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FERMAT'S AND HUYGENS' PRINCIPLES, AND HYPERBOLIC EQUATIONS AND THEIR EQUIVALENCE IN WAVEFRONT CONSTRUCTION

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ABSTRACT. Consider propagation of a wavefront in a medium. According to Fermat's principle a ray, travelling from one point P_0 to another point P_1 in space, chooses a path such that the time of transit is stationary. Given initial position of a wavefront Ω_0, we can use rays to construct the wavefront Ω_t at any time t. Huygens' method states that all points of a wavefront Ω_0 at $t = 0$ can be considered as point sources of *spherical* secondary wavelets and after time t the new position Ω_t of the wavefront is an envelope of these secondary wavelets. The equivalence of the two methods of construction of a wavefront Ω_t in a medium governed by a general hyperbolic system of equations does not seem to have been proved. Hyperbolic equations have their own method of construction of a wavefront. We shall discuss this still open (as far as I know) problem for a general hyperbolic system and briefly sketch the relation between the three methods for a particular case when the medium is governed by Euler equations of a polytropic gas in free space [16].

AMS (MOS) Subject Classification. 39A10.

1. Introduction

We represent a wavefront at any time t by $\Omega_t : \varphi(\mathbf{x}, t) = 0$, $\mathbf{x} \,\varepsilon\, \mathbb{R}^m$. There are three important methods of construction of a wavefront Ω_t at time t starting from its initial position Ω_0:

1. Huygens method[1](1676-78) of construction of an envelop Ω_t of spherical secondary wavelets with centres on the primary wavefront Ω_0.
2. Using a nonlinear first order partial differential equation (PDE)[2], called *eikonal equation*, for the function $\varphi(\mathbf{x}, t)$:

$$Q(\mathbf{x}, t,\ \nabla\varphi, \varphi_t) = 0 \tag{1.1}$$

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[1] Also known as Huygens's principle. Huygens formulated what became the guiding principle in theory of wave propagation and which is of great interest in mathematics [3]. It has inspired research of deep mathematical value and many areas of mathematical physics.

[2] Theory was developed mainly by Lagrange (1774), Charpit(1784), Monge (1809) and Cauchy (1819), see [7]

in constructing the secondary wavelets or using it to derive the ray equations to construct Ω_t. The equation (1.1) is in a slightly different form in which an eikonal equation is written.

3. Fermat's variational formulation[3] leading to a ray, which is a path such that the time of transit along the path is stationary for a given ray velocity[4] $\boldsymbol{\chi} = (\chi_1, \chi_2, \cdots, \chi_n)$ at a point $\mathbf{x} \,\varepsilon\, \mathbb{R}^m$ and then using the rays to construct Ω_t.

All the three methods are discussed in detail in [14]. In Huygens method, wavefront is the primary object and existence of rays is not postulated. In PDE method also, we can completely bypass use of rays by constructing the solutions of (1.1) with the help of complete integrals, which form the secondary wavelets. In Fermat's method ray is the primary object and the wavefront Ω_t at time t is obtained as locus of tips of rays drawn from various points of Ω_0. Tips here refer to the end points of the rays at time t. We can obtain rays also from (1.1) as projections on $\mathbf{x}$-space of the of the characteristic curves of (1.1) in space-time and then use these rays to construct Ω_t. We shall explain briefly these results in the next section. In this article we discuss our attempt to prove equivalence of Fermat's and Huygens' methods of wavefront construction in a medium governed by a hyperbolic system and having no boundaries.

2. Basic mathematical steps in proof of equivalence in construction of a wavefront Ω_t

Discussion of Huygens's and Fermat's methods requires a medium in which the wave propagates. We shall choose a medium whose motion is governed by a hyperbolic system of n first order partial differential equations in $m+1$ independent variables $(\mathbf{x}, t)$:

$$A(\mathbf{u}, \mathbf{x}, t)\mathbf{u}_t + B^{(\alpha)}(\mathbf{u}, \mathbf{x}, t)\mathbf{u}_{x_\alpha} + \mathbf{C}(\mathbf{u}, \mathbf{x}, t) = 0 \tag{2.1}$$

where $\mathbf{u} \in \mathbb{R}^n$, $A \in \mathbb{R}^{n\times n}$, $B^{(\alpha)} \in \mathbb{R}^{n\times n}$ and $\mathbf{C} \in \mathbb{R}^n$. The characteristic equation (which we also call eikonal equation) of (2.1) is

$$Q(\mathbf{x}, t; \nabla\phi, \phi_t) := \det(A\phi_t + B^{(\alpha)}\phi_{x_\alpha}) = 0. \tag{2.2}$$

[3]It is an outcome of Fermat's principle, which is dated as 1660 in Holliday and Resnick's book "Physics". In 1662 Fermat wrote to C. de la Chambre on his investigation of the refraction of light "...from the principle, so common and so well-established, that Nature always acts in a shortest ways." The Principle is an important mathematical formulation, which precedes calculus of variations, one of the very central field of analysis [5].

[4]Note that not every vector $\boldsymbol{\chi}$ qualifies to be a ray velocity (see footnote 5 on page 3).

We choose a wavefront Ω_t corresponding to a specific family of characteristic field with characteristic velocity c and unit normal $\mathbf{n}$:

$$c = -\frac{\varphi_t}{|\nabla\varphi|}, \ \mathbf{n} = \frac{\nabla\varphi}{|\nabla\varphi|}. \tag{2.3}$$

Let $\mathbf{l}$ and $\mathbf{r}$ be the corresponding left and right eigenvectors of the characteristic matrix $[n_\alpha B^{(\alpha)} - cA]$. Then the ray velocity $\boldsymbol{\chi}$ and the wavefront velocity c are given by

$$\chi_\alpha \quad = \frac{\mathbf{l}B^{(\alpha)}\mathbf{r}}{\mathbf{l}A\mathbf{r}} \quad and \quad c = \langle \mathbf{n}, \boldsymbol{\chi} \rangle. \tag{2.4}$$

The expression for χ_α in (2.4) was derived in [5].

A point $\mathbf{x}$ on a ray moving with above ray velocity satisfies[5]

$$\frac{dx_\alpha}{dt} = \frac{\mathbf{l}\mathbf{B}^{(\alpha)}\mathbf{r}}{\mathbf{l}\mathbf{A}\mathbf{r}} \equiv \chi_\alpha. \tag{2.5}$$

The time rate of change (along the ray) of the components of the unit normal $\mathbf{n}$ to Ω_t was given by Prasad in 1993 (see [14] and [15] for the original reference and a derivation) as

$$\frac{dn_\alpha}{dt} = -\frac{1}{\mathbf{l}A\mathbf{r}}\mathbf{l}\left\{ n_\beta \left(n_\gamma \frac{\partial B^{(\gamma)}}{\partial \eta^\alpha_\beta} - c\frac{\partial A}{\partial \eta^\alpha_\beta} \right) \right\} \mathbf{r} = \Psi_\alpha, \quad say, \tag{2.6}$$

where

$$\frac{\partial}{\partial \eta^\alpha_\beta} = n_\beta \frac{\partial}{\partial x_\alpha} - n_\alpha \frac{\partial}{\partial x_\beta} \tag{2.7}$$

are tangential derivatives on the wavefront Ω_t. (2.5) and (2.6) are the ray equations derived from the eikonal equation (2.2) of a hyperbolic system (2.1). Once the rays have been calculated, we can construct Ω_t from Ω_0 as described in Fermat's method at the end of the last section.

We now indicate very briefly two methods of construction of *spherical wavefronts*, $t = W(\mathbf{x}, \mathbf{x}_0)$ with centre at $\mathbf{x}_0$ on Ω_0, for a general eikonal equation (1.1) and in particular for the hyperbolic system (2.1).

1. We solve $\varphi(\mathbf{x}, t) = 0$ for t, write the equation of the wavefront in the form $t = \psi(\mathbf{x})$ and derive from (1.1) a first order nonlinear PDE for ψ. A suitable $m-1$ parameter family of complete integrals of this equation can be found as the required spherical wavefronts (see [14], section 3.2.2). The parameters are chosen as the $n-1$ surface coordinates on Ω_0 (see the statement in theorem 3.1).

[5] Not every vector $\boldsymbol{\chi}$ qualifies to be a ray velocity. The equations (2.5) for a general $\boldsymbol{\chi}$ contain some additional terms but for the velocity $\boldsymbol{\chi}$ of a hyperbolic system, given by (2.4), these additional terms disappear and now $\boldsymbol{\chi}$ qualifies as a ray velocity [15].

2. The second method uses the characteristic curves of the (1.1) or the bicharacteristics or rays of the hyperbolic system (2.1). The tips of all the rays at time t starting from $\mathbf{x}_0$ at time 0 generate the spherical wavefront (see equation (3.2.9), [14]) with centre at $\mathbf{x}_0$.

Let us use Fermat's method to derive now the equation for $\mathbf{n}$ in a medium in *steady state*. The medium need not be governed by a hyperbolic system. We start with a general ray velocity $\boldsymbol{\chi}$, not necessarily given by (2.4) i.e., we just take

$$\frac{d\mathbf{x}}{dt} = \boldsymbol{\chi}(\mathbf{x}, \mathbf{n}), \tag{2.8}$$

where we have also assumed $\boldsymbol{\chi}$ to be independent of t. For a given medium in steady motion the ray velocity $\boldsymbol{\chi}$ depends on the point $\mathbf{x}$ on Ω_t and the normal direction $\mathbf{n}$ of Ω_t. For example, in gas dynamics $\boldsymbol{\chi} = \mathbf{q} + \mathbf{n}a$, where $\mathbf{q}$ is fluid velocity and a is local sound velocity. We should be able to express $\mathbf{n}$ in terms $\mathbf{x}$ and the ray direction $\mathbf{N} = \boldsymbol{\chi}/|\boldsymbol{\chi}|$.

Consider a point $P(\mathbf{x}(\mathbf{t}))$ moving according to (2.8) with velocity $\boldsymbol{\chi}$. The time of transit T along a path joining two points $P_0(\mathbf{x_0})$ and $P_1(\mathbf{x_1})$ in $\mathbb{R}^m$ is

$$T = \int_{P_0}^{P_1} \frac{ds}{|\boldsymbol{\chi}|} = \int_{P_0}^{P_1} \frac{|\mathbf{x}'|dt}{|\boldsymbol{\chi}|} \tag{2.9}$$

where ds is an element of distance along the path and

$$\mathbf{x}' = \frac{d\mathbf{x}}{dt}. \tag{2.10}$$

Let μ be a parametrisation of this path such that

$$\mathbf{x}(\mu = 0) = \mathbf{x}_0, \quad \mathbf{x}(\mu = 1) = \mathbf{x}_1 \tag{2.11}$$

and let

$$\dot{\mathbf{x}} = \frac{d\mathbf{x}}{d\mu}, \tag{2.12}$$

then $\mathbf{N} = \dot{\mathbf{x}}/|\dot{\mathbf{x}}|$. The expression T in (2.9) becomes

$$T = \int_0^1 \frac{|\dot{\mathbf{x}}|}{|\boldsymbol{\chi}|} d\mu = \int_0^1 F(\mathbf{x}, \dot{\mathbf{x}}, \mu) d\mu, \quad F(\mathbf{x}, \dot{\mathbf{x}}, \mu) = \frac{|\dot{\mathbf{x}}|}{|\boldsymbol{\chi}(\mathbf{x}, \dot{\mathbf{x}})|}. \tag{2.13}$$

The Euler equations for the variational problem, which makes T stationary are

$$\frac{d}{d\mu}\left(\frac{\partial F}{\partial \dot{x}_\alpha}\right) = \frac{\partial F}{\partial x_\alpha}, \ \alpha = 1, 2, \cdots, m. \tag{2.14}$$

Given the expression for $\boldsymbol{\chi}$, we should be able to derive from (2.14) the equation for $\mathbf{n}$, which along with (2.8) would complete the ray equation by Fermat's principle.

To discuss relation between Fermat's and Huygens' methods of wavefront construction in a medium governed by the hyperbolic system (2.1), we begin with the following two propositions.

Proposition 2.1. *For $\boldsymbol{\chi}$ given by (2.5), deduce equations (2.6) from the Euler equations (2.14). This means Fermat's method implies the wavefront construction by eikonal equation method of the hyperbolic system.*

Proposition 2.2. *The method using spherical wavefronts from the eikonal equation of the hyperbolic system implies Huygens' method. In the case of a steady state solution (see solution $\mathbf{u}_0(\mathbf{x})$ of (3.9)) these two methods are equivalent i.e., eikonal equation method of a hyperbolic system and Huygens' method are equivalent.*

Remark 2.3. We note that proposition 2.1 contains only one way implication. The first part of the proposition 2.2 also contains only one way implication, where as its second part implies two way implication.

Remark 2.4. A wavefront construction with the help rays of the eikonal equation (2.2) is unique. Hence proposition 2.1 and the first part of the proposition 2.2 imply that given Ω_0 the only wavefront Ω_t obtained by Fermat's method is also a wavefront by Huygens' method. Please note that we use here **a** wavefront and not **the** wavefront by Huygens' method. In the case of wave propagation in a steady medium both methods always give the same wavefront.

3. **Some results on the three methods**

There is no doubt of the equivalence of the famous and old methods of Fermat and Huygens for a wavefront construction in an isotropic medium (like light waves and sound waves in a gas at rest). In this case mathematical proof is very simple, see [6] and [14]. The proof of equivalence is not available for a wavefront in anisotropic medium governed by a general hyperbolic system. We discuss some results in this direction.

3.1. **Huygens's and eikonal equation methods for a hyperbolic system.** We briefly elaborate the proposition 2.2 by two theorems. The proofs, inspired by related results in [6], is available in detail in sections 3.2.2 and 3.2.3, [14]. In the first theorem, we deduce Huygens' method with the help of rays of the eikonal equation (2.2):

Theorem 3.1. *Consider a parametric representation $\mathbf{x}_0(\eta_1, \eta_2, ..., \eta_{m-1})$ of points on the primary wavefront Ω_0 and spherical wavefronts of radius t and centres at $\mathbf{x}_0$, obtained from the eikonal equation (2.2). The spherical wavefronts are defined in terms of the metric determined by the hyperbolic system. Then the envelop of these spherical wavefronts is Ω_t. This is the Huygens' method of construction the wavefront Ω_t at time t.*

Thus with the help of the eikonal equation (2.2) we have deduced the Huygens' method of wavefront construction.

In the second theorem we deduce the rays of (2.1) starting from Huygens' method. As we mentioned in the introduction, Huygens' method does not assume existence of rays - even in construction of the spherical wavefronts. Assuming that the coefficient metrices A and $B^{(\alpha)}$ do not depend on t but only on $\mathbf{x}$ and further assuming existence of spherical wavefronts of the form $t = W(\mathbf{x}, \mathbf{x}_0)$ with $\mathbf{x}_0$ lying on the initial wavefront Ω_0, the following result has been proved (stated slightly differently from the statement at the end of section 3.2.3, [14]).

Theorem 3.2. *From Huygens' method of wavefront construction we can deduce the ray equations of the hyperbolic system (2.1) when A and B are independent of t.*

The above two theorems prove the equivalence of the Huygens' method and the eikonal method completely for a system when the coefficients A and B are independent of t.

Let us now go to discussion of Fermat's method. Proof of the proposition 2.1 i.e.,

$$\{Wavefront\, obtained\, from\, rays\, from\, Fermat's\, principle\} \Rightarrow$$

$$\{Wavefront\, obtained\, from\, rays\, of\, the\, corresponding\, hypebolic\, system\} \tag{3.1}$$

has not yet been achieved in this generality.

It appears that this equivalence has been proved for elastodynamics by Epstein and Sniatycky, 1992. It has been very clearly proved for Euler equations of a polytropic gas by us in 1993 ([16], also mentioned in [14]) but we did not publish it because we hoped that we could prove it for a general hyperbolic system. The equivalence for Euler equation has now been written and will be sent for publication. We describe it very briefly in section 3.2.

As mentioned in proposition 2.1 the proof of (3.1) in most general form reduces to proving the equivalence of (2.6) and (2.14) for $F(\mathbf{x}, \dot{\mathbf{x}}, \mu) = \frac{|\dot{\mathbf{x}}|}{\boldsymbol{\chi}(\mathbf{x},\dot{\mathbf{x}})}$, *where* $\chi_\alpha = \frac{\mathbf{l B^{(\alpha)} r}}{\mathbf{l A r}}$ *is given.*

3.2. **Proof of (3.1) for Euler equations of a polytropic gas in steady motion.**

Let us consider motion of a polytropic gas, for which we denote mass density by ρ, particle velocity by $\mathbf{q} = (q_1, q_2, q_3)$ and pressure by p. The sound velocity a is given by $a^2 = \frac{\gamma p}{\rho}$, where γ is a constant. For a forward facing gas we can write characteristic equation (or eikonal equation) as

$$Q := \varphi_t + \langle \mathbf{q}, \nabla\varphi \rangle + a|\nabla\varphi| = 0. \tag{3.2}$$

The ray equations are

$$\frac{d\mathbf{x}}{dt} = \mathbf{q} + \mathbf{n}a = \boldsymbol{\chi}, \text{ say,} \tag{3.3}$$

and

$$\frac{d\mathbf{n}}{dt} = -\mathbf{L}a - n_\beta \mathbf{L} q_\beta = \mathbf{\Psi}, \text{ say,} \tag{3.4}$$

where

$$\mathbf{L} = \nabla - \mathbf{n}\langle \mathbf{n}, \nabla \rangle \tag{3.5}$$

is a tangential derivative on the wavefront $\Omega_t : \phi(\mathbf{x}, t) = 0$.

We consider a steady solution $(\rho_0, \mathbf{q}_0, p_0)$ of the Euler equations, then the function $F(\mathbf{x}, \dot{\mathbf{x}}, \mu)$ appearing in the variational problem (2.13) is $F(\mathbf{x}, \dot{\mathbf{x}}, \mu) = \frac{\dot{\mathbf{x}}}{|\mathbf{q}_0 + \mathbf{n}a_0|}$ and Euler variational equation (2.14) becomes

$$\frac{d}{d\mu}\left\{\frac{\partial}{\partial \dot{\mathbf{x}}_\alpha}\left(F = \frac{|\dot{\mathbf{x}}|}{|\mathbf{q_0} + \mathbf{n}a_0|}\right)\right\} = \frac{\partial}{\partial x_\alpha}\left(F = \frac{|\dot{\mathbf{x}}|}{|\mathbf{q}_0 + \mathbf{n}a_0|}\right). \tag{3.6}$$

Note that the function F is to be expressed in terms of $\mathbf{x}$ and $\dot{\mathbf{x}}$, which requires expressing $\mathbf{n}$ as a function of $N = \frac{\dot{\mathbf{x}}}{|\dot{\mathbf{x}}|}$. This is one of the most difficult part of the proof and requires many pages of calculations. One of intermediate steps in these calculations leads to equations

$$\frac{d}{d\mu}\left(\frac{n_\alpha}{c}\right) = \frac{\partial}{\partial x_\alpha}\left(\frac{|\dot{\mathbf{x}}|}{|\boldsymbol{\chi}|}\right) = |\dot{\mathbf{x}}|\frac{\partial}{\partial x_\alpha}\left(\frac{1}{|\boldsymbol{\chi}|}\right) \tag{3.7}$$

where $c = a_0 + \langle \mathbf{n}, \mathbf{q}_0 \rangle$, or

$$\frac{d}{dt}\left(\frac{n_\alpha}{c}\right) = |\boldsymbol{\chi}|\frac{\partial}{\partial x_\alpha}\left(\frac{1}{|\boldsymbol{\chi}|}\right). \tag{3.8}$$

After some more lengthy calculations, we can derive (3.4) (with a and $\mathbf{q}$ replaced by a_0 and $\mathbf{q}_0$ respectively) from (3.8). Thus for the ray velocity (3.3) of a polytropic gas in steady motion, we have proved that the rays obtained by Fermat's principle are indeed the rays of Euler equations of a polytropic gas. *Since the ray equations of Euler equations are unique, this means that in a steady motion the wave front Ω_t obtained by Fermat's method and that obtained by using the rays of the equations of motion of the polytropic gas are the same.*

3.3. **Fermat's principle for deducing ray equations of hyperbolic system representing a steady state of a medium.** The quantities $\mathbf{n}, c$, and $\boldsymbol{\chi}$ appearing in the equation (3.8) do not refer to gasdynamic equations alone but to a general hyperbolic system. They are given in terms of the coefficients A and $B^{(\alpha)}$ for a given mode of propagation (i.e., a characteristic field) under consideration. A beautiful, simple, elegant and general result like this, that too after a long and involved calculations from a particular case, must be true for a general system. This leads me to make me a conjecture, which we shall state soon.

For many physical systems, the equations of motion of media is governed by hyperbolic systems in which $A, B^{(\alpha)}$, and C do not depend explicitly on t i.e., by a hyperbolic system of the form

$$A(\mathbf{u}, \mathbf{x})\mathbf{u}_t + B^{(\alpha)}(\mathbf{u}, \mathbf{x})\mathbf{u}_{x_\alpha} + \mathbf{C}(\mathbf{u}, \mathbf{x}) = 0. \tag{3.9}$$

This system can have both steady solution $\mathbf{u}_0(\mathbf{x})$ and an unsteady solution $\mathbf{u}(\mathbf{x}, t)$. When the latter is substituted in coefficients in (3.9), they depend on t.

For a steady solution $\mathbf{u}_0(\mathbf{x})$ of (3.9), the ray equations (2.5) and (2.6) become

$$\frac{dx_\alpha}{dt} = \left(\frac{\mathbf{l}\mathbf{B}^{(\alpha)}\mathbf{r}}{\mathbf{l}\mathbf{A}\mathbf{r}}\right)_{u_0} = \chi_{\alpha 0}, \quad \text{say}, \tag{3.10}$$

and

$$\frac{dn_\alpha}{dt} = -\left(\frac{1}{\mathbf{l}A\mathbf{r}}\mathbf{l}\left\{n_\beta\left(n_\gamma\frac{\partial B^{(\gamma)}}{\partial\eta_\beta^\alpha} - c\frac{\partial A}{\partial\eta_\beta^\alpha}\right)\right\}\mathbf{r}\right)_{u_0} = \Psi_{\alpha 0}, \quad \textit{say}. \tag{3.11}$$

Now we state the conjecture

Conjecture 3.3. *The Fermat's variational principle, with ray velocity χ_0 given by (3.10), leads to the second part of the ray equations (3.11) of hyperbolic system (3.9) with its solution representing a steady state of a medium.*

I have attempted to prove this conjecture many times since 1993 but have not succeeded. We have seen this result to be true for steady motion of a polytropic gas.

4. Extended Fermat's principle and a lemma

The equation (2.13) for the time of transit T has been deduced for a ray velocity $\boldsymbol{\chi}(\mathbf{x}, \mathbf{n})$ in a stationary medium (see the line just below(2.8)). In a non-stationary medium $\boldsymbol{\chi}$ depends also on time t and the time of transit from $P_0(\mathbf{x}_0)$ to $P_1(\mathbf{x}_1)$ is

$$T = \int_0^1 F(\mathbf{x}, \dot{\mathbf{x}}, t)d\mu, \;\; F = \frac{|\dot{\mathbf{x}}|}{|\boldsymbol{\chi}(\mathbf{x}, \dot{\mathbf{x}}, \mathbf{t})|}. \tag{4.1}$$

It has been shown (see section 3.2.4, [14]) that the variational problem for (4.1) is ill posed.

One of the reasons of illposedness is the following. Fermat's principle was first formulated for a stationary medium in which any two distinct points in space could be connected at two different times by a ray and hence also the end points P_0 and P_1 in $\mathbf{x}$-space could be connected by a ray. The formulation (4.1) given above for two arbitrary points $\tilde{P}_0(\mathbf{x}_0, t_0)$ and $\tilde{P}(\mathbf{x}_1, t_1)$ in space-time requires examination whether $\tilde{P}_0$ and $\tilde{P}_1$ can also be connected by a ray. This is not true for an arbitrary pair $(\tilde{P}_0, \tilde{P}_1)$ in space-time. This was observed by Kovner (1990) followed by a demonstration by Nityananda and Samuel (1992) in general relativity, who restricted the points $\tilde{P}_0$ and $\tilde{P}_1$ to points which can be joined by null curves. The point noted by Kovner is relevant not only in general relativity but in a physical system governed by any hyperbolic system. We should consider only those points $\tilde{P}_0$ and $\tilde{P}_1$ in space-time which can be connected by a bicharacteristic curve i.e., $\tilde{P}_1$ should be a point on the forward characteristic conoid of the point $\tilde{P}_0$. When $\tilde{P}_0$ and $\tilde{P}_1$ are such points, it is now clear that the value of T should be stationary with respect to paths which lie on the characteristic conoid at $\tilde{P}_0$; such paths need not be bicharacteristics but the path which makes T stationary should turn out to be the bicharacteristic curve joining $\tilde{P}_0$ and $\tilde{P}_1$.

Now we give a general formulation of the Fermat's principle, which we call *extended Fermat's principle* in a nonstationary medium governed by a hyperbolic system. Let the equation of the forward characteristic conoid of the point $\tilde{P}_0(\mathbf{x}_0, t_0)$ be given by

$$t = \psi(\mathbf{x}) \tag{4.2}$$

and let $\tilde{P}_1(\mathbf{x}_1, t_1)$ be such that $t_1 = \psi(\mathbf{x}_1)$. We now define a function $\bar{F}$ on the characteristic conoid by

$$\bar{F}(\mathbf{x}, \dot{\mathbf{x}}) = \frac{|\dot{\mathbf{x}}|}{\bar{\boldsymbol{\chi}}(\mathbf{x})}\ , \quad \bar{\boldsymbol{\chi}}(\mathbf{x}) = \boldsymbol{\chi}(\mathbf{x}, \psi(\mathbf{x})). \tag{4.3}$$

Extended Fermat's principle says that a ray is defined to be a path which makes the integral

$$T = \int_0^1 \bar{F}(\mathbf{x}, \dot{\mathbf{x}}) d\sigma,\ \bar{F} = \frac{|\dot{\mathbf{x}}|}{\bar{\boldsymbol{\chi}}(\mathbf{x})} \tag{4.4}$$

stationary with respect to variations in the paths, which now obviously lie on the characteristic conoid at the point $\tilde{P}_0$.

The Euler's equations corresponding to the variational problem (4.4) are

$$\frac{d}{d\mu}\left(\frac{\partial \bar{F}}{\partial \dot{x}_\alpha}\right) = \frac{\partial \bar{F}}{\partial x_\alpha}\ ,\ \ \alpha = 1, 2, \ldots, m \tag{4.5}$$

or

$$\frac{d}{d\mu}\left(\frac{\partial F}{\partial \dot{x}_\alpha}\right) = \frac{\partial F}{\partial x_\alpha} + \psi_{\chi_\alpha}\frac{\partial F}{\partial t}. \tag{4.6}$$

Note that the $\frac{\partial}{\partial x_\alpha} + \psi_{x_\alpha}\frac{\partial}{\partial t}$ is an operator, which represents differentiation in a direction of a tangent to the characteristic conoid $t = \psi(\mathbf{x})$.

We denote $\boldsymbol{\chi}$ and Ψ on the right hand sides of (2.5) and (2.6) by $\boldsymbol{\chi}(A, B, \mathbf{n})$ and $\Psi(A, B, \mathbf{n})$ respectively. Let $\Omega_0 : \varphi_0(\mathbf{x}) = 0$ be the initial position of a surface represented parametrically in the form $\mathbf{x} = \mathbf{x}_0(\eta)$, $\eta = (\eta_1, \eta_2, ..., \eta_{m-1})$ and let $\mathbf{n}_0(\eta)$ be the unit normal vector calculated on Ω_0. We now prove the following lemma.

Lemma 4.1. *Let $A(\mathbf{x},t)$ and $B(\mathbf{x},t) = (B^{(1)},\cdots,B^{m})$ be $\mathcal{C}^1$ functions. Let Ω_t : $\varphi(\mathbf{x},t)=0$ be a surface generated by the end points (at time t) the integral curves (let us call them rays) of the initial value problem*

$$\frac{d\mathbf{x}}{dt} = \boldsymbol{\chi}(A,B,\mathbf{n}), \;\; \frac{d\mathbf{n}}{dt} = \boldsymbol{\Psi}(A,B,\mathbf{n}) \tag{4.7}$$
$$\mathbf{x}(0) = \mathbf{x}_0(\eta), \quad \mathbf{n}(0) = \mathbf{n}_0(\eta).$$

Further let us represent Ω_t in the form $t=\psi(\mathbf{x})$ and set

$$\bar{A}(\mathbf{x}) = A(\mathbf{x},\psi(\mathbf{x})), \;\; \bar{B} = B(\mathbf{x},\psi(\mathbf{x})). \tag{4.8}$$

Then the surface generated by the end points (at time t) of the integral curves of initial value problem

$$\frac{d\mathbf{x}}{dt} = \boldsymbol{\chi}(\bar{A},\bar{B},\mathbf{n}), \;\; \frac{d\mathbf{n}}{dt} = \boldsymbol{\Psi}(\bar{A},\bar{B},\mathbf{n}) \tag{4.9}$$
$$\mathbf{x}(0) = \mathbf{x}_0(\eta), \quad \mathbf{n}(0) = \mathbf{n}_0(\eta)$$

coincides with surface Ω_t.

Proof. The expression for Ψ_α given on the right hand side of (2.6), does not contain any derivative of $\mathbf{n}$. Therefore, the expressions for $\boldsymbol{\chi}$ and Ψ on the right hand side of (4.7) are $\mathcal{C}^1$ functions of $\mathbf{x}, t$ and $\mathbf{n}$. We can find local solutions (depending on $m-1$ parameters in η) of the problem (4.7). From the tips of these rays we can generate the surface Ω_t and find the function $t=\psi(\mathbf{x})$. With this function $\psi(\mathbf{x})$, the initial value problem (4.9) is well defined and we can be solve it locally.

The operators $\frac{\partial}{\partial\eta_\beta^\alpha}$ appearing in (2.6) or (4.9) are tangential derivatives on the surface Ω_t. Hence, at any point $(\mathbf{x},t)$ on $t=\psi(\mathbf{x})$

$$\frac{\partial A}{\partial\eta_\beta^\alpha} = \frac{\partial\bar{A}}{\partial\eta_\beta^\alpha} \quad \text{and} \quad \frac{\partial B^{(\gamma)}}{\partial\eta_\beta^\alpha} = \frac{\partial\bar{B}^{(\gamma)}}{\partial\eta_\beta^\alpha}. \tag{4.10}$$

(4.10) implies that the solution of the problem (4.9) i.e., *rays* given by (4.9) coincides with those of (4.7). Therefore the tips of the *rays* of (4.7) and (4.9) at time t generate the same surface Ω_t. This completes a proof of the lemma. □

4.1. **Derivation of ray equations of a hyperbolic system with time dependent coefficients from extended Fermat's principle.** Let us use the above lemma to derive the ray equations (2.5)-(2.6) of the (2.1) with time dependent coefficients and having an unsteady solution $\mathbf{u}(\mathbf{x},t)$. Consider a wavefront Ω_t : $\varphi(\mathbf{x},t)=0$ in the state represented by this solution at a time t. $\varphi(\mathbf{x},t)$ satisfies the eikonal equation (2.2). Let $\varphi(\mathbf{x},t)=0$ be expressed in the form $t=\psi(\mathbf{x})$. As in the lemma, we define functions $\bar{A}(\mathbf{x}) = A(\mathbf{u}(\mathbf{x},\psi(\mathbf{x})),\mathbf{x},\psi(\mathbf{x})), \bar{B}(\mathbf{x}) = B(\mathbf{u}(\mathbf{x},\psi(\mathbf{x})),\mathbf{x},\psi(\mathbf{x}))$. Assuming that the conjecture (3.3) is true, we drive from it the equations $\frac{d\mathbf{n}}{dt} = \boldsymbol{\Psi}(\bar{A},\bar{B},\mathbf{n})$ in (4.9) from the expression of ray velocity $\boldsymbol{\chi}(\bar{A},\bar{B},\mathbf{n})$. From relations (4.10) it follows that $\boldsymbol{\Psi}(\bar{A},\bar{B},\mathbf{n}) = \boldsymbol{\Psi}(A,B,\mathbf{n})$. Thus using the extended Fermat's principle, we have derived the ray equations (2.5)-(2.6) of the hyperbolic system (2.1) with time dependent coefficients.

4.2. **Fermat's principle for derivation of ray equations of a weakly nonlinear ray theory (WNLRT).** Consider a steady solution $\mathbf{u} = \mathbf{u}_0(\mathbf{x})$ of (3.9) and a perturbation $\mathbf{v}(\mathbf{x}, t)$ of $\mathbf{u}_0(\mathbf{x})$. Assume that

$$\mathbf{u} = \mathbf{u}_0(\mathbf{x}) + \mathbf{v}(\mathbf{x}, t), \ |\mathbf{v}| << 1 \tag{4.11}$$

satisfies (3.9) in high frequency approximation [14]. Then

$$\mathbf{v} \simeq \mathbf{r}_0(\mathbf{x}, \mathbf{n}) w(\mathbf{x}, t, \mathbf{n}), \quad |w| << 1, \tag{4.12}$$

where right eigenvector $\mathbf{r}_0$ satisfies

$$[n_\alpha B^{(\alpha)}(\mathbf{u}_0(\mathbf{x}), \mathbf{x}) - cA(\mathbf{u}_0(\mathbf{x}), \mathbf{x})]\mathbf{r}_0 = 0 \tag{4.13}$$

and the amplitude w of the nonlinear wave satisfies a transport equation (chapter 4, [14]), which we do not write here.

The ray velocity $\boldsymbol{\chi}$ of the nonlinear wavefront, retaining only the first order terms, is

$$\begin{aligned} \boldsymbol{\chi} &\simeq \boldsymbol{\chi}(A(\mathbf{u}_0 + \mathbf{r}_0 w, \mathbf{x}), B(\mathbf{u}_0 + \mathbf{r}_0 w, \mathbf{x}), \mathbf{n}) \\ &\simeq \boldsymbol{\chi}_0 + \boldsymbol{\chi}_1 w, \ \ \boldsymbol{\chi}_1 = (\nabla_u \boldsymbol{\chi})_0 \mathbf{r}_0. \end{aligned} \tag{4.14}$$

We now use the conjecture (3.3), that the Fermat's principle gives the ray equations of a hyperbolic system and also use the extended Fermat's principle to derive the equation (2.6) with $\mathbf{u}$ given by (4.11) and (4.12). To the first order we get the ray diffraction rate Ψ

$$\begin{aligned} \Psi &\simeq \Psi(A_0 + (\nabla_u A)_0 \mathbf{r}_0 w, \ B_0 + (\nabla_u B)_0 \mathbf{r}_0 w, \mathbf{n}) \\ &\simeq \Psi_0 + \Psi_1 w, \ \Psi_1 = (\nabla_u \Psi)_0 \mathbf{r}_0, \end{aligned} \tag{4.15}$$

where Ψ_0 and Ψ_1 contain differential operators in directions tangential to the nonlinear wavefront Ω_t.

The ray equations of WNLRT for a hyperbolic system (3.9), with solutions given by (4.11) and (4.12), are

$$\frac{d\mathbf{x}}{dt} = \boldsymbol{\chi}_0 + \boldsymbol{\chi}_1 w \tag{4.16}$$

and

$$\frac{d\mathbf{n}}{dt} = \boldsymbol{\Psi}_0 + \Psi_1 w. \tag{4.17}$$

These two equations are exactly the same as the ray equations (4.3.28) and (4.3.29) of [14]. Derivation of the transport equation, which couples the amplitude equation for w to the ray equations leading to WNLRT forms a difficult mathematical problem. Its various formal derivations started in 1975 and ended in 2000 (see

[14]). An alternative formal derivation following [4] (see also [8]), in which the ray diffraction term $\Psi_1 w$ due to nonlinearity is missing and which is valid as long as the nonlinear rays do not deviate significantly from the linear rays (see [14] for a detailed discussion), does not satisfy Fermat's principle. Our WNLRT follows the nonlinear rays, is valid over a long distance far beyond a caustic region and gives many physically realistic results ([10] and [2]). There is no doubt of validity of the WNLRT, however a rigorous mathematical proof of its validity is extremely difficult and is still an open problem.

4.2.1. *Derivation of ray equation of WNLRT in a polytropic gas.* We can use the Fermat's principle to derive the ray equations WNLRT in a forward facing wave in polytropic gas at rest and in uniform state: ($\rho_0 =$ constant, $\mathbf{q} = 0$ and $p_0 =$ constant). The small amplitude perturbation from (4.12) is given by

$$\rho - \rho_0 = (\rho_0/a_0)w, \;\; \mathbf{q} = \mathbf{n}w, \;\; p - p_0 = \rho_0 a_0 w, \quad |w| << 1. \tag{4.18}$$

The equations (4.16) and (4.17) reduce to (see equations (6.1.2), [14])

$$\frac{d\mathbf{x}}{dt} = \left(a_0 + \frac{\gamma+1}{2}w\right)\mathbf{n}, \;\; \frac{d\mathbf{n}}{dt} = -\frac{\gamma+1}{2}\mathbf{L}w. \tag{4.19}$$

These equations can be also be obtained from (3.3) and (3.4) using (4.18).

An important use of extended Fermat's principle is not only in a simple and elegant derivation of the nonlinear rays but also in convincing us that these ray equations, obtained by a different method, are correct and physically realistic. Applications of these equations are available in [2] *and* [1].

4.3. **Derivation of shock ray equations in a polytropic gas.** It is interesting to use the extended Fermat's principle to derive the shock ray equations. The eikonal equation in this case is the shock manifold partial differential equation (SME), discussed in detail in [13]:

$$Q := \varphi_t + \langle \mathbf{q}_r, \nabla\varphi\rangle + S|\nabla\varphi| = 0, \tag{4.20}$$

where $\mathbf{q}_r$ is the fluid velocity ahead of the shock and S is the shock velocity. S depends on the state on two sides of the shock and does not depend on its unit normal N. The equation (4.20) is exactly the same as (3.2) except that $\mathbf{q}$ is replaced by $\mathbf{q}_r$ and a is replaced by S. The shock ray equations obtained in [13] are

$$\frac{d\mathbf{x}}{dt} = \mathbf{q}_r + \mathbf{N}S \tag{4.21}$$

and

$$\frac{d\mathbf{N}}{dt} = -\mathbf{L}S - N_\beta \mathbf{L}\mathbf{q}_{r\beta} \tag{4.22}$$

where $\mathbf{L}$ here is same as that in (3.5) except that $\mathbf{n}$ is replaced by $\mathbf{N}$. Since S is well defined on the shock front, its tangential derivative $\mathbf{L}S$ is also well defined. Unless a shock front is stationary, a flow containing a shock is always unsteady.

Following the derivation of the ray equations (3.3) and (3.4) from (extended) Fermat's principle in section 4.2, we can derive the shock ray equations (4.21) and (4.22) also from Fermat's principle. The relation discussed between the Huygens's method and eikonal equation method remains true also for the construction a shock front.

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REFERENCES

[1] K.R. Arun, *A Numerical Scheme for three-dimensional Front propagation and control of Jordan mode*, SIAM J. Sci. Comput., **34**:B148-178, 2012.

[2] K.R. Arun, M. Lukacova-Medvidova, Phoolan Prasad and S.V. Raghurama Rao, *An application of 3-D kinematical conservation laws: propagation of a three dimensional wavefront*, SIAM J. Appl. Math., **70**:2604-2626, 2010.

[3] H. Blok, H.A. Ferwenda and and H.K. Kuiken, *Huygens' Principle 1690-1990: Theory and Applications*, North-Holland, 1990.

[4] Y. Choquet-Bruhat, Ondes asymptotiques et approchees pour de systems d'equations aux derive′es partialles nonline′aires *J.Math Pures et Appl*, **48**:117-58, 1969.

[5] R. Courant and D. Hilbert,*Methods of Mathematical Physics*, vol 1. John Wiley, 1953.

[6] R. Courant and D. Hilbert, *Methods of Mathematical Physics*, vol 2: Partial Differential Equations, Interscience Publishers, 1962.

[7] S. S. Demidov, *The Study of Partial Differential Equations of the First Order in the 18 th and 19 th Centuries*. Archive for History of Exact Sceinces, **26**:325-350, 1982.

[8] J. K. Hunter, *Asymptotic equations for nonlinear hyperbolic waves, Surveys in Applied Mathematics, vol II*, edited by J.B.Keller, W. McLaughlin and C. Papanicolaou, Plenum Press, 1995.

[9] Christiaan Huygens, *Traité de la Lumière*, 1676-78. For a picture of the title page see [3].

[10] N. K. R. Kevlahan, *The propagation of weak shocks in non-uniform flow*, J. Fluid Mech., **327**:167-97, 1996.

[11] I. Kovner, *Fermat principle in arbitrary gravitational fields*. Astrophysical J., **351**:114-20, 1990.

[12] R. Nityananda and J. Samuel. *Fermat's principle in general relativity*, Phys. Review D, **45**:3862-64, 1992.

[13] Phoolan Prasad, *Kinematics of a multi-dimensional shock of arbitrary strength in an ideal gas*, Acta Mechanica, **45**:167-176, 1982.

[14] Phoolan Prasad, *Nonlinear Hyperbolic Waves in Multi-dimensions*, Monographs and Surveys in Pure and Applied Mathematics, Chapman and Hall/CRC, Vol 121, 2001.

[15] Phoolan Prasad, *Ray theories for hyperbolic waves, kinematical conservation laws (KCL), and applications*, Indian J. of pure and appl. Math., **38**:467-490, 2007.

[16] Phoolan Prasad and G. Russo, *Fermat's and Huygens' Principles and their Equivalence in Wavefront Construction in a polytropic gas*, first worked out in 1993, written in 2013.