NONEXISTENCE OF ALMOST COMPLEX STRUCTURES
ON GRASSMANN MANIFOLDS

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Abstract. In this paper we prove that, for $3 < k < n-3$, none of the oriented Grassmann manifolds, $\tilde{G}_{n,k}$—except for $\tilde{G}_{6,3}$—admits a weakly almost complex structure. The result for $k = 1, 2, n-1, n-2$ are well known and classical. The proofs make use of basic concepts in $K$-theory, the property that $\tilde{G}_{n,k}$ is $(n-k)$-universal, known facts about $K(\mathbb{H}P^k)$, and characteristic classes.

1. Introduction

For $1 < k < n$, let $\tilde{G}_{n,k}$ denote the oriented Grassmann manifold of oriented $k$-dimensional vector subspaces of $\mathbb{R}^n$. $\tilde{G}_{n,k}$ is a smooth manifold of dimension $k(n-k)$. Note that $\tilde{G}_{n,1} \cong S^{n-1}$, the $(n-1)$-sphere, and that $\tilde{G}_{n,k} \cong \tilde{G}_{n,n-k}$ under the diffeomorphism that sends an oriented $k$-plane $V$ to $V^\perp$ together with that orientation on $V^\perp$ which induces the standard orientation on $V \oplus V^\perp = \mathbb{R}^n$.

Recall that a smooth manifold $M$ is said to be (weakly) almost complex if its tangent bundle $TM$ is (stably) isomorphic to a complex vector bundle over $M$. For example, $\tilde{G}_{n,1} \cong S^{n-1}$ is weakly almost complex for all $n$, but is almost complex only when $n = 3$ or $7$. (See [14, p. 217] and [5, 15.1].) It is a classical result that $\tilde{G}_{n,2} \cong SO(n)/(SO(2) \times SO(n-2))$ is a Hermitian symmetric space, and is therefore almost complex for all $n$.

In this paper, we investigate which of the remaining $\tilde{G}_{n,k}$'s are weakly almost complex. Since $\tilde{G}_{n,k} \cong \tilde{G}_{n,n-k}$, we assume, without loss of generality, that $2k \leq n$. Our main result is

1.1. Theorem. Let $3 \leq k \leq n/2$. Then

(i) $\tilde{G}_{n,k}$ is not weakly almost complex if $n$ is odd or if $(n-k) \geq 16$.

(ii) $\tilde{G}_{6,3}$ is weakly almost complex. $\tilde{G}_{6,3} \times \tilde{G}_{6,3}$ is almost complex.
Attempts by the author to settle the remaining cases for which \( n \) is even and \( 3 \leq k \leq (n - k) \leq 15 \), left unanswered in the above theorem, have failed.

Let \( \gamma_{n,k} \) denote the canonical \( k \)-plane bundle over \( \tilde{G}_{n,k} \), and let \( \beta_{n,k} \) be its orthogonal complement, whose fiber over \( V \in \tilde{G}_{n,k} \) is the vector space \( V^\perp \subset \mathbb{R}^n \). One has the bundle equivalence

\[
(1.2) \quad \gamma_{n,k} \oplus \beta_{n,k} \cong n\epsilon,
\]

where \( \epsilon \) denotes a trivial line bundle.

It is well known that the tangent bundle \( \tau\tilde{G}_{n,k} \) of \( \tilde{G}_{n,k} \) has the following description (see [9]):

\[
(1.3) \quad \tau\tilde{G}_{n,k} \cong \gamma_{n,k} \otimes_{\mathbb{R}} \beta_{n,k}.
\]

Using (1.3) or [6], we obtain

\[
(1.4) \quad \tau\tilde{G}_{n,k} \oplus (\gamma_{n,k} \otimes_{\mathbb{R}} \gamma_{n,k}) \cong n\gamma_{n,k}.
\]

For a topological space \( X \), let \( r : KO(X) \to KO(X) \) denote the homomorphism (of Abelian groups) gotten by restriction of scalars to \( \mathbb{R} \), and let \( c : KO(X) \to K(X) \) denote the complexification, \( c[\xi] = [\xi \otimes_{\mathbb{R}} \mathbb{C}] \), which is a ring homomorphism.

One has the following identities:

\[
(1.5) \quad r \circ c(x) = 2x \quad \forall x \in KO(X)
\]

\[
(1.6) \quad c \circ r(y) = y + y' \quad \forall y \in K(X),
\]

where \( y' \) stands for complex conjugation of \( y \).

Note that a smooth manifold \( M \) is weakly almost complex if and only if \( [\tau M] + \delta \) is in the image of \( r \), where \( \delta = 0 \) or \( [\epsilon] \) according to whether \( \dim M \) is even or odd.

2. \( K \)-THEORY OF QUATERNIONIC PROJECTIVE SPACES

Let \( \mathbb{H}P^n \) denote the quaternionic projective \( n \)-space. Let \( \eta_n \) denote the canonical right \( \mathbb{H} \)-vector bundle over \( \mathbb{H}P^n \), and let \( \xi_n \) denote the complex vector bundle obtained from \( \eta_n \) by restriction of scalars to \( \mathbb{C} \). Let \( v = -c_2(\xi_n) \in H^4(\mathbb{H}P^n ; \mathbb{Z}) \subset H^4(\mathbb{H}P^n ; \mathbb{Q}) \). Then \( v \) generates the ring \( H^*(\mathbb{H}P^n ; \mathbb{Q}) \). The Chern character \( ch(\xi_n) \) of \( \xi_n \) is

\[
ch(\xi_n) = \exp(y_1) + \exp(y_2),
\]

where \( (1 + y_1)(1 + y_2) = 1 - v = c(\xi_n) \), the total Chern class of \( \xi_n \). Hence \( y_1 + y_2 = 0 \), and \( y_1 y_2 = -v \). Therefore \( y_1 = -y_2 \), and \( y_1^2 = +v \). Thus

\[
(2.1) \quad ch(\xi_n) = \exp(\sqrt{v}) + \exp(-\sqrt{v}).
\]
2.2. **Proposition.** The Chern character \( \text{ch} : K(\mathbb{H}P^n) \to H^*(\mathbb{H}P^n ; \mathbb{Q}) \) is a monomorphism. The image of the Chern character is freely generated over the integers by \( \{1, w, w^2, \ldots, w^n\} \), where

\[
w = 2 \cosh(\sqrt{v}) - 2 = v + \frac{2v^2}{4!} + \cdots + \frac{2v^n}{(2n)!}.
\]

The proof can be found in [11]. (See also [2, 3.1].)

Note that \( w^{n+1} = 0 \).

Let \( \eta_n^* = \text{Hom}_\mathbb{H}(\eta_n, \mathbb{H}) \) be the dual of \( \eta_n \), which is a left \( \mathbb{H} \)-vector bundle of rank 1. Consider the bundle \( \omega = \eta_n \otimes_\mathbb{H} \eta_n^* \) over \( \mathbb{H}P^n \). Then \( \omega \) is a real vector bundle whose rank is 4. The map \( (q) \mapsto q \otimes f_q \), where \( f_q : q\mathbb{H} = \langle q \rangle \to \mathbb{H} \) is the \( \mathbb{H} \)-linear map defined by \( f_q(q) = 1 \), is a well-defined, continuous, nowhere vanishing section of the bundle \( \omega \). Hence \( \omega \) splits as \( \omega \simeq \varepsilon \oplus \zeta \), where \( \zeta \) is a 3-plane bundle. \( \zeta \) is necessarily orientable, since \( \mathbb{H}P^n \) is simply connected.

2.3. **Lemma.** Let \( n \geq 4 \). Then \( [\zeta \otimes \varepsilon \zeta] + [\varepsilon] \in KO(\mathbb{H}P^n) \) is not in the image of \( r : K(\mathbb{H}P^n) \to KO(\mathbb{H}P^n) \).

**Proof.** Note that the rank of \( \zeta \otimes \varepsilon \zeta \) is 9. Also, \( [\omega \otimes \varepsilon \omega] = [\zeta \otimes \varepsilon \zeta] + [\varepsilon] + 2[\zeta] \), and \( 2[\zeta] \in \text{Im}(r) \) by (1.5). Thus it suffices to show that \( [\omega]^2 \notin \text{Im}(r) \).

Let, if possible, \( [\omega]^2 = r(y) \), for some \( y \in K(\mathbb{H}P^n) \). Then, by (1.6),

\[
c([\omega]^2) = c \circ r(y) = y + y'.
\]

That is,

\[
(2.4) \quad c([\omega]^2) = y + y'.
\]

Now \( c[\omega] = c[\eta_n \otimes_\mathbb{H} \eta_n^*] = [\xi_n \otimes_\mathbb{C} \xi_n^*] \), where \( \xi_n^* \) is the complex bundle obtained from \( \eta_n^* \) by restricting the scalars to \( \mathbb{C} \). The last equality follows from the fact that there exists a functorial isomorphism between \( (V \otimes_\mathbb{H} W) \otimes_\mathbb{R} \mathbb{C} \) and \( V \otimes_\mathbb{C} W \), where \( V \) is a right \( \mathbb{H} \)-vector space, \( W \) is a left \( \mathbb{H} \)-vector space and they are regarded as \( \mathbb{C} \)-vector spaces by restriction of scalars. (See [1, 3.7–3.9].) Taking Chern characters on both sides of (2.4), we get

\[
\text{ch}(y) + \text{ch}(y') = \text{ch}(c[\omega]^2) = (\text{ch}(c[\omega]))^2 = \text{ch}([\xi_n])^2 \text{ch}([\xi_n^*])^2.
\]

Since \( H^j(\mathbb{H}P^n ; \mathbb{Q}) = 0 \) unless \( j \equiv 0 \mod 4 \), the total Chern class \( c(y) \) equals \( c(y') \), and \( c(\xi_n) = c(\xi_n^*) \). Therefore we get

\[
\text{ch}(y) = 1/2(\text{ch}([\xi_n]))^4 = 1/2(2 \cosh(\sqrt{v}))^4
\]

\[
= 1/2w^4 + \text{terms involving lower powers of } w.
\]

This is a contradiction because, by Proposition 2.2, \( \text{ch}(y) \) must be an integral linear combination of powers of \( w \) and \( w^4 \neq 0 \) for \( n \geq 4 \). This proves the lemma.
3. Proof of Theorem 1.1

Let \( 3 \leq k \leq n/2 \). The manifold \( \tilde{G}_{6,3} \) is parallelizable [13]. Therefore, it is weakly almost complex. (Note that \( \dim \tilde{G}_{6,3} \) is odd.) It also follows that \( \tilde{G}_{6,3} \times \tilde{G}_{6,3} \) is almost complex.

Let \( n \) be odd. As \( 3 \leq k \leq n/2 \), we have \( n \geq 7 \). Let \( \pi : \tilde{G}_{n,k} \to G_{n,k} \) be the double covering map onto the Grassmann manifold \( G_{n,k} \) of \( k \)-planes in \( \mathbb{R}^n \). It can be seen from [3, Theorem 1.1] that the Stiefel-Whitney class \( w_3(\tilde{G}_{n,k}) = \pi^*(w_3(G_{n,k})) = w_3(\gamma_{n,k}) \), for \( n \) odd and \( k \geq 3 \). Using the Gysin sequence of the double covering map \( \pi \), and the knowledge of \( H^*(G_{n,k}; \mathbb{Z}_2) \), it can be shown that \( w_3(\gamma_{n,k}) \neq 0 \) for \( n \geq 7 \). Since the odd-dimensional Stiefel-Whitney classes of any weakly almost complex manifold must vanish [12], it follows that \( \tilde{G}_{n,k} \) is not weakly almost complex.

Now let \( n \) be even, \( k > 3 \), and \( n - k > 16 = \dim \mathbb{H}P^4 \). Since \( \tilde{G}_{n,k} \) is \((n - k)\)-universal for orientable \( k \)-plane bundles, there exists a map \( f : \mathbb{H}P^4 \to \tilde{G}_{n,k} \) such that \( f^*(\gamma_{n,k}) = \zeta \oplus m\varepsilon \), where \( m = k - 3 \), and \( \zeta \) is the orientable 3-plane bundle of \( \mathbb{O} \). One has

\[
    f^*(\gamma_{n,k} \otimes \mathbb{R} \gamma_{n,k}) \approx (\zeta \otimes \mathbb{R} \zeta) \oplus 2m\zeta \oplus m^2\varepsilon.
\]

Using (1.4), (1.5), Lemma 2.3, and the fact that \( n \) is even, we see that \( \tilde{G}_{n,k} \) is not weakly almost complex. This completes the proof of Theorem 1.1.

We now turn to the Grassmann manifolds. Denote the unique (up to bundle equivalence) nontrivial line bundle over \( G_{n,k} \) by \( \psi \). Note that the tangent bundle of the real projective space \( \mathbb{R}P^{n-1} \cong G_{n,1} \) is stably isomorphic to the bundle \( n\psi \). Hence \( G_{n,1} \) is weakly almost complex for \( n \) even. For \( n \) odd \( G_{n,1} \) is not orientable and hence it is not weakly almost complex.

3.1. Lemma. Let \( 2 \leq k \leq n/2 \).

(i) (Borel-Hirzebruch [4, p. 526]) For \( n \geq 5 \), \( G_{n,2} \) is not almost complex.

(ii) \( G_{n,k} \) is not weakly almost complex if \( n \) is odd, or if \( k \geq 3 \), and \( n - k \geq 16 \).

(iii) \( G_{4,2} \) and \( G_{6,3} \) are weakly almost complex. \( G_{4,2} \) is not almost complex.

Proof of (ii). For \( n \) odd, \( G_{n,k} \) is not orientable, and is therefore not weakly almost complex. The remaining cases now follow from Theorem 1.1, the naturality of the homomorphism \( r \), and the observation that \( r\tilde{G}_{n,k} \approx \pi^*(rG_{n,k}) \), where \( \pi : \tilde{G}_{n,k} \to G_{n,k} \) is the double covering map.

Proof of (iii). Denote by \( \gamma \) and \( \beta \) the canonical \( k \)- and \((n - k)\)-plane bundles over \( G_{n,k} \). According to Lam [9], a stable normal bundle to \( \tilde{G}_{n,k} \) is \( \lambda^2(\gamma) \oplus \lambda^2(\beta) \). When \( n = 4 \) and \( k = 2 \), \( \lambda^2(\gamma) \approx \lambda^2(\beta) \approx \psi \). Hence the stable normal bundle in this case is in the image of \( r \). It follows that \( G_{4,2} \) is weakly almost complex.
On $G_{6,3}$, $\lambda^2(\gamma) \approx \gamma \otimes_{\mathbb{R}} \psi$, and $\lambda^2(\beta) \approx \beta \otimes_{\mathbb{R}} \psi$ as can be shown using [7, Proposition 10.3, Chapter 12] the Hodge duality $\lambda^{n-1}(\nu) \approx \nu$ for orientable $n$-plane bundles, and $\theta \otimes_{\mathbb{R}} \theta \approx \varepsilon$ for any line bundle $\theta$. Thus, on $G_{6,3}$,

$$\lambda^2(\gamma) \otimes \lambda^2(\beta) \approx (\gamma \otimes_{\mathbb{R}} \psi) \otimes (\beta \otimes_{\mathbb{R}} \psi) \approx (\gamma \otimes \beta) \otimes_{\mathbb{R}} \psi \approx 6 \varepsilon \otimes_{\mathbb{R}} \psi \approx 6 \psi,$$

which is in the image of $r$. It follows that $G_{6,3}$ is weakly almost complex.

To see that $G_{4,2}$ is not almost complex, notice, first, that $H^i(G_{4,2}; \mathbb{Q}) \cong \mathbb{Q}$ for $i = 0, 4$, and is zero for $i \neq 0, 4$. Since $2\psi$ is a stable normal bundle for $G_{4,2}$, we have the following formula for the rational Pontrjagin class $p_1(G_{4,2}) \in H^4(G_{4,2}; \mathbb{Q})$:

$$p_1(G_{4,2}) = -p_1(2\psi) = c_2(2\psi \otimes_{\mathbb{R}} \mathbb{C}) = (c_1(\psi \otimes_{\mathbb{R}} \mathbb{C}))^2 = 0$$

because $H^2(G_{4,2}; \mathbb{Q}) = 0$. On the other hand, if $\tau = \tau G_{4,2}$ were a complex vector bundle, then [12],

$$-p_1(G_{4,2}) = 2c_2(\tau) - (c_1(\tau))^2 = 2c_2(\tau).$$

Since the top Chern class of $\tau$ is its Euler class, we must have $-\frac{1}{2} \langle p_1(G_{4,2}), [G_{4,2}] \rangle = (c_2(\tau), [G_{4,2}] = \chi(G_{4,2}) = 2$, the Euler characteristic of $G_{4,2}$. Hence $p_1(G_{4,2}) \neq 0$, contradicting (3.2). This shows that $G_{4,2}$ is not almost complex.

As a corollary to the above theorem and Theorem 1.1., we obtain the following:

3.3. **Theorem.**

(i) A product of any finite number of oriented Grassmann manifolds, $\tilde{G}_{n,k}$'s, (resp. the Grassmann manifolds $G_{m,p}$'s) is not weakly almost complex if, for one of the factors, $n \geq 7$ is odd, $3 \leq k \leq n - 3$; or $k \geq 3$, $(n - k) \geq 16$; (resp. $m$ is odd; or $p \geq 3$, $(m - p) \geq 16$).

(ii) The oriented flag manifold

$$\tilde{G}(n_1, n_2, \ldots, n_s) = SO(n)/\left(SO(n_1) \times SO(n_2) \times \cdots \times SO(n_s)\right),$$

where $n = n_1 + \cdots + n_s$ is not weakly almost complex if, for some $i \neq j$, any one of the following holds: (a) $n_i + n_j$ is odd, $n_i, n_j \geq 3$; (b) $n_i \geq 3$, $n_j \geq 16$.

(iii) (Korbaš) Let $s \geq 3$. The flag manifold

$$G(n_1, n_2, \ldots, n_s) = O(n)/\left(O(n_1) \times \cdots \times O(n_s)\right)$$

is weakly almost complex if and only if $n_1 = \cdots = n_s = 1$. $M = G(1, \ldots, 1)$ is almost complex if $\dim M = \left(\frac{s}{2}\right)$ is even.

**Proof.** (i), (ii), and parts of (iii) follow from the observation that in each of the cases the manifold in question can be regarded as the total space of a differentiable bundle with fiber, an oriented Grassmann manifold, or a Grassmann manifold which is not weakly almost complex by Theorem 1.1 or 3.1 (cf. [13]).
Korbas proves (iii) by a Stiefel-Whitney class argument. The positive results follow from the fact that $G(1, \ldots, 1)$ is parallelizable. See [15] for details.

3.4. **Remarks.**

(i) M. Markl [10] has observed using his ‘$J$-genus’ that $(\tilde{G}_7, 3)$ is not almost complex. He shows, also, that none of the quaternionic flag manifolds other than $\mathbb{H}G(1, \ldots, 1)$ admits a weakly almost complex structure, using the corresponding negative results of Hsiang and Szczarba [6] for quaternionic Grassmannians.

(ii) J. Korbas [8] has shown that $\tilde{G}(n_1, \ldots, n_s)$, $s \geq 2$, $n_1 \equiv \cdots \equiv n_{s-1} \equiv 0 \mod 4$, $n_2 \equiv 1 \mod 2$, is not almost complex. For $n_2 \geq 3$, this is weaker than the result of Theorem 3.3(ii), whereas our theorem does not cover completely the case $n_3 = 1$.

**References**


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