PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 113, Number 1, September 1991

NONEXISTENCE OF ALMOST COMPLEX STRUCTURES ON GRASSMANN MANIFOLDS

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(Communicated by Frederick R. Cohen)

ABSTRACT. In this paper we prove that, for $3 \le k \le n-3$, none of the oriented Grassmann manifolds, $\widetilde{G}_{n,k}$—except for $\widetilde{G}_{6,3}$, and a few as yet undecided cases—admits a weakly almost complex structure. The result for $k = 1, 2, n-1, n-2$ are well known and classical. The proofs make use of basic concepts in K-theory, the property that $\widetilde{G}_{n,k}$ is $(n-k)$-universal, known facts about $K(\mathbb{H}P^4)$, and characteristic classes.

1. INTRODUCTION

For $1 \le k < n$, let $\widetilde{G}_{n,k}$ denote the oriented Grassmann manifold of oriented k-dimensional vector subspaces of $\mathbb{R}^n$. $\widetilde{G}_{n,k}$ is a smooth manifold of dimension $k(n-k)$. Note that $\widetilde{G}_{n,1} \cong S^{n-1}$, the $(n-1)$-sphere, and that $\widetilde{G}_{n,k} \cong \widetilde{G}_{n,n-k}$ under the diffeomorphism that sends an oriented k-plane V to $V^\perp$ together with that orientation on $V^\perp$ which induces the standard orientation on $V \oplus V^\perp = \mathbb{R}^n$.

Recall that a smooth manifold M is said to be (weakly) almost complex if its tangent bundle τM is (stably) isomorphic to a complex vector bundle over M. For example, $\widetilde{G}_{n,1} \cong S^{n-1}$ is weakly almost complex for all n, but is almost complex only when $n = 3$ or 7. (See [14, p. 217] and [5, 15.1].) It is a classical result that $\widetilde{G}_{n,2} \cong SO(n)/(SO(2) \times SO(n-2))$ is a Hermitian symmetric space, and is therefore almost complex for all n.

In this paper, we investigate which of the remaining $\widetilde{G}_{n,k}$'s are weakly almost complex. Since $\widetilde{G}_{n,k} \cong \widetilde{G}_{n,n-k}$, we assume, without loss of generality, that $2k \le n$. Our main result is

1.1. **Theorem.** *Let* $3 \le k \le n/2$. *Then*

(i) $\widetilde{G}_{n,k}$ *is not weakly almost complex if* n *is odd or if* $(n-k) \ge 16$.

(ii) $\widetilde{G}_{6,3}$ *is weakly almost complex.* $\widetilde{G}_{6,3} \times \widetilde{G}_{6,3}$ *is almost complex.*

Received by the editors January 8, 1990.
1980 *Mathematics Subject Classification* (1985 *Revision*). Primary 57R15.
This work was done when the author was at the Institute of Mathematical Sciences, C.I.T. Campus, Madras.

Attempts by the author to settle the remaining cases for which n is even and $3 \leq k \leq (n-k) \leq 15$, left unanswered in the above theorem, have failed.

Let $\gamma_{n,k}$ denote the canonical k-plane bundle over $\widetilde{G}_{n,k}$, and let $\beta_{n,k}$ be its orthogonal complement, whose fiber over a $V \in \widetilde{G}_{n,k}$ is the vector space $V^{\perp} \subset \mathbb{R}^n$. One has the bundle equivalence

$$(1.2) \qquad \gamma_{n,k} \oplus \beta_{n,k} \approx n\varepsilon,$$

where ε denotes a trivial line bundle.

It is well known that the tangent bundle $\tau\widetilde{G}_{n,k}$ of $\widetilde{G}_{n,k}$ has the following description (see [9]):

$$(1.3) \qquad \tau\widetilde{G}_{n,k} \approx \gamma_{n,k} \otimes_{\mathbb{R}} \beta_{n,k}.$$

Using (1.3) or [6], we obtain

$$(1.4) \qquad \tau\widetilde{G}_{n,k} \oplus (\gamma_{n,k} \otimes_{\mathbb{R}} \gamma_{n,k}) \approx n\gamma_{n,k}.$$

For a topological space X, let $r : K(X) \to KO(X)$ denote the homomorphism (of Abelian groups) gotten by restriction of scalars to $\mathbb{R}$, and let $c : KO(X) \to K(X)$ denote the complexification, $c[\xi] = [\xi \otimes_{\mathbb{R}} \mathbb{C}]$, which is a ring homomorphism.

One has the following identities:

$$(1.5) \qquad r \circ c(x) = 2x \qquad \forall x \in KO(X)$$

$$(1.6) \qquad c \circ r(y) = y + y' \qquad \forall y \in K(X),$$

where y' stands for complex conjugation of y.

Note that a smooth manifold M is weakly almost complex if and only if $[\tau M] + \delta$ is in the image of r, where $\delta = 0$ or $[\varepsilon]$ according to whether $\dim M$ is even or odd.

2. K-theory of quaternionic projective spaces

Let $\mathbb{H}P^n$ denote the quaternionic projective n-space. Let η_n denote the canonical right $\mathbb{H}$-vector bundle over $\mathbb{H}P^n$, and let ξ_n denote the complex vector bundle obtained from η_n by restriction of scalars to $\mathbb{C}$. Let $v = -c_2(\xi_n) \in H^4(\mathbb{H}P^n; \mathbb{Z}) \subset H^4(\mathbb{H}P^n; \mathbb{Q})$. Then v generates the ring $H^*(\mathbb{H}P^n; \mathbb{Q})$. The Chern character $\mathrm{ch}(\xi_n)$ of ξ_n is

$$\mathrm{ch}(\xi_n) = \exp(y_1) + \exp(y_2),$$

where $(1+y_1)(1+y_2) = 1 - v = c(\xi_n)$, the total Chern class of ξ_n. Hence $y_1 + y_2 = 0$, and $y_1 y_2 = -v$. Therefore $y_1 = -y_2$, and $y_1^2 = +v$. Thus

$$(2.1) \qquad \mathrm{ch}(\xi_n) = \exp(\sqrt{v}) + \exp(-\sqrt{v}).$$

2.2. **Proposition.** *The Chern character* $\mathrm{ch}: K(\mathbb{H}P^n) \to H^*(\mathbb{H}P^n; \mathbb{Q})$ *is a monomorphism. The image of the Chern character is freely generated over the integers by* $\{1, w, w^2, \dots, w^n\}$, *where*

$$w = 2\cosh(\sqrt{v}) - 2 = v + \frac{2v^2}{4!} + \cdots + \frac{2v^n}{(2n!)}.$$

The proof can be found in [11]. (See also [2, 3.1].)

Note that $w^{n+1} = 0$.

Let $\eta_n^* = \mathrm{Hom}_{\mathbb{H}}(\eta_n, \mathbb{H})$ be the dual of η_n, which is a left $\mathbb{H}$-vector bundle of rank 1. Consider the bundle $\omega = \eta_n \otimes_{\mathbb{H}} \eta_n^*$ over $\mathbb{H}P^n$. Then ω is a real vector bundle whose rank is 4. The map $\langle q \rangle \mapsto q \otimes f_q$, where $f_q : q\mathbb{H} = \langle q \rangle \to \mathbb{H}$ is the $\mathbb{H}$-linear map defined by $f_q(q) = 1$, is a well-defined, continuous, nowhere vanishing section of the bundle ω. Hence ω splits as $\omega \approx \varepsilon \oplus \zeta$, where ζ is a 3-plane bundle. ζ is necessarily orientable, since $\mathbb{H}P^n$ is simply connected.

2.3. **Lemma.** *Let* $n \geq 4$. *Then* $[\zeta \otimes_{\mathbb{R}} \zeta] + [\varepsilon] \in KO(\mathbb{H}P^n)$ *is not in the image of* $r : K(\mathbb{H}P^n) \to KO(\mathbb{H}P^n)$.

Proof. Note that the rank of $\zeta \otimes_{\mathbb{R}} \zeta$ is 9. Also, $[\omega \otimes_{\mathbb{R}} \omega] = [\zeta \otimes_{\mathbb{R}} \zeta] + [\varepsilon] + 2[\zeta]$, and $2[\zeta] \in \mathrm{Im}(r)$ by (1.5). Thus it suffices to show that $[\omega]^2 \notin \mathrm{Im}(r)$.

Let, if possible, $[\omega]^2 = r(y)$, for some $y \in K(\mathbb{H}P^n)$. Then, by (1.6),

$$c([\omega]^2) = c \circ r(y) = y + y'.$$

That is,

$$c([\omega]^2) = y + y'. \tag{2.4}$$

Now $c[\omega] = c[\eta_n \otimes_{\mathbb{H}} \eta_n^*] = [\xi_n \otimes_{\mathbb{C}} \xi_n^*]$, where ξ_n^* is the complex bundle obtained from η_n^* by restricting the scalars to $\mathbb{C}$. The last equality follows from the fact that there exists a functorial isomorphism between $(V \otimes_{\mathbb{H}} W) \otimes_{\mathbb{R}} \mathbb{C}$ and $V \otimes_{\mathbb{C}} W$, where V is a right $\mathbb{H}$-vector space, W is a left $\mathbb{H}$-vector space and they are regarded as $\mathbb{C}$-vector spaces by restriction of scalars. (See [1, 3.7–3.9].) Taking Chern characters on both sides of (2.4), we get

$$\mathrm{ch}(y) + \mathrm{ch}(y') = \mathrm{ch}(c[\omega]^2) = (\mathrm{ch}(c[\omega]))^2 = \mathrm{ch}([\xi_n])^2\,\mathrm{ch}([\xi_n^*])^2.$$

Since $H^j(\mathbb{H}P^n; \mathbb{Q}) = 0$ unless $j \equiv 0 \mod 4$, the total Chern class $c(y)$ equals $c(y')$, and $c(\xi_n) = c(\xi_n^*)$. Therefore we get

$$\begin{aligned} \mathrm{ch}(y) &= 1/2(\mathrm{ch}([\xi_n]))^4 = 1/2(2\cosh(\sqrt{v}))^4 \\ &= 1/2w^4 + \text{terms involving lower powers of } w. \end{aligned}$$

This is a contradiction because, by Proposition 2.2, $\mathrm{ch}(y)$ must be an *integral* linear combination of powers of w and $w^4 \neq 0$ for $n \geq 4$. This proves the lemma.

3. Proof of Theorem 1.1

Let $3 \le k \le n/2$.

The manifold $\widetilde{G}_{6,3}$ is parallelizable [13]. Therefore, it is weakly almost complex. (Note that $\dim \widetilde{G}_{6,3}$ is odd.) It also follows that $\widetilde{G}_{6,3} \times \widetilde{G}_{6,3}$ is almost complex.

Let n be odd. As $3 \le k \le n/2$, we have $n \ge 7$. Let $\pi : \widetilde{G}_{n,k} \to G_{n,k}$ be the double covering map onto the Grassmann manifold $G_{n,k}$ of k-planes in $\mathbb{R}^n$. It can be seen from [3, Theorem 1.1] that the Stiefel-Whitney class $w_3(\widetilde{G}_{n,k}) = \pi^*(w_3(G_{n,k})) = w_3(\gamma_{n,k})$, for n odd and $k \ge 3$. Using the Gysin sequence of the double covering map π, and the knowledge of $H^*(G_{n,k}; \mathbb{Z}_2)$, it can be shown that $w_3(\gamma_{n,k}) \ne 0$ for $n \ge 7$. Since the odd-dimensional Stiefel-Whitney classes of any weakly almost complex manifold must vanish [12], it follows that $\widetilde{G}_{n,k}$ is not weakly almost complex.

Now let n be even, $k \ge 3$, and $n - k \ge 16 = \dim \mathbb{H}P^4$. Since $\widetilde{G}_{n,k}$ is $(n-k)$-universal for orientable k-plane bundles, there exists a map $f : \mathbb{H}P^4 \to \widetilde{G}_{n,k}$ such that $f^*(\gamma_{n,k}) = \zeta \oplus m\varepsilon$, where $m = k - 3$, and ζ is the orientable 3-plane bundle of §2. One has

$$f^*(\gamma_{n,k} \otimes_{\mathbb{R}} \gamma_{n,k}) \approx (\zeta \otimes_{\mathbb{R}} \zeta) \oplus 2m\zeta \oplus m^2\varepsilon.$$

Using (1.4), (1.5), Lemma 2.3, and the fact that n is even, we see that $\widetilde{G}_{n,k}$ is not weakly almost complex. This completes the proof of Theorem 1.1.

We now turn to the Grassmann manifolds. Denote the unique (up to bundle equivalence) nontrivial line bundle over $G_{n,k}$ by ψ. Note that the tangent bundle of the real projective space $\mathbb{R}P^{n-1} \cong G_{n,1}$ is stably isomorphic to the bundle $n\psi$. Hence $G_{n,1}$ is weakly almost complex for n even. For n odd $G_{n,1}$ is not orientable and hence it is not weakly almost complex.

3.1. **Lemma.** *Let* $2 \le k \le n/2$.

(i) (Borel-Hirzebruch [4, p. 526]) *For* $n \ge 5$, $G_{n,2}$ *is not almost complex.*

(ii) $G_{n,k}$ *is not weakly almost complex if* n *is odd, or if* $k \ge 3$, *and* $n - k \ge 16$.

(iii) $G_{4,2}$ *and* $G_{6,3}$ *are weakly almost complex.* $G_{4,2}$ *is not almost complex.*

Proof of (ii). For n odd, $G_{n,k}$ is not orientable, and is therefore not weakly almost complex. The remaining cases now follow from Theorem 1.1, the naturality of the homomorphism r, and the observation that $\tau\widetilde{G}_{n,k} \approx \pi^*(\tau G_{n,k})$, where $\pi : \widetilde{G}_{n,k} \to G_{n,k}$ is the double covering map.

Proof of (iii). Denote by γ and β the canonical k- and $(n-k)$-plane bundles over $G_{n,k}$. According to Lam [9], a stable normal bundle to $G_{n,k}$ is $\lambda^2(\gamma) \oplus \lambda^2(\beta)$. When $n = 4$ and $k = 2$, $\lambda^2(\gamma) \approx \lambda^2(\beta) \approx \psi$. Hence the stable normal bundle in this case is in the image of r. It follows that $G_{4,2}$ is weakly almost complex.

On $G_{6,3}$, $\lambda^2(\gamma) \approx \gamma \otimes_{\mathbb{R}} \psi$, and $\lambda^2(\beta) \approx \beta \otimes_{\mathbb{R}} \psi$ as can be shown using [7, Proposition 10.3, Chapter 12] the Hodge duality $\lambda^{n-1}(\nu) \approx \nu$ for orientable n-plane bundles, and $\theta \otimes_{\mathbb{R}} \theta \approx \varepsilon$ for any line bundle θ. Thus, on $G_{6,3}$,

$$\lambda^2(\gamma) \oplus \lambda^2(\beta) \approx (\gamma \otimes_{\mathbb{R}} \psi) \oplus (\beta \otimes_{\mathbb{R}} \psi) \approx (\gamma \oplus \beta) \otimes_{\mathbb{R}} \psi \approx 6\varepsilon \otimes_{\mathbb{R}} \psi \approx 6\psi,$$

which is in the image of r. It follows that $G_{6,3}$ is weakly almost complex.

To see that $G_{4,2}$ is not almost complex, notice, first, that $H^i(G_{4,2}; \mathbb{Q}) \cong \mathbb{Q}$ for $i = 0, 4$, and is zero for $i \neq 0, 4$. Since 2ψ is a stable normal bundle for $G_{4,2}$, we have the following formula for the rational Pontrjagin class $p_1(G_{4,2}) \in H^4(G_{4,2}; \mathbb{Q})$:

$$p_1(G_{4,2}) = -p_1(2\psi) = c_2(2\psi \otimes_{\mathbb{R}} \mathbb{C}) = (c_1(\psi \otimes_{\mathbb{R}} \mathbb{C}))^2 = 0 \tag{3.2}$$

because $H^2(G_{4,2}; \mathbb{Q}) = 0$. On the other hand, if $\tau = \tau G_{4,2}$ were a complex vector bundle, then [12],

$$-p_1(G_{4,2}) = 2c_2(\tau) - (c_1(\tau))^2 = 2c_2(\tau).$$

Since the top Chern class of τ is its Euler class, we must have $-\frac{1}{2}\langle p_1(G_{4,2}), [G_{4,2}]\rangle = \langle c_2(\tau), [G_{4,2}]\rangle = \chi(G_{4,2}) = 2$, the Euler characteristic of $G_{4,2}$. Hence $p_1(G_{4,2}) \neq 0$, contradicting (3.2). This shows that $G_{4,2}$ is not almost complex.

As a corollary to the above theorem and Theorem 1.1., we obtain the following:

3.3. **Theorem.**

(i) *A product of any finite number of oriented Grassmann manifolds, $\widetilde{G}_{n,k}$'s, (resp. the Grassmann manifolds $G_{m,p}$'s) is not weakly almost complex if, for one of the factors, $n \geq 7$ is odd, $3 \leq k \leq n-3$; or $k \geq 3$, $(n-k) \geq 16$; (resp. m is odd; or $p \geq 3$, $(m-p) \geq 16$).*

(ii) *The oriented flag manifold*

$$\widetilde{G}(n_1, n_2, \dots, n_s) = SO(n)/(SO(n_1) \times SO(n_2) \times \cdots \times SO(n_s)),$$

where $n = n_1 + \cdots + n_s$ is not weakly almost complex if, for some $i \neq j$, any one of the following holds: (a) *$n_i + n_j$ is odd, $n_i, n_j \geq 3$;* (b) *$n_i \geq 3$, $n_j \geq 16$.*

(iii) (Korbaš) *Let $s \geq 3$. The flag manifold*

$$G(n_1, n_2, \dots, n_s) = O(n)/(O(n_1) \times \cdots \times O(n_s))$$

is weakly almost complex if and only if $n_1 = \cdots = n_s = 1$. $M = G(1, \dots, 1)$ is almost complex if $\dim M = \binom{n}{2}$ *is even.*

Proof. (i), (ii), and parts of (iii) follow from the observation that in each of the cases the manifold in question can be regarded as the total space of a differentiable bundle with fiber, an oriented Grassmann manifold, or a Grassmann manifold which is not weakly almost complex by Theorem 1.1 or 3.1 (cf. [13]).

Korbaš proves (iii) by a Stiefel-Whitney class argument. The positive results follow from the fact that $G(1, \ldots, 1)$ is parallelizable. See [15] for details.

3.4. *Remarks.*

(i) M. Markl [10] has observed using his 'J-genus' that $(\widetilde{G}_{7,3})^n$ is not almost complex. He shows, also, that none of the quaternionic flag manifolds other than $\mathbb{H}G(1, \ldots, 1)$ admits a weakly almost complex structure, using the corresponding negative results of Hsiang and Szczarba [6] for quaternionic Grassmannians.

(ii) J. Korbaš [8] has shown that $\widetilde{G}(n_1, \ldots, n_s)$, $s \geq 2$, $n_1 \equiv \cdots \equiv n_{s-1} \equiv 0 \bmod 4$, $n_2 \equiv 1 \bmod 2$, is not almost complex. For $n_s \geq 3$, this is weaker than the result of Theorem 3.3(ii), whereas our theorem does not cover completely the case $n_s = 1$.

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