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Dirac, Harish-Chandra and the unitary representations of the Lorentz group

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Harish-Chandra began his research career working with Homi Bhabha at the Indian Institute of Science, Bangalore, on problems in theoretical physics. Soon after, he went to Cambridge University in England, to be guided by Paul Dirac towards a PhD degree. Dirac suggested that he investigate the unitary irreducible representations (UIRs) of the homogeneous Lorentz group $SO(3, 1)$ (and of its universal covering group $Sl(2, C)$). As it turned out, this work of Harish-Chandra done under Dirac's supervision pretty much determined the major interests that he pursued throughout his life. In this article I would like to describe this work in perspective, and take the opportunity to recall some personalities and events of half a century ago. It may also not be out of place to indicate to younger readers and aspirants today that there continues to be a need and room for heroes that we may admire and attempt to emulate.

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The importance of the linear representations of the groups $SO(3, 1)$ and $Sl(2, C)$ in special relativistic problems is well known. The superposition principle of quantum mechanics, in the context of relativistic quantum mechanics and field theory, only enhances this fact. Since these groups are noncompact, there is a great difference between their finite and infinite dimensional irreducible representations. Namely (always excluding the trivial one-dimensional representation) every finite dimensional representation is necessarily nonunitary; and every unitary representation is necessarily infinite dimensional. Of course there can be (and there are) infinite dimensional irreducible representations which are nonunitary[1].

These properties are in striking contrast to the situation with the three dimensional rotation group $SO(3)$, and its universal covering group $SU(2)$, both of which are compact and which are so important in the quantum theory of angular momentum. Thus each irreducible representation of $SU(2)$ is known to be both finite dimensional and unitary. These representations are labelled by the 'spin value j' which can take values $j = 0, 1/2, 1, 3/2,\ldots$ and are of dimension $(2j+1)$: for half odd integral j we have genuinely spinorial representations of $SU(2)$ (also called double-valued representations of $SO(3)$), while for integral j we have the vector and tensor representations of $SO(3)$ of various ranks[1].

The finite dimensional irreducible representations (IRs) of the Lorentz group $SO(3)$ and of $Sl(2, C)$ are quite familiar. They are the subject matter of spinor and tensor analysis. Each such representation is labelled by a pair of spin values, (j_1, j_2) say, and is of dimension $(2j_1 + 1)(2j_2 + 1)$. These 'quantum numbers' j_1 and j_2 can independently take on any value in the range 0, 1/2, 1, 3/2,.... If $j_1 + j_2$ is half odd integral we have basically a spinorial representation of $Sl(2, C)$, otherwise an $SO(3, 1)$ tensor of suitable type. Within an $\mathrm{IR}(j_1, j_2)$, the spectrum of angular momentum values (i.e. UIRs of $SU(2)$ present) goes from the minimum value $j_0 = |j_1 - j_2|$ in steps of unity up to the maximum of $j_1 + j_2$; and each of these appears exactly once.

Here are some simple examples of these finite dimensional nonunitary IRs. In special relativistic mechanics and electrodynamics, the space-time position four-vector, the energy-momentum four-vector and the charge current density all belong to the IR (1/2, 1/2). The ten-component energy momentum tensor of any system belongs to the sum of the two IRs (1,1) and (0,0), the latter being the trivial one dimensional or scalar representation. The electromagnetic field strengths belong to the sum of the two IRs (1,0) and (0,1) – these correspond to the complex combinations $\mathbf{E} \pm i\mathbf{B}$ of electric and magnetic field vectors. The Klein-Gordon wave equation involves of course a scalar field on space-time.

The first time spinor quantities entered into physics was with Dirac's discovery in 1928 of the relativistic wave equation for the electron[2]. Written in modern notation this reads

$$(-i\gamma^{\mu}\partial_{\mu} + m)\psi(x) = 0. \tag{1}$$

Here the γ^{μ} are certain 4×4 matrices – the 'Dirac gammas' – and $\psi(x)$ is the Dirac wave function having four components. It belongs to the reducible representation $(1/2, 0) \oplus (0, 1/2)$ of $Sl(2, C)$. Even though spinors were known within mathematics somewhat earlier, through the work of Elie Cartan, their use in physics came as a total surprise. Indeed Dirac has recalled that Eddington could not believe there could be nontensorial quantities at all, while von Neumann was surprised that there could be four-component objects in relativity which were not four-vectors! Soon after Dirac's discovery, Weyl constructed wave equations for massless spin one half particles – neutrinos – using two-component wave functions belonging to either the IR (1/2, 0) or the IR (0, 1/2) alone[3]. Another slightly later example is the Rarita-Schwinger wave equation for particles of spin 3/2, which was based on the representations (1, 1/2), (1/2, 1), (1/2, 0) and (0, 1/2) all put together[4]. In the thirties generally the study of relativistic wave equations was quite an industry – and later both Bhabha and Harish-Chandra were to contribute to it too.

The detailed group-theoretic and algebraic properties of the finite dimensional IRs of $Sl(2, C)$ were brilliantly exploited by Wolfgang Pauli in his 1940 proof of the spin-statistics theorem – integral (half odd integral) spin particles must necessarily obey Bose (Fermi) statistics[5].

Now let us turn to the story of the unitary representations (URs). Probably the earliest occurrence of these was in 1932 in a remarkable paper by the talented Italian physicist Ettore Majorana[6]. At the time of Majorana's work the positron had not yet been experimentally discovered, and the negative energy solutions of the Dirac equation (1) were felt to be an embarrassment. Therefore Majorana set out to devise a relativistic wave equation which would completely avoid negative energy solutions! He constructed two-wave equations – the Majorana wave equations – which may be written, in the pattern of equation (1), as

$$(-i\Gamma^{\mu}\partial_{\mu} + \kappa)\psi(x) = 0. \tag{2}$$

In contrast to equation (1), however, the wave-function $\psi(x)$ here has infinitely many components, and the matrices Γ_{μ} are infinite dimensional and hermitian. And under a Lorentz transformation, $\psi(x)$ changes according to an infinite dimensional UIR of $Sl(2, C)$. The two Majorana equations describe respectively particles of all integral spins $j = 0, 1, 2, \ldots$, and of all half odd integral spins $j = 1/2, 3/2, 5/2, \ldots$, once each; and the two UIRs of $Sl(2, C)$ concerned (which we shall identify later) are very special indeed. Apart from these features, while γ^0 is indefinite, the hermitian Γ^0 is positive definite – this was the key to the avoidance of negative energy solutions.

However these equations of Majorana ran into several problems:

1. the positron was soon discovered;
2. each equation describes a 'tower' of particles with steadily increasing spin but decreasing mass, which is unphysical, namely the mass spin relation was of the form
$$m(j) = \frac{\kappa}{j + 1/2}; \tag{3}$$
3. much later Valentine Bargmann[7] pointed out that the equation (2) possesses solutions for space-like energy momenta (tachyons), and in this way negative energies reappear.

For all these reasons, not much interest in Majorana's work remained, until Yoichiro Nambu in the late 1960s revived for a while the whole subject of infinite component relativistic wave equations in the context of the strong interactions of elementary particles[8]. But what remains remarkable is that as early as 1932 Majorana had constructed two (albeit rather special) UIRs of $Sl(2, C)$.

For many years after Majorana's work, there was no systematic study of the URs and UIRs of $SO(3, 1)$ and $Sl(2, C)$. It was in 1945 that Dirac picked up this problem again, and constructed a certain class of URs of the Lorentz group, hoping that they might be used in physical problems[9]. According to Dirac, his ideas were inspired by the algebraic treatment of the harmonic oscillator problem in quantum mechanics due to Vladimir Fock. In his article on Harish-Chandra[10], Langlands quotes these sentences from the Dirac paper: 'The Lorentz group is the group of linear transformations of four real variables $\xi_0, \xi_1, \xi_2, \xi_3$ such that $\xi_0^2-\xi_1^2-\xi_2^2-\xi_3^2$ is invariant. The finite representations of the group ... are all well known and are dealt with by the usual tensor analysis and its extension spinor analysis. None of them is unitary. The group has also some infinite representations which are unitary. These do not seem to have been studied much, in spite of their possible importance for physical applications', and remarks that 'This is as close as one comes to the source of the theory of infinite dimensional representations of semisimple and reductive groups...'.

A somewhat more detailed and technical account of Dirac's work (followed by Harish-Chandra's) was presented elsewhere[11]. Here we limit ourselves to a qualitative outline. In essence Dirac's idea – something uniquely his own – was to start from the finite dimensional nonunitary symmetric tensor representations (j, j) of $SO(3, 1)$, well known from tensor analysis, and *analytically continue* the *rank* of the tensor to *complex* values until he arrived at infinite dimensional *unitary* representations. This is in a sense like 'lifting the lid' off the sequence of spin values 0, 1, 2, ..., $2j$ present in (j, j), and letting it run on to infinity. The key step was to consider formal infinite series expressions in the components of a four-vector, involving positive powers of the space components but negative powers of the time component; and to regard the coefficients of the terms in such expressions as the components of a new kind of unitarily transforming object with respect to $SO(3, 1)$.

The URs constructed by Dirac contained integral spins alone, hence they were representations of $SO(3, 1)$. They were also highly reducible. That is to say, Dirac's work was very far indeed from determining UIRs of $SO(3, 1)$. He gave the name 'expansors' to infinite component fields on space-time belonging to his URs of $SO(3, 1)$. He showed that they could be realized in a rather simple fashion as follows. Consider complex-valued square integrable functions $\phi(x)$ on a fictitious 'space-time' for which the Hilbert space norm $\|\phi\|$ and the action of Lorentz transformations Λ are given by

$$\|\phi\|^2 = \int d^4x |\phi(x)|^2 < \infty,$$

$$\Lambda \in SO(3,1): \phi(x) \to \phi'(x) = \phi(\Lambda^{-1}x). \qquad (4)$$

That one has a representation here, and that it is unitary, are both trivial observations. (Realize that the 'wave function' ϕ here is a scalar, i.e. it is a single-component object, and it is *not* subject to any equation of motion or wave equation.) Dirac showed that his expansor UR, in the simplest case, was just the UR(4), described in a particular way by introducing a Fock-like basis for harmonic oscillators with respect to the four variables x^μ. In spite of this 'triviality' of the expansor UR, Dirac's work was important in that he directed attention to a neglected problem, and took the first steps towards its solution.

At this point the story shifts to India. Harish-Chandra completed his M Sc in physics at Allahabad University during 1941–43. Here he grew very close to Professor K. S. Krishnan, and obtained much encouragement from him. He also studied Dirac's 'Principles of Quantum Mechanics' and as he later said this '... prompted in me a strong desire to devote my life to theoretical physics'! After the M Sc, at Krishnan's urging, he moved to the Indian Institute of Science at Bangalore, with the idea of working in the physics school established by C. V. Raman, and more particularly with Homi J. Bhabha.

Bhabha had been a student of Dirac's in Cambridge in the thirties, (though he obtained his Ph D under R. H. Fowler) and after some time spent at various centres in Europe he came back to India in 1939 for a holiday. However, due to the outbreak of war, he could not return to Europe and so decided to stay on in India. In 1940 Raman offered him a special Readership in Theoretical Physics at the Indian Institute of Science. Bhabha worked at the Institute up to 1945, then moved on to Bombay where he set up the Tata Institute of Fundamental Research and then the Atomic Energy Establishment at Trombay. A not too well-known fact is that the Tata Institute of Fundamental Research was actually 'born' within the Indian Institute of Science at Bangalore, 'lived' there for about six months, and was then 'carried' to Bombay!

When Harish-Chandra first came to Bangalore he stayed with the Kale family on the Institute campus. He had known them at Allahabad, from where Kale, a botanist, had moved to the Institute as its Librarian. Mrs Kale, who was Polish, taught foreign languages to the Institute students, and their daughter Lalitha later became Harish-Chandra's wife. Even now each year the Students' Gymkhana at the Institute organizes a Kale Memorial Table Tennis Tournament, and one cannot help thinking that if the connection to Harish-Chandra were better known, more mathematics students and faculty might participate and feel the better for it!

Harish-Chandra's work with Bhabha at Bangalore was largely inspired by some ideas of Dirac[12] in his paper 'Classical Theory of Radiating Electrons' published in 1938. Here Dirac had succeeded in obtaining an equation of motion for the classical structureless point electron based only on relativistic covariance and *the fundamental conservation laws* of energy and momentum. In this context he introduced the rather novel 'half advanced plus half retarded' solutions of Maxwell's equations, and reproduced the radiation reaction terms of classical electrodynamics. His aim was to try to put the classical theory on as sound a basis as possible, before going on to tackle the divergences of the quantum theory. On the way to deriving the so-called Dirac–Lorentz equation, he even brought in the concept of mass-renormalization, already at the classical level. In their work, Harish-Chandra and Bhabha extended Dirac's results by considering also the effects of nontrivial internal structure for the classical relativistic particle, and the consequences of conservation of relativistic angular momentum[13]. Dirac's ideas were pursued not only at Bangalore but also by Wheeler and Feynman at Princeton; but ultimately they could not be pushed as far as Dirac might have hoped.

Around 1945 both Bhabha and K. S. Krishnan recommended Harish-Chandra to Dirac for further work at Cambridge. About this time Dirac had just completed his theory of expansors recounted earlier, so it was quite natural for him to suggest that Harish-Chandra examine the general problem of constructing infinite irreducible representations of the Lorentz group, determine which ones were unitary, and so on. In particular, Dirac asked Harish-Chandra to find the half integral spin analogue to expansors.

In the period he spent at Cambridge, up to about 1947, Harish-Chandra succeeded in doing all this and more[14]. He first used Lie algebraic methods to show that any IR of the group $Sl(2, C)$ is determined by a pair $\{j_0, \mu\}$, where j_0 is the lowest 'spin' (smallest UIR of $SU(2)$) present, and μ is a complex number. (The notations are slightly different from those in Harish-Chandra's paper.) He originally assumed (and subsequently proved) that the 'spin spectrum' in the $IR\{j_0, \mu\}$ consists of the sequence $j_0, j_0+1, j_0+2, \ldots$ in a multiplicity-free manner; namely each UIR of $SU(2)$ that is present occurs just once. The situations when the $\mathrm{IR}\{j_0, \mu\}$ collapses to a finite dimensional one, when it is unitary (and so necessarily infinite dimensional), and when nonunitary infinite dimensional, were all carefully traced. In all this Harish-Chandra admirably exploited both the basic commutation relations among the generators of $Sl(2, C)$, and the existence of two independent Casimir invariants which reduce to pure numbers in any IR. The UIRs were shown by Harish-Chandra to come in two families, which were later named the Principal Series and the Supplementary or Exceptional Series respectively[1]:

1. Principal Series UIRs of $Sl(2, C)$: $\{j_0, \mu\}$ where $j_0 = 0, 1/2, 1, \ \ , \mu$ real (≥ 0 if $j_0 = 0$);
2. Supplementary Series UIRs of $Sl(2, C)$: $\{0, \mu\}$, where $\mathrm{Re}\,\mu = 0$, $0 < \mathrm{Im}\,\mu < 1$.

In this notation the two UIRs discovered by Majorana in 1932 are $\{1/2, 0\}$ from the Principal Series and $\{0, i/2\}$ from the Supplementary Series.

Following this infinitesimal analysis of the IRs and UIRs of $Sl(2, C)$, Harish-Chandra proceeded to construct 'spinor operators' with respect to $Sl(2, C)$: these are analogues of two-component spinor operators at the $SU(2)$ level which can couple any spin j to the two 'neighbouring' spin values $j \pm 1/2$. In the $Sl(2, C)$ case there are two kinds of spinor operators to consider, belonging (like Weyl fields) to one of the two IRs (1/2, 0) and (0, 1/2). And such operators can couple a general IR $\{j_0, \mu\}$ to the four 'neighbouring' IRs $\{j_0 \pm 1/2, \mu \pm i/2\}$. In this part of his paper, Harish-Chandra displays considerable skill and ingenuity in handling the algebra – one must remember that the so-called Racah–Wigner methods for angular momentum in quantum mechanics were not yet so well-known and widely used as they are today, and here was Harish-Chandra essentially extending them (for the case of spinor operators) to the Lorentz group!

Finally Harish-Chandra turned to the problem posed by Dirac of finding half integral spin analogues to expansors. Here he succeeded in achieving several things. He introduced function spaces made up of functions of components of two-component spinors of types (1/2, 0), (0, 1/2), having definite complex degrees of homogeneity with respect to each kind of spinor, and carrying general IRs of $Sl(2, C)$. The concept of complex degrees of homogeneity was similar in spirit to Dirac's analytic continuation of the rank of a tensor; and the action of $Sl(2, C)$ was given directly for finite elements of the group and not for infinitesimal transformations alone. Harish-Chandra called elements of these function spaces 'expinors' – unlike Dirac's expansors which are based on four-vectors and which provide highly reducible representations and so in a sense are 'very bulky', expinors being based on spinors are much more 'primitive' and 'lean'; so they immediately lead to irreducible representations. In the unitary cases, he also constructed the appropriate conserved Hilbert space inner product.

All in all, Harish-Chandra gave a complete and thorough answer to the problems suggested to him by Dirac. (In their respective papers both Dirac and Harish-Chandra attempted to use the representations constructed by them to give new relativistic wave equations, and it is interesting that on one aspect of the physical interpretation Harish-Chandra clearly expresses his disagreement with Dirac.) Around the same time, essentially the same work was done both by Gelfand and

Naimark in the (erstwhile) Soviet Union[15], and by V. Bargmann at Princeton, but the latter's results remained unpublished. However, Bargmann's work on the UIRs of the three dimensional Lorentz group $SO(2, 1)$ was published and remains a classic – it is here that the famous discrete series unitary representations first made their appearance[16].

Soon after completing this work at Cambridge, Harish-Chandra went to Princeton, and then wrote to Dirac to say that he was concerned at the lack of rigour in his work. To this Dirac responded: 'I am not interested in proofs but only in what nature does'.

Throughout his life Harish-Chandra retained a great sense of 'awe and reverence' for Dirac. Later these were also 'mixed with a feeling of gratitude and affection'. For a volume in honour of Dirac's eightieth birthday Harish-Chandra wrote: 'I have often pondered over the roles of knowledge or experience, on the one hand, and imagination or intuition, on the other, in the process of discovery. I believe that there is a certain fundamental conflict between the two and knowledge, by advocating caution, tends to inhibit the flight of imagination. Therefore a certain naivete, unburdened by conventional wisdom, can sometimes be a positive asset. I regard Dirac's discovery of the relativistic equation of the electron as a shining example of such a case.'

As is well known, Dirac was a master craftsman in the art of theoretical physics, creating mathematical tools appropriate for his needs with ease and elegance. Langland's words on Harish-Chandra[10] – '... by and large it is not too much of an exaggeration to say that he manufactured his own tools as the need arose, and that one of the grand mathematical theories of this century has been constructed with the skills with which one leaves a course in advanced calculus' – show that in this respect master and pupil were alike.

Subsequent developments in physics have made rather limited use of the UIRs of the Lorentz group, and their significance has been greater as the origin of a great and beautiful chapter in modern mathematics. To Dirac and Harish-Chandra will always go the credit for having jointly initiated this development.

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