusoidal wave of length $2\pi/k$ travels with a velocity c in the horizontal direction, where

$$c^2 = gh\frac{\tanh kh}{kh} \le gh \qquad (1)$$

where g is the acceleration due to gravity; h is the undisturbed flow depth of water; k is the wave number. It may be noted that for long waves, such as tides, $c^2 = gh$ and for short waves, $c^2 < gh$. The governing field equation in the above theory is the Laplace's equation $\nabla^2\phi = 0$ for the velocity potential ϕ.

In shallow water depth, when the ratios of wave amplitude to flow depth, a/h, and flow depth to characteristic length scale, h/l, are small, from the governing Laplace's equation using the boundary conditions at the free surface and the bed, it can be shown that the wave elevation satisfies the nondimensional *Korteweg-de Vries* (KdV) equation

$$\frac{\partial\hat{\eta}_1}{\partial\hat{t}} + \frac{\partial\hat{\eta}_1}{\partial\hat{x}} + \frac{3}{2}\hat{\eta}_1\frac{\partial\hat{\eta}_1}{\partial\hat{x}} + \frac{1}{6}\cdot\frac{\partial^3\hat{\eta}_1}{\partial\hat{x}^3} = 0 \qquad (2)$$

where $\hat{\eta}_1 = \eta_1/a$; η_1 is the elevation of wave with respect to undisturbed free surface; a is the wave amplitude; $\hat{x} = x/l$; x is the horizontal distance; l is the characteristic length scale (say a wave length of $2\pi/k$); $\hat{t} = (gh)^{0.5}t/l$; and t is the time. In Eq. (2), the $\hat{\eta}_1$, $\hat{x}$ and $\hat{t}$ are the nondimensional variables. The solution of the above equation is known to be given by the elliptic function *cn*; and the wave is called a *cnoidal wave*. The wave is characterized by the sharper crests and the flatter troughs than in a sinusoidal wave. As a particular case, it is also known that a single crest of sech^2 form can propagate on water and is known as *solitary waves*.

In potential and cnoidal wave theories, the viscosity of fluid and the turbulence generated from the bed friction and other disturbances during the initial motion are not taken into account. Mader (2004) in modeling water waves introduced viscosity and used the Navier-Stokes equations. However, the turbulence based theory appears to be lacking in literature dealing with water waves. In free surface flows, steady and unsteady turbulent flow profiles were extensively reported in literature related to hydraulic engineering (Chow 1959; Henderson 1966; Chaudhry 1994). Based on the integration of the continuity and the momentum equations, unsteady flows were studied by Strelkoff (1969), Yen (1973) and Basco (1987). Reynolds averaged Navier-Stokes equations (RANS) were used by Dey and Lambert (2005) to study accelerated and decelerated flows down the bed slopes. Subsequently, Bose and Dey (2007) systematically investigated the two-dimensional, curvilinear free surface flows by using RANS equations. Introducing reasonable assumptions on turbulence, they obtained explicit equations for the depth-averaged flow velocity and the surface elevation, which satisfied the generalized form of Saint Venant type of equation for unsteady free surface flows. Later, the equations were generalized to develop theories for dune and antidune propagation (Bose and Dey 2009), hydraulic jump for flows down a slope (Bose et al. 2012), formation of sand ripples (Bose and Dey 2012) and surge travelling on an adverse bed slope (Bose and Dey 2013).

In this paper, the basic equations given by Bose and Dey (2007) are used to develop a theory of the surface gravity waves on a layer of water over a horizontal bed, in which the turbulence that is generated in the flow is taken into account through the RANS equations. Having used the governing equations from Bose and Dey (2013), a third-order nonlinear ordinary differential equation is derived for the elevation of a propagating wave. Neglecting the bed friction term that is subsequently shown to be insignificantly small, the third-order equation was integrated, once a nonlinear second-order ordinary differential equation is obtained. The latter equation is approximately integrated for the elevation in the form of nondimensional elevation of wave with respect to undisturbed free surface $\hat{\eta} = a \cos k\hat{\xi}/(1 - b \cos k\hat{\xi})$, where $\hat{\xi} = \xi/h$ and $\xi = x - ct$. Evidently the wave form is periodic, but exhibits the property of a cnoidal wave, that is the sharper crests and the flatter troughs. The effects of flowing water in and against the travelling wave direction are also examined.

Basic Equations for Gravity Waves in Turbulent Flow

Figure 1 schematically depicts the propagation of a two-dimensional gravity wave in a straight horizontal channel having a flow depth h, in which it is assumed that the bed friction and other possible disturbances generate turbulence in the medium. The x-axis is taken along the bed and the y-axis vertically upwards. The free surface elevation η above the bed level is supposed to propagate in the positive direction of the x-axis with a velocity such that η is a function of $x - ct$. Accounting for the generated turbulence, Bose and Dey (2007, 2013) gave a pair of equations for η and the depth averaged forward velocity U obtained by systematically treating the RANS equations of motion in two-dimensions, under an appropriate flow and a turbulence closure assumption. The continuity equation is

$$\frac{\partial \eta}{\partial t} + \frac{\partial}{\partial x}(\eta U) = 0 \tag{3}$$

and the momentum equation in x-direction is

$$\eta \frac{\partial U}{\partial \eta} + \eta U \frac{\partial U}{\partial x} + \frac{2}{5} \cdot \frac{\partial}{\partial x}\left(\eta^2 U^2 \frac{\partial^2 \eta}{\partial x^2}\right) - \frac{7}{22} \cdot \frac{\partial}{\partial x}\left(\eta^3 \frac{\partial^2 U}{\partial x \partial t} - \frac{\eta^2}{7} \cdot \frac{\partial \eta}{\partial t} \cdot \frac{\partial U}{\partial x}\right) + g\eta \frac{\partial \eta}{\partial x} + \frac{g m^2 U^2}{\eta^{1/3}} = 0 \tag{4}$$

In Eq. (4), the constant fractional coefficients 2/5, 7/22 and 1/7 appear due to an assumption of the 1/7th power law of the forward velocity distribution with the vertical distance y from the bed. The last term on the left hand side of Eq. (4) represents the bed resistance, assumed to be given by the Manning's equation, where m is the Manning's roughness coefficient.

Equations for Propagating Waves

For propagating waves in x-direction with velocity c, the free surface elevation η and the depth-averaged forward velocity U are both functions of a single variable ξ (= $x - ct$, say). Thus,

$$\eta = \tilde{\eta}(\xi), \; U = \tilde{U}(\xi) \tag{5}$$

Since the operators are $\partial/\partial x = d/d\xi$ and $\partial/\partial t = -cd/d\xi$, Eq. (3) yields on integration

$$\tilde{\eta}(c - \tilde{U}) = q \tag{6}$$

where q is a discharge per unit width due to the passage of the wave over a stream flowing with a velocity U. Equation (6) suggests that

$$\tilde{U} = c - \frac{q}{\tilde{\eta}}, \; \tilde{U}' = \frac{q}{\tilde{\eta}^2}\tilde{\eta}', \; \tilde{U}'' = \frac{q}{\tilde{\eta}^3}(\tilde{\eta}\tilde{\eta}'' - 2\tilde{\eta}'^2) \tag{7}$$

where primes denote differentiation with respect to the single variable ξ. Using Eqs. (5) and (7) into Eq. (4), yields

$$\frac{d}{d\xi}\left[\left(\tilde{\eta}^2-\frac{53}{44}\cdot\frac{q}{c}\tilde{\eta}+\frac{q^2}{c^2}\right)\tilde{\eta}''-\frac{75}{44}\cdot\frac{q}{c}\tilde{\eta}'^2+\frac{5}{4}\cdot\frac{g}{c^2}\tilde{\eta}^2+\frac{5}{2}\cdot\frac{q^2}{\tilde{\eta}}\right]=-\frac{gm^2}{\tilde{\eta}^{1/3}}\left(c-\frac{q}{\tilde{\eta}}\right)^2 \tag{8}$$

The right hand side of Eq. (8) is due to bed resistance that can be estimated by $\Phi\tilde{U}^2$, where $\Phi = gm^2/h^{1/3}$, termed *channel characteristic parameter* whose typically value can be approximated as 2.5×10^{-4}. In addition, the flow velocity $\tilde{U}$ may be either zero or almost a small fraction of the wave propagation velocity c. Hence, the term under consideration can be neglected for moderate lengths of propagation; and the equation can be integrated as

$$\left(\tilde{\eta}^2-\frac{53}{44}\cdot\frac{q}{c}\tilde{\eta}+\frac{q^2}{c^2}\right)\tilde{\eta}''-\frac{75}{44}\cdot\frac{q}{c}\tilde{\eta}'^2+\frac{5}{4}\cdot\frac{g}{c^2}(\tilde{\eta}^2-h^2)-\frac{5}{2}\cdot\frac{q^2}{c^2h\tilde{\eta}}(\tilde{\eta}-h)=0 \tag{9}$$

In Eq. (9) it is observed that in particular $\tilde{\eta} = h$ is a solution that represents streaming flow without any surface wave propagation. Equation (7) can be made nondimensional by setting

$$\tilde{\eta}=h(1+\hat{\eta}),\ \xi=h\hat{\xi} \tag{10}$$

Using two transformations Eq. (10) in Eq. (9), one obtains the nondimensional equation for surface wave propagation under turbulent flow condition as

$$\left[\hat{\eta}^2+\left(2-\frac{53}{44}Fr\right)\hat{\eta}+\left(1-\frac{53}{44}Fr+Fr^2\right)\right]\hat{\eta}''-\frac{75}{44}Fr\hat{\eta}'^2+\frac{5}{4}\cdot\frac{gh}{c^2}(\hat{\eta}+2)\hat{\eta}-\frac{5}{2}Fr^2\frac{\hat{\eta}}{1+\hat{\eta}}=0 \tag{11}$$

where $Fr = q/ch$, that is the flow Froude number; and primes denote the differentiation with respect to $\hat{\xi}$. In Eq. (11), there also appears a parameter gh/c^2, which is the reciprocal of the square root of the Froude number of the propagating wave.

Approximate Solution of Eq. (11)

Equation (11) can be treated numerically for the suitable values of Froude numbers Fr and $c/(gh)^{0.5}$. However, approximate closed form solutions for these parameters can be obtained first by treating the equation approximately dropping the second and the fourth terms on the left hand side of Eq. (11). Such approximation procedure is intended from the fact that the nondimensional elevation $\hat{\eta}$ and the slope $\hat{\eta}'$ may not be large, but eventually their contributions are taken into account numerically. Dropping temporarily the two terms and writing $\hat{\eta}'' = d(0.5\hat{\eta}'^2)/d\hat{\eta}$, Eq. (11) can be integrated as

$$\hat{\eta}'^2 = -\frac{5}{2}\cdot\frac{gh}{c^2}\left[C + \int \frac{(\hat{\eta}+2)\hat{\eta}}{\hat{\eta}^2 + \left(2-\frac{53}{44}Fr\right)\hat{\eta} + \left(1-\frac{53}{44}Fr + Fr^2\right)} d\hat{\eta}\right] \tag{12}$$

where C is an integrating constant. The indefinite integral can be evaluated exactly and one obtains

$$\hat{\eta}'^2 = -\frac{5}{2}\cdot\frac{gh}{c^2}\left[C + \hat{\eta} + \frac{53}{88}\ln\left\{\hat{\eta}^2 + \left(2-\frac{53}{44}Fr\right)\hat{\eta} + \left(1-\frac{53}{44}Fr + Fr^2\right)\right\}\right.$$
$$\left. -\frac{352}{281}\left(1+\frac{11}{40}Fr^2\right)\arctan\left\{\frac{352}{281}\left(\hat{\eta}+1-\frac{53}{88}Fr\right)Fr\right\}\right] \tag{13}$$

Note that in the last term of Eq. (13), the simplified fraction 11/40 is written replacing the unwielding fraction 56339/205216, when the replaced fraction is correct up to the three decimal places.

To progress further, approximate factorization of the quantity within the square brackets of Eq. (13) is attempted by locating the real zeros of the function for particular values of Fr. The following three cases are considered in defining q in Eq. (6):

(a) Wave propagating on a still water surface: $\tilde{U} = 0$ and Fr (= q/ch) = 1;

(b) Wave propagating on flowing water surface in the forward direction: $\tilde{U} = c/5$ and $Fr = 4/5$;

(c) Wave propagating on flowing water surface in the backward direction: $\tilde{U} = -c/5$ and $Fr = 6/5$.

In this way fixing the values of Fr, the real zeros of the function within the square brackets of Eq. (13) with $C = 0$ are sought. It is however found that there are three such zeros. The constant C is then adjusted so that one of the zeros (corresponding to $\hat{\eta}' = 0$) yields the height of a crest. If it is assumed that the nondimensional height of the crest is $\hat{\eta} = 0.2$, then C is evaluated accordingly. The values of C obtained in three cases are thus (a) $C = 0.39773$; (b) $C = 1.39199$; and (c) $C = 0.37513$. In matching the values of the function of Eq. (13) with factored expressions corresponding to the three real zeros, it is recognized that the factor corresponding to the farthest zero from $\hat{\eta} = 0$

must be repeated twice so that the polynomial approximation is a quortic, rather than a cubic. Using the least square fitting with the computed data of the function, the polynomial approximations of the two functions within the square brackets of Eq. (13) are

Case (a): 0.03882 ($\hat{\eta}$ + 4.8826)^2 ($\hat{\eta}$ + 0.18785) ($\hat{\eta}$ – 0.2)

Case (b): 0.03251 ($\hat{\eta}$ + 5.63605)^2 ($\hat{\eta}$ + 0.17611) ($\hat{\eta}$ – 0.2)

Case (c): 0.04062 ($\hat{\eta}$ + 4.42727)^2 ($\hat{\eta}$ + 0.19879) ($\hat{\eta}$ – 0.2)

The approximations in the above cases are compared with the computed functions in Figs. 2 – 4. It is noted in Cases (a) – (c) that since $\hat{\eta}'^2$ is proportional to these expressions, the crest height is $\hat{\eta}'$ = 0.2 and the trough depths are slightly lesser in magnitude being 0.18785, 0.17611 and 0.19879 in the respective three cases.

In Case (a), that is *Fr* = 1, replacing the function in the square brackets, polynomial approximation for Case (a) yields the following approximation:

$$\hat{\eta}'^2 = \frac{5}{2} \cdot \frac{gh}{c^2} \times 0.03882(\hat{\eta} + 4.8826)^2 (\hat{\eta} + 0.18785)(0.2 - \hat{\eta}) \tag{14}$$

Then, by integration, it is

$$\hat{\xi} = \pm 3.20998 \frac{c}{(gh)^{0.5}} \int \frac{1}{(\hat{\eta} + 4.8826)[(\hat{\eta} + 0.18785)(0.2 - \hat{\eta})]^{0.5}} d\hat{\eta} + C' \tag{15}$$

where C' is a constant of integration. Setting $\hat{\eta}$ + 4.8826 = 1/*z*, the indefinite integral can be evaluated as

$$\hat{\xi} = \pm 0.65713 \arccos\left(\frac{z - 0.20488}{0.00799}\right) + C' \tag{16}$$

So that in terms of $\hat{\eta}$, one can express it as

$$\hat{\eta} = -\frac{0.19042 \cos\left[1.52177 \frac{(gh)^{0.5}}{c} \hat{\xi} - C'\right]}{1 + 0.039 \cos\left[1.52177 \frac{(gh)^{0.5}}{c} \hat{\xi} - C'\right]} \tag{17}$$

If we suppose that a wave crest $\hat{\eta}$ = 0.2 lies at ξ = 0, then cosC' ≈ –1 or C' ≈ π. Hence, for Case (a) and similarly for Cases (b) and (c), one gets

$$\hat{\eta} = \frac{0.19042\cos\left[1.52177\frac{(gh)^{0.5}}{c}\hat{\xi}\right]}{1-0.039\cos\left[1.52177\frac{(gh)^{0.5}}{c}\hat{\xi}\right]} \tag{18a}$$

$$\hat{\eta} = \frac{0.19492\cos\left[1.60927\frac{(gh)^{0.5}}{c}\hat{\xi}\right]}{1-0.03458\cos\left[1.60927\frac{(gh)^{0.5}}{c}\hat{\xi}\right]} \tag{18b}$$

$$\hat{\eta} = \frac{0.1972\cos\left[1.4096\frac{(gh)^{0.5}}{c}\hat{\xi}\right]}{1-0.04454\cos\left[1.4096\frac{(gh)^{0.5}}{c}\hat{\xi}\right]} \tag{18c}$$

Equations (18a) – (18c) are of the form

$$\hat{\eta} = \frac{a\cos(k\hat{\xi})}{1-b\cos(k\hat{\xi})},\ \ a, b > 0 \tag{19}$$

Equation (19) is a periodic wave like function with wave length $2\pi/k$. The profile of the wave has the crests with a maximum height of $a/(1 - b)$ at $\xi = 2n\pi/k$, $n = 0, \pm1, \pm2, \cdots$ and the troughs with a maximum depth of $a/(1 + b)$ at $\xi = (2n + 1)\pi/k$, $n = 0, \pm1, \pm2, \cdots$. Thus, the wave crests have greater height than the trough depths, resulting in the sharper crests and the flatter troughs, as is characterized by cnoidal wave.

Equations (18a) – (18c) are plotted in Figs. 2(a – c) – 4(a – c), respectively, for the suitable values of gh/c^2. For Case (a), gh/c^2 is taken as 2, 3 and 5; for Case (b), it is taken as 1, 2 and 3; and for Case (c), the chosen values are 2.2, 3 and 5. The lowest value of gh/c^2 is chosen so that the computed value of the profile $\hat{\eta}$ from the full Eq. (9) possesses periodicity. This means that for Case (a) $c/(gh)^{0.5} \geq 1/2^{0.5} = 0.70711$; for Case (b) $c/(gh)^{0.5} \geq 1$; and for Case (c) $c/(gh)^{0.5} > 1/2.2^{0.5} = 0.6742$.

Modification in *k*, *a*, and *b* for Full Wave Equation

In Figs. 2(a – c) – 4(a – c), the full equation of wave propagation given by Eq. (11) is also numerically computed for the initial conditions: $\xi = 0$, $\hat{\eta} = 0.2$ (crest height) and $\hat{\eta}' = 0$. For this purpose, the second-order differential equation is first converted into a pair of first-order differential equations by introducing the phase variables $\hat{\eta}_1 = \hat{\eta}$ and $\hat{\eta}_2 = \hat{\eta}'$ and using a subroutine RK4 for the pair of equations with a step length $h = 0.1$ (Bose 2010, page 255). The computed curves resemble the profiles given by Eqs. (18a) – (18c) save for some departures in wave length and the amplitudes. In order to match the computed data with profiles of the form obtained from Eq. (19), the wave number k is first estimated by equating the seven successive zeros of a computed profile with that of the theoretical curve obtained from Eq. (19), namely, $(2n + 1)\pi/2k$, $n = 0, 1, 2, \cdots, 6$. The equations yield seven estimates of k. After the estimation of k, the amplitudes a and b are estimated by writing Eq. (19) as

$$a\cos(k\hat{\xi}) + \hat{\eta}b\cos(k\hat{\xi}) = \hat{\eta} \tag{20}$$

Using the computed values of $\hat{\eta}$ for a number of values of ξ with a spacing of 0.1, the values of a and b are estimated by least square fitting. The results of estimation are furnished in Table. 1. The modified forms of Eq. (19) for the parameters k, a and b show excellent agreement with the computed curves for $\hat{\eta}$ in Figs. 2(a – c) – 4(a – c). In the above theoretical development of gravity wave propagation, the right hand side of Eq. (8) was neglected. Its magnitude under the nondimensional transformation of Eq. (10) leading to Eq. (11), termed *error*, is

$$\text{error} = \frac{5}{2}\Phi\left(1 - \frac{Fr}{1+\hat{\eta}}\right)^2 \frac{1}{(1+\hat{\eta})^{1/3}} \tag{21}$$

where $\Phi = gm^2/h^{3/2}$, that is the characteristic parameter, whose typical value is taken as 2.5×10^{-3}. The maximum errors of the terms for varying $\hat{\eta}$ in the different cases are given in the last column of the Table 1. These data show the smallness of the errors in the approximation of Eq. (8).

Comparison of Figs. 2(a – c) – 4(a – c) shows that as gh/c^2 increases, the wave length decreases. For waves with a flow [Figs. 2(a – c) and 3(a – c)], the crests and troughs become wider if the waves propagate in the direction of the flow. On the other hand, if the waves propagate opposite to the direction of flow, the crests and troughs become narrower due to the opposing by the flow.

Conclusions

A theory of propagation of surface gravity waves in a channel with a flow is developed here, taking into account the turbulence generated in the flow. The theory is based on the continuity and the momentum equations given by Bose and Dey (2007, 2012) derived from the RANS equations. For waves propagating with a constant velocity and neglecting the contribution of the small bed resistance to the flow, the equation for the surface elevation can be integrated once to yield a second-order nonlinear ordinary differential equation whose approximate analytical solution is of the form given by Eq. (19). The profile of a wave has taller and narrower crests compared to shallower and wider troughs. In a channel with a flow, if the surface wave propagates in the direction of flow, the crests and troughs get wider. On the other hand, if the waves propagate opposite to the direction of flow, the crests and troughs get narrower.

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Notation

The following symbols are used in this paper:

a, b = constant amplitudes;

c = wave velocity;

Fr = flow Froude number;

g = acceleration due to gravity;

h = undisturbed flow depth;

k = wave number;

l = characteristic length scale;

m = Manning's roughness coefficient;

q = discharge per unit width;

t = time;

$\hat{t}$ = $(gh)^{0.5}t/l$;

U = depth-averaged forward velocity;

x = horizontal distance;

$\hat{x}$ = x/l;

y = vertical distance;

Φ = channel characteristic parameter;

ϕ = velocity potential;

η = elevation of free surface wave from bed;

$\hat{\eta}$ = nondimensional elevation of wave with respect to undisturbed free surface;

η_1 = elevation of wave with respect to undisturbed free surface;

$\hat{\eta}_1$ = η/a;

ξ = $x - ct$; and

$\hat{\xi}$ = dimensionless horizontal distance, ξ/h.

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Table 1. Computed Results

Fr	gh/c^2	k	a	b	Maximum error in Eq. (21)
Case (a)					
1	2	1.70394	0.16464	0.17851	1.71×10^{-2}
	3	2.46442	0.16068	0.18315	1.63×10^{-2}
	5	3.42747	0.15382	0.26498	1.63×10^{-2}
Case (b)					
4/5	1	1.06625	0.17554	0.08959	6.53×10^{-2}
	2	2.18597	0.15372	0.27402	6.53×10^{-2}
	3	2.83324	0.14927	0.29426	6.53×10^{-2}
Case (c)					
6/5	2.2	1.18311	0.1972	0.04455	1.43×10^{-2}
	3	1.9141	0.16898	0.14535	1.08×10^{-2}
	5	2.927	0.16118	0.21191	9.82×10^{-2}

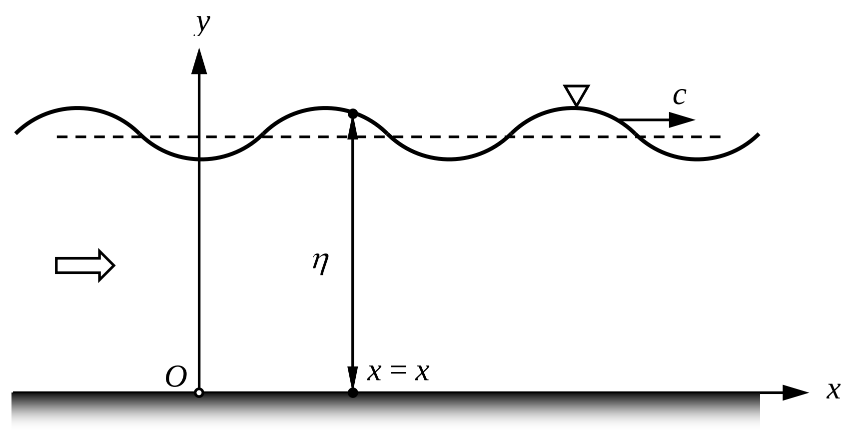

Fig. 1. Definition sketch of a progressive wave in an open channel flow

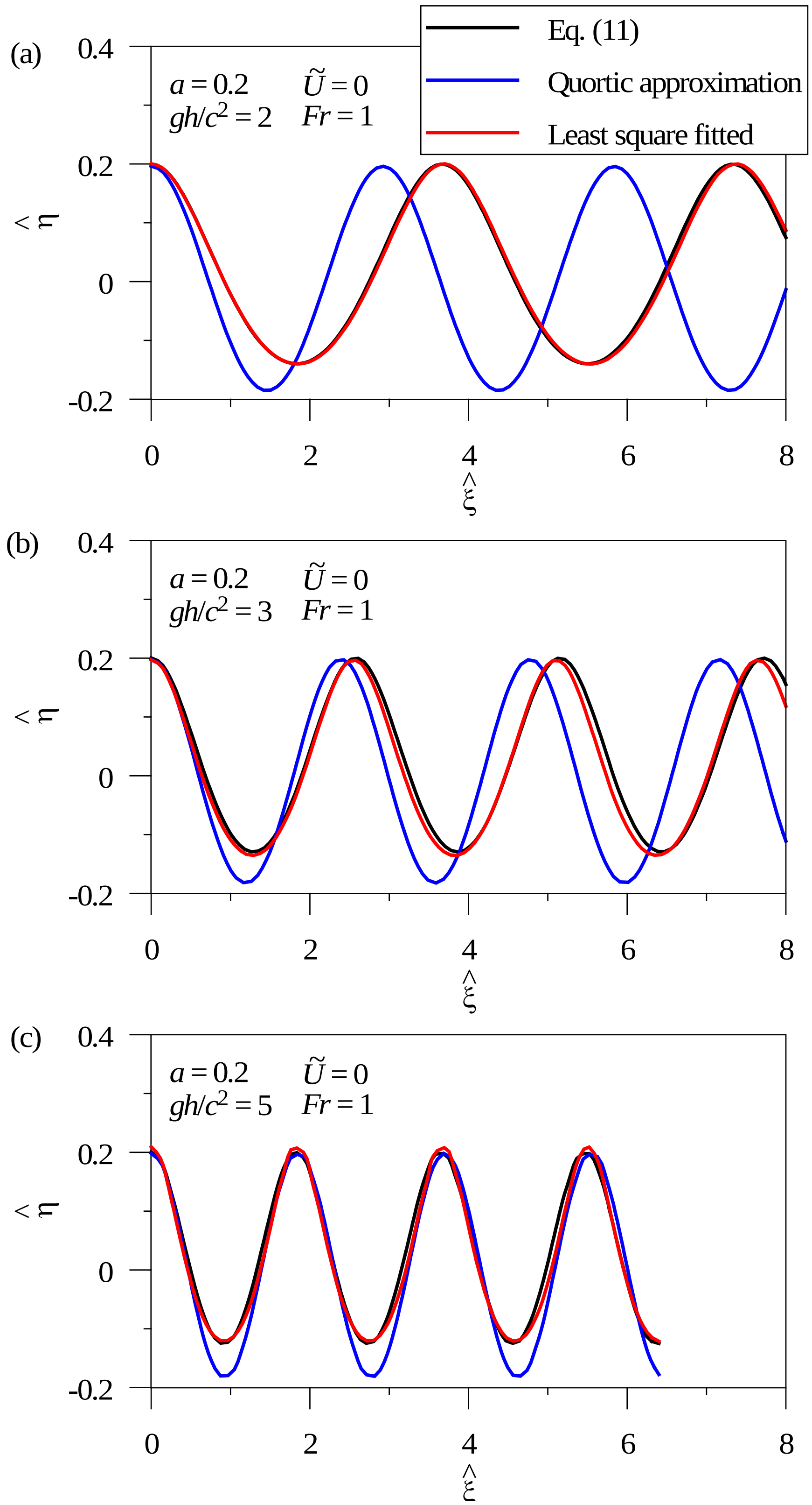

Fig. 2. Nondimensional progressive wave profiles: (a) $gh/c^2 = 2$, (b) $gh/c^2 = 3$ and (b) $gh/c^2 = 5$

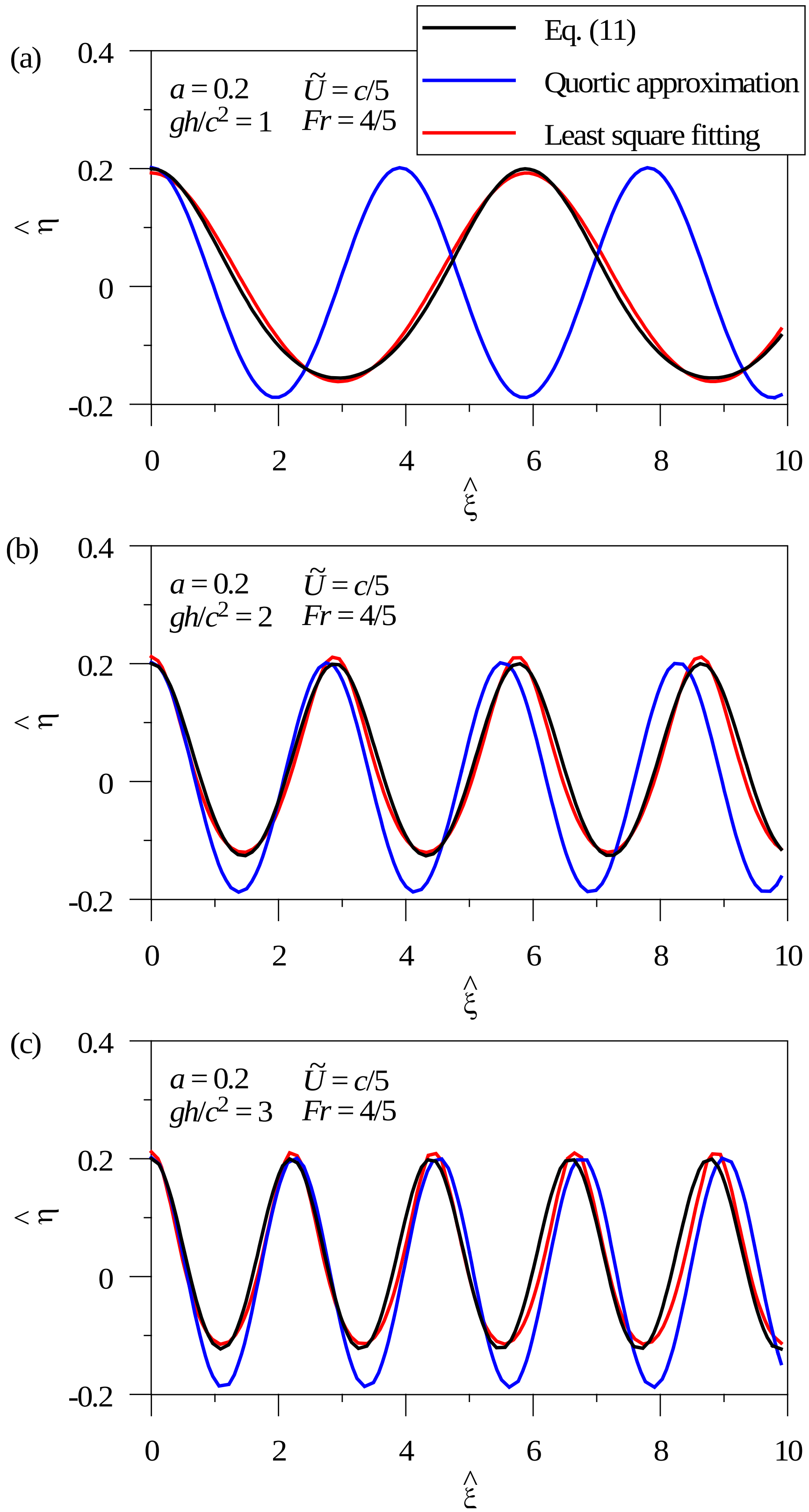

Fig. 3. Nondimensional progressive wave profiles: (a) $gh/c^2 = 1$, (b) $gh/c^2 = 2$ and (b) $gh/c^2 = 3$

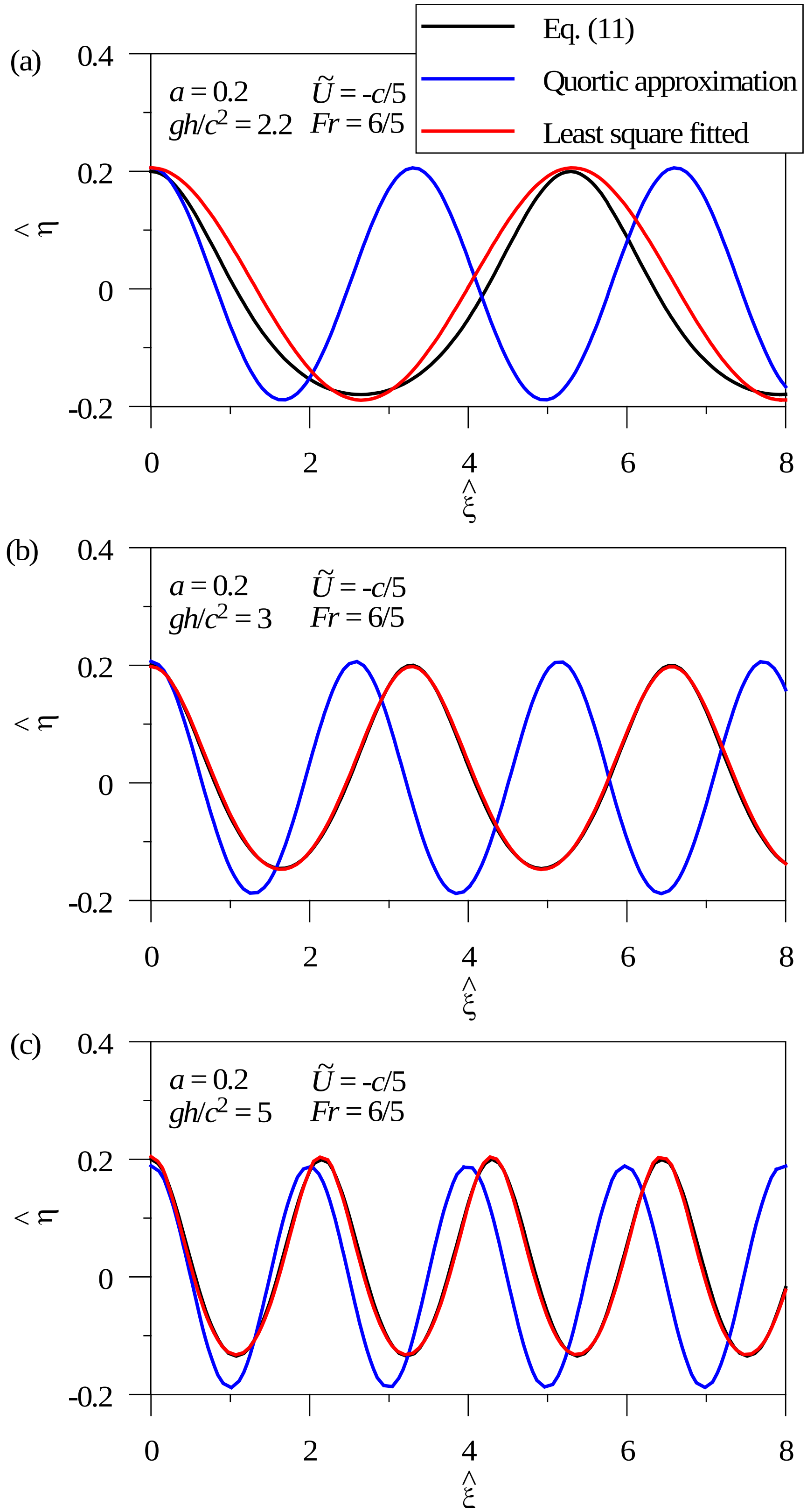

Fig. 4. Nondimensional progressive wave profiles: (a) $gh/c^2 = 2.2$, (b) $gh/c^2 = 3$ and (b) $gh/c^2 = 5$