Time-delay interferometry and the relativistic treatment of LISA optical links

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Abstract.

In order to attain the requisite sensitivity for LISA - a joint space mission of the ESA and NASA- the laser frequency noise must be suppressed below the secondary noises such as the optical path noise, acceleration noise etc. By combining six appropriately time-delayed data streams containing fractional Doppler shifts - a technique called time delay interferometry (TDI) - the laser frequency noise may be adequately suppressed. Here we investigate the problem of TDI in the general case of unequal up-down links and also include the effect of the Earth on the spacecraft and the optical links. We show that there are symmetries in the physics which can be successfully used to simplify the algebra of the TDI. We finally give the example of the first generation modified Sagnac observable in which the laser frequency noise is suppressed because of the symmetries.

1. Introduction

LISA - Laser Interferometric Space Antenna - is a proposed mission which will use coherent laser beams exchanged between three identical spacecraft forming a giant (almost) equilateral triangle of side 5×10^6 kilometres to observe and detect low frequency cosmic GW [1].

Laser frequency noise dominates the other secondary noises, such as optical path noise, acceleration noise by 7 or 8 orders of magnitude, and must be removed if LISA is to achieve the required sensitivity of $h \sim 10^{-22}$, where h is the metric perturbation caused by a gravitational wave. In LISA, six data streams arise from the exchange of laser beams between the three spacecraft approximately 5 million km apart. These six streams produce redundancy in the data which can be used to suppress the laser frequency noise by the technique called time-delay interferometry (TDI) in which the six data streams are combined with appropriate time-delays [2]. A mathematical foundation for the TDI problem for LISA was given in [3], by showing that the data combinations constituted an algebraic structure; the data combinations cancelling laser frequency noise formed the *module of syzygies* over the polynomial ring of time-delay operators.

We extend the work of [3] in which we had a commutative polynomial ring over the three time-delay operators. Because in the general case, the up-down links are unequal and because of the flexing of the arms, we have a non-commutative ring in six indeterminates. We show in this paper that the algebraic structure is still a module over this non-commutative polynomial ring. Secondly, for the LISA model that we consider, which has been optimised to reduce flexing, we have found symmetries which could be used to simplify the algebra. We discuss these symmetries and their implications; the symmetries can be used to construct a smaller ring - the original non-commutative ring is quotiented by the ideal generated by commutators which capture the symmetries. For the model of LISA we use the orbits given in [4] in which the orbits have been computed in the Sun and Earth's field (we have ignored the effects of Jupiter and other planets). We show that optimising the spacecraft orbits for minimal flexing is conducive to suppressing the laser frequency noise. For some TDI observables their modified first generation form may prove adequate; in this context we discuss the Sagnac observable. These computations will be useful in the development of a LISA simulators, the LISACode for instance [5].

2. The flexing of LISA's arms in the Sun's and Earth's field

The orbits of the spacecraft are chosen as follows [4]: first, we choose a set of Keplerian orbits in the Sun's field only which give a minimum flexing of the arms - the peak to peak variation in armlengths is the least ~ 48000 km, see [6]. In the second step, on these base orbits we linearly add the perturbations due to the Earth's gravitational field. This choice is so, because we believe, that the optimum solution which yields the minimum flexing of the arms in the combined field of the Sun and Earth, lies close to the optimum solution already found in the field of the Sun only. We briefly describe the orbits below: we choose the Sun as the origin with Cartesian coordinates $\{X, Y, Z\}$ as follows: the ecliptic plane is the $X - Y$ plane and we consider a circular reference orbit of radius $R = 1$ A. U. centred at the Sun. Let $\delta_0 = 5\alpha/8$ where $\alpha = L_0/2R$ and $L_0 \sim 5,000,000$ km represents the nominal distance between two spacecraft of the LISA configuration. We choose the tilt of the plane of the LISA triangle to be $\delta = \pi/3 + \delta_0$ which has been shown in [6] to yield minimum flexing of the arms. We choose spacecraft 1 to be at its lowest point (maximum negative Z) at $t = 0$. This means that at this point, $Y = 0$ and $X \simeq R(1 - e)$. The orbit of spacecraft 1 is an ellipse with inclination angle $\sim \alpha$ with the ecliptic, eccentricity $e \sim \alpha/\sqrt{3}$. This fixes the orbit of spacecraft 1. The orbits of of spacecraft 2 and 3 are obtained by rotating the orbit of spacecraft 1 about the Z-axis by the angles 120° and 240° respectively, their phases are adjusted so that they maintain a distance of $\sim L_0$ between each other. To the first order in the eccentricity, the spacecraft form a rigid equilateral triangle of side L_0. It is only at the second order in the α (or eccentricity) that the flexing of the arms occurs.

The Earth's field is now included perturbatively via the CW framework [7]. The CW frame is chosen as follows: We take the reference particle to be orbiting in a circle of radius R with constant Keplerian angular velocity Ω. Then the transformation to the CW frame $\{x, y, z\}$ from the barycentric frame $\{X, Y, Z\}$ is given by,

$$x = X \cos \Omega t + Y \sin \Omega t - R, \; y = -X \cos \Omega t + Y \sin \Omega t, \; z = Z. \qquad (1)$$

Consistent with the Keplerian orbits described above and to the first order in eccentricity, for the kth spacecraft, we have the following position coordinates in the CW frame:

$$x_k = -\frac{1}{2}\rho_0 \cos\phi_k, \ y_k = \rho_0 \sin\phi_k, \ z_k = -\frac{\sqrt{3}}{2}\rho_0 \cos\phi_k, \tag{2}$$

where $\rho_0 = L_0/\sqrt{3}$ and $\phi_k = \Omega t - 2\pi(k-1)/3 - \phi_0$.

LISA follows the Earth $20°$ behind. We consider the model where the centre of the Earth leads the origin of the CW frame by $20°$ - thus in our model, the 'Earth' or the centre of force representing the Earth, follows the circular reference orbit of radius 1 A. U. Also the Earth is at a fixed position vector $\mathbf{r}_\oplus = (x_\oplus, y_\oplus, z_\oplus)$ in the CW frame. We find that $x_\oplus = -R(1 - \cos 20°) \sim -9 \times 10^6$ km, $y_\oplus = R \sin 20° \sim 5.13 \times 10^7$ km and $z_\oplus = 0$. We introduce the small parameter ϵ in terms of the quantity $\omega_\oplus^2 = GM_\oplus/d_\oplus^3$, where $d_\oplus$ is the distance of the Earth from the origin of the CW frame, $M_\oplus \sim 5.97 \times 10^{24}$ kg is the mass of the Earth and $G = 6.67 \times 10^{-11}$ kg^{-1}m^{3}sec^{-2} Newton's gravitational constant. We find $d_\oplus \sim 5.2 \times 10^7$ km. We define $\epsilon = \omega_\oplus^2/\Omega^2 \simeq 7.16 \times 10^{-5}$ which is just the ratio of the tidal forces due to the Earth and the Sun. The CW equations including the Earth's field take the form:

$$\begin{aligned}
\ddot{x} - 2\Omega\dot{y} - 3\Omega^2 x + \epsilon\Omega^2(x - x_\oplus) &= 0, \\
\ddot{y} + 2\Omega\dot{x} + \epsilon\Omega^2(y - y_\oplus) &= 0, \\
\ddot{z} + \Omega^2(1 + \epsilon)z &= 0.
\end{aligned} \tag{3}$$

Note that the compounded flexing due to the combined field of Earth and Sun is a nonlinear problem; it is infact a three body problem. We however solve this problem approximately. Assuming that both effects are small we may linearly add the flexing vectors due to the Sun and Earth; the nonlinearities appear at higher orders in α and ϵ. These would modify the flexing but we may neglect this effect because of the smallness.

We then seek perturbative solutions to Eq. (3) to the first order in ϵ. We write, $x = x_0 + \epsilon x_1, y = y_0 + \epsilon y_1, z = z_0 + \epsilon z_1$ where x_0, y_0, z_0 are solutions at the zeroth order given by Eq.(2). With the initial conditions: $x_1 = y_1 = z_1 = \dot{x}_1 = \dot{y}_1 = \dot{z}_1 = 0$ at $t = 0$, we solve the equations for x_1, y_1 and z_1. We then add the perturbation $\epsilon\mathbf{r}_1 = \epsilon(x_1, y_1, z_1)$ to the Keplerian orbit of each spacecraft to obtain the orbits in the combined gravitational field of the Sun and Earth [4]. The time-delay required for the TDI needs to be known to at least to 1 part in 10^8 - about few metres - for the laser frequency noise to be suppressed. In order to guarantee such level of accuracy, we numerically integrate the null geodesics followed by the laser ray. Our numerical scheme is accurate to better than 10 metres. In figure 1 we plot the rate of change of armlengths (flexing) for the six optical links. Including the Earth's field the flexing still remains $\lesssim 6$ m/sec in the first two years and increases to $\lesssim 8$ m/sec in the third year. Earlier estimates were ~ 10 m/sec.

3. Algebraic approach to TDI: polynomial equations and the module of syzygies

For stationary LISA in flat spacetime, there are only three delay operators corresponding to the three armlengths and the time-delay operators commute. However, in a more
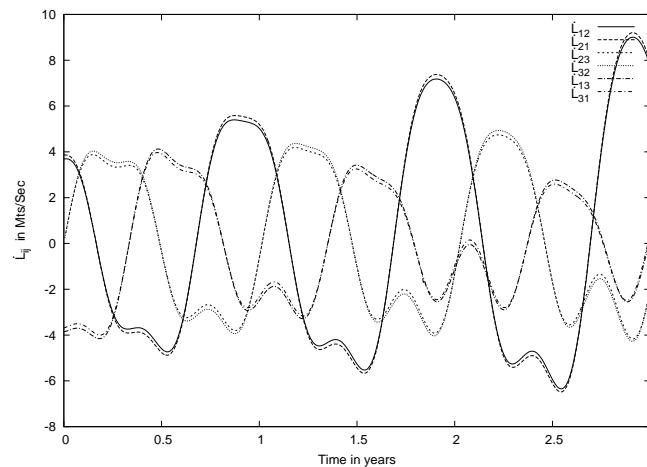
Figure 1. The rate of change of armlengths for the six links is shown in units of m/sec for $\phi_0 = 0$. This rate of change is less than 6 m/sec upto the second year and increases to a maximum of about 8 m/sec in the third year.

realistic LISA model the arms flex and the up-down links are unequal which gives six non-commuting time-delay operators - the second generation TDI.

We follow the notation and conventions of [8] and [3] which are the simplest for our purpose. The six links are denoted by $U^i, V^i, i = 1, 2, 3$. The time-delay operator for the link U^2 from S/C 1 to S/C 2 or $1 \longrightarrow 2$ is denoted by x in [8] and so on in a cyclic fashion. The delay operators in the other sense are denoted by l, m, n; the link $-V^1$ from $2 \longrightarrow 1$ by l and similarly the links V^2, V^3 are defined through cyclic permutation. Figure 2 depicts the optical links as described.
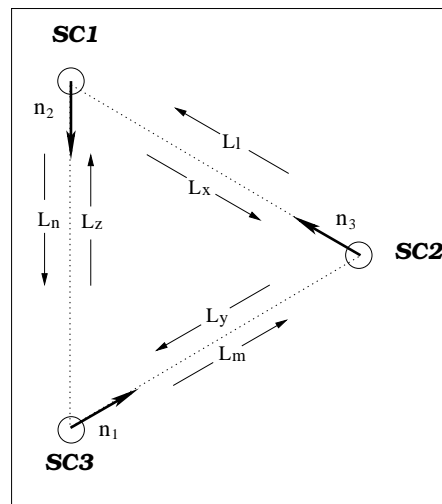

Figure 2. A schematic diagram of LISA's optical links.

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Let $C_i(t) = \Delta\nu_i(t)/\nu_0$ represent the laser frequency noise in S/C i. Let j be the delay operator corresponding to the variable armlength $L_j(t)$, i.e. $jC_i(t) = C_i(t - L_j(t))$. Then we have, $U^1 = C_1 - zC_3$, $V^1 = lC_2 - C_1$. The other links in terms of $C_i(t)$ are obtained by cyclic permutations. Also in the U^i, V^i we have not included contributions from the secondary noises, gravitational wave signal etc. since here our aim is to deal with laser frequency noise only. Any observable X is written as: $X = p_i V^i + q_i U^i$, where $p_i, q_i, i = 1, 2, 3$ are polynomials in the variables x, y, z, l, m, n. Thus X is specified by giving the six tuple polynomial vector (p_i, q_i). Writing out the (V_i, U_i) in terms of the laser noises $C_i(t)$, and in order that the laser frequency noise cancel for arbitrary functions $C_i(t)$, the polynomials (p_i, q_i) must satisfy the equations:

$$
\begin{aligned}
p_1 - q_1 + q_2 x - p_3 n &= 0, \\
p_2 - q_2 + q_3 y - p_1 l &= 0, \\
p_3 - q_3 + q_1 z - p_2 m &= 0.
\end{aligned}
\tag{4}
$$

The solutions to these equations as realised in earlier works are important, because they consist of polynomials with lowest possible degrees and thus are simple [2].

Since these are linear equations they define a homomorphism of modules as follows: Eliminating p_1 and p_2 from the three equations (4) while respecting the order of the variables we get:

$$
\psi(x, y, z, l, m, n) \equiv p_3(1 - nlm) + q_1(z - lm) + q_2(xl - 1)m + q_3(ym - 1) = 0 \,.
\tag{5}
$$

Consider the polynomial ring $\mathcal{Q}(x, y, z, l, m, n) \equiv \mathcal{K}$, in general non-commutative, of polynomials in the six variables x, y, z, l, m, n and coefficients in the rational field $\mathcal{Q}$. The operators x, y, z, l, m, n play the role of indeterminates. Eq. (5) defines a homomorphism $\varphi : \mathcal{K}^4 \longrightarrow \mathcal{K}$ where any polynomial vector $(p_3, q_1, q_2, q_3) \in \mathcal{K}^4$ is mapped to the polynomial $\psi(x, y, z, l, m, n) \in \mathcal{K}$. The set of noise free TDI combinations is just the kernel of this homomorphism $\varphi^{-1}(0) \subset \mathcal{K}^4$ which is a submodule of $\mathcal{K}^4$. This is called the *first module of syzygies* over the polynomial ring $\mathcal{K}$. This homomorphism can be extended to $\mathcal{K}^6$ via the elimination equations for p_1 and p_2. Thus one obtains a module of noisefree TDI observables $\mathcal{M} \subset \mathcal{K}^6$ which is isomorphic to $\varphi^{-1}(0)$.

In case of non-commuting operators the problem is far more complex than the commutative case. If we follow on the lines of the commutative case, the first step would be to find a Gröbner basis for the ideal generated by the coefficients appearing in Eq. (5), namely, the set of polynomials $\{1 - nlm, z - lm, (xl - 1)m, ym - 1\}$. Here we do not even know whether the Gröbner basis for this ideal would be finite - the algorithm for finding the basis may not terminate. However, here we show that simplifications are possible because of the inherent symmetries in the problem and so the ring $\mathcal{K}$ can be quotiented by a certain ideal, simplifying the algebraic problem. One then needs to deal with a 'smaller' ring which is perhaps simpler. We now estimate the level of non-commutativity of these operators in the context of our LISA model.

4. The role of symmetries and the construction of the quotient ring
The level of non-commutativity can be found by computing commutators which occur in several of the well known TDI observables like the Michelson, Sagnac etc. We find

that given our model of LISA, we require to go only upto the first order in $\dot{L}$; we find for our model $\ddot{L} \sim 10^{-6}$ metres/sec^2 and thus even if one considers say 6 successive optical paths, that is, about $\Delta t \sim 100$ seconds of light travel time, $\Delta t^2 \ddot{L} \sim 10^{-2}$ metres. This is well below few metres and thus can be neglected in the residual laser noise computation. Moreover, $\dot{L}^2$ terms (and higher order) can be dropped since they are of the order of $\lesssim 10^{-15}$ (they come with a factor $1/c^2$) which is much smaller than 1 part in 10^8. The calculations which follow neglect these terms.

Applying the operators twice in succession and dropping higher order terms as explained above,

$$\begin{aligned} k_2 k_1 C &= C(t - L_{k_1}(t - L_{k_2}) - L_{k_2}), \\ &\approx C(t - L_{k_1} - L_{k_2}) + L_{k_2}\dot{L}_{k_1}\dot{C}(t - L_{k_1} - L_{k_2}). \end{aligned} \qquad (6)$$

The above formula can be easily generalised by induction to n operators. In several of the TDI observables, commutators play a major role. In the Sagnac observable, for example, the following commutator occurs:

$$\begin{aligned} lmnxyz - xyzlmn &= (L_l + L_m + L_n)(\dot{L}_x + \dot{L}_y + \dot{L}_z) \\ &\quad - (L_x + L_y + L_z)(\dot{L}_l + \dot{L}_m + \dot{L}_n), \end{aligned} \qquad (7)$$

where it is understood that the LHS acts on C while the RHS multiplies $\dot{C}$ at an appropriately delayed time. Note that the term in C cancels out on the RHS. Further, we observe the following approximate symmetries in our model:

$$\dot{L}_x \approx \dot{L}_l, \quad \dot{L}_y \approx \dot{L}_m, \quad \dot{L}_z \approx \dot{L}_n, \qquad (8)$$

In our model, $|\dot{L}_x - \dot{L}_l| \lesssim 0.8$ m/sec and the sum $|(\dot{L}_x + \dot{L}_y + \dot{L}_z) - (\dot{L}_l + \dot{L}_m + \dot{L}_n)| \lesssim 1$ m/sec upto the first three years in our model. Thus the commutators:

$$[x, l] \approx 0, [y, m] \approx 0, [z, n] \approx 0, \quad [xyz, lmn] \approx 0. \qquad (9)$$

The near vanishing of the above commutators implies that a vast simplification in the algebra is possible. In particular the pairs $(x, l), (y, m), (z, n)$ approximately commute.

Apart from the above approximate symmetries there are other 'exact' symmetries (after dropping terms in $\dot{L}^2$ and $\ddot{L}$ and higher order) which lead to vanishing commutators. Consider the commutator $[x_1 x_2 ... x_n, y_1 y_2 ... y_n]$ where $n \geq 2$ and x_k or y_m represents any one of the delay operators x, y, z, l, m, n. Upto the order of approximation we are working in:

$$[x_1 x_2 ... x_n, y_1 y_2 ... y_n] = \sum_{k=1}^{n} L_{x_k} \sum_{m=1}^{n} \dot{L}_{y_m} - \sum_{m=1}^{n} L_{y_m} \sum_{k=1}^{n} \dot{L}_{x_k}. \qquad (10)$$

From this equation it immediately follows that if the operators $y_1, y_2, ..., y_n$ are a permutation of the operators $x_1, x_2, ..., x_n$, and then the commutator,

$$[x_1 x_2 ... x_n, y_1 y_2 ... y_n] = 0. \qquad (11)$$

We describe below how these vanishing commutators can be used to simplify the algebra.

The commutators can be used to construct an ideal $\mathcal{U} \subset \mathcal{K}$. We first construct the ideal $\mathcal{U}$ generated by the commutators such as those given in Eq. (9) and Eq. (11), that is, the hybrid and the exact ones. Then we quotient the ring $\mathcal{K}$ by $\mathcal{U}$, thereby constructing a smaller ring $\mathcal{K}/\mathcal{U} \equiv \bar{\mathcal{K}}$. Further, we can translate the equations given in Eq. (4) to ones over $\bar{\mathcal{K}}$ as follows: Each polynomial $\bar{p} \in \bar{\mathcal{K}}$ is an equivalence class of polynomials $p + \mathcal{U} \subset \mathcal{K}$. The solution set of such polynomial vectors, which we now denote by $(\bar{p}_i, \bar{q}_i) \in \bar{\mathcal{K}}^6$ still form a module over $\bar{\mathcal{K}}$. A future goal would be to construct this module or obtain the generators of this module.

The simplification in the ring $\mathcal{K}$ can be seen as follows: Consider just two of the operators x and l and consider the possible polynomials or monomials (polynomials with just one term) in them in $\mathcal{K}$ and $\bar{\mathcal{K}}$ for different degrees. We have $xl = lx + xl - lx = lx + [x, l] = lx$ in $\bar{\mathcal{K}}$ (we have dropped the overbars to avoid clutter). So we gain by the fact that the number of monomials in $\bar{\mathcal{K}}$ are reduced, we can identify lx with xl as just a single monomial. The higher the degree of the monomials, the greater is the number of identifications and more is the simplification in the algebra.

5. Residual laser frequency noise in the Sagnac observable

By the time LISA flies the expectations are for the laser frequency noise estimate to reduce to say $\widetilde{\Delta\nu} \sim 10\mathrm{Hz}/\sqrt{\mathrm{Hz}}$ [9]. If we divide this number by the laser frequency $\nu_0 \sim 3 \times 10^{14}$ Hz, we obtain the noise estimate in the fractional Doppler shift C with the power spectral density (PSD), $S_C(f) = \langle |\tilde{C}(f)|^2 \rangle \sim 10^{-27}$ Hz^{-1}, where $\tilde{C}(f)$ is the Fourier transform of $C(t)$.

The modified Sagnac first generation TDI observable α is given by the six tuple polynomial vector $\alpha = (\kappa, \kappa l, \kappa lm, \eta, \eta zy, \eta z)$, where $\kappa = 1 - zyx$ and $\eta = 1 - lmn$. The flexing leads to the residual term $\Delta C = [zyx, lmn]C_1 = \Delta t \dot{C}_1$ where by Eq. (7):

$$\Delta t(t) = \frac{1}{c^2}[(L_x + L_y + L_z)(\dot{L}_l + \dot{L}_m + \dot{L}_n) - (L_l + L_m + L_n)(\dot{L}_x + \dot{L}_y + \dot{L}_z)], \qquad (12)$$

Then the PSD of ΔC is $S_{\Delta C}(f; t) = 4\pi^2 \Delta t(t)^2 f^2 S_C(f)$. This noise must be compared with the secondary noise [1] which is given by $S_\alpha(f) = 4\sin^2(3\pi fL_0)\{[8\sin^2 3\pi fL_0 + 16\sin^2 \pi fL_0]S_{acc} + 6S_{opt}\}$, where $S_{acc} = 2.5 \times 10^{-48}(f/1\mathrm{Hz})^{-2}\mathrm{Hz}^{-1}$ and $S_{opt} = 1.8 \times 10^{-37}(f/1\mathrm{Hz})^2\mathrm{Hz}^{-1}$. In the Figure 3 we plot $S_\alpha(f)$ and $S_{\Delta C}(f; t)$ at three epochs an year apart. For more details see [4].

We see that, clearly the residual laser frequency noise is few orders of magnitude below the secondary noises. Analogous result was demonstrated in [10]. Since the other Sagnac variables β and γ are obtained by cyclic permutations of the spacecraft, the residual laser noise is similarly suppressed in them. Note that the only polynomial that occurs in ΔC is the commutator $[zyx, lmn]$ which is in $\mathcal{U}$ and so α satisfies the three equations (4) approximately, therefore is an element of the module we are seeking.

6. Concluding remarks

We have described in this paper the algebraic approach to the problem of TDI in the general case of the LISA model in which the up and down links are unequal and the arms flex. This gives rise to a polynomial ring in six variables which in general do not
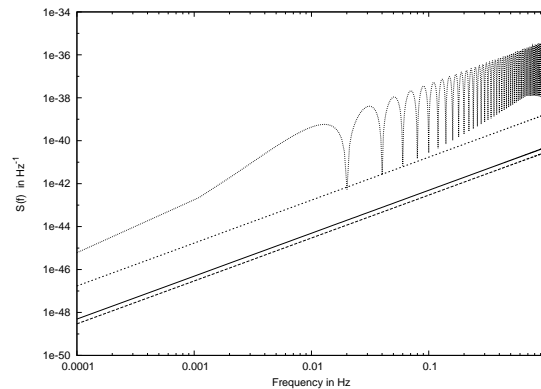
Figure 3. The 'top' curve shows the PSD $S_\alpha(f)$ of the secondary noises. The straight lines are the PSDs of the residual noise at three epochs chosen an year apart. Clearly the residual laser noise is seen to be adequately below the secondary noises.

commute. However, we emphasise that symmetries exist in the physics and show how they could be exploited to simplify the algebra -the symmetries essentially simplify the ring. It may be possible to extend the Gröbner basis methods which worked so well in the commutative case.

We have computed the residual laser frequency noise in one of the important TDI variables, namely, the Sagnac. The residual noise is satisfactorily suppressed because of the optimised orbits and the symmetries. Thus by optimising LISA's orbits it may be possible to suppress the laser frequency noise in other of TDI observables as well [4].

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